On normal space : OR, Og

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Abstract :- In this paper we will study OR-normal space and Og-normal space and get to know their characteristics from one hand and their relation with normal space under certain condition on the other hand. Also, we will justify the relation between Og-normal space and Og-regular space under certain condition.

Keywords: OR-normal space and Og-normal space and Og-regular space and symmetric space and g-closed set and rg-open set

1 Introduction

We studied two other types of normal space and regular space in this paper, so we introduced OR-normal space as a generalization for normal space and studied some of its properties using the g-open set and the g-closed rg-open set .

Under certain conditions, we also justified its relationship with normal space .

We also introduced the concept of Og-normal space and its relationship with normal space, OR-normal space, and Og-regular space. Finally, we introduced a chart that justified the relationship between the spaces mentioned above.

2 Preliminaries

Definition 2.1[1] Let (X, T) be a topological space. Then the space (X, T) is called a Normal Space if and only if for each pair of disjoint closed subsets $A \text{ and } B$ of X, there exist open sets $U \text{ and } V$**,** such that $A \subseteq U \text{ and } B \subseteq V$

Definition 2.2[2] A topological space X is said to be regular space if and only if $\forall F$ closed, $\forall p \notin F, \quad \exists G \text{ and } H \in T: p \in G \text{ and } F \subseteq H, \ G \cap H = \emptyset$

Proposition 2.3 Every regular closed set is closed set.

Proposition 2.4 [3] Let X be a topological space, if A a subset in X then: $(A^{\circ})^c = \overline{(A^c)}$.

Proposition 2.5 Each closed set be g-closed set

Corollary 2.6 Every regular closed set be g-closed set.

3 OR, Og-normal space

Definition 3.1 Let X be a topological space, then X is said to be OR-normal space, for each two disjoint regular closed set A and B in X there exists a two disjoint open sets $U \text{ and } V$ in X, such that $A \subseteq U \text{ and } B \subseteq V$

Example 3.2 Let $X = \{a, b, c, d\}$ and
$T_x = \{x, \emptyset, \{a\}, \{b, c, d\}, \{a, c, d\}, \{a, b, c\}, \{b, c\}, \{a, c\}, \{c\}, \{c, d\}$
Be a topological space in X
Regular closed set in $X = \{x, \emptyset, \{a\}, \{b, c, d\}\}$
Let $A = \{a\} \text{ and } B = \{b, c, d\} \longrightarrow A \cap B = \emptyset$ (regular closed)
Let $U = \{a\} \text{ and } V = \{b, c, d\} \longrightarrow U \cap V = \emptyset$ (open set)
Such that $A \subseteq U \text{ and } B \subseteq V$
Thus $X$is OR-Normal space.
The following theorem justify the relation between OR-Normal space and Normal space

Theorem 3.3 Every normal space is OR-normal space.
Proof: Let X is normal space
And let $A \text{ and } B$ are two regular closed set in X.

Such that $A \cap B = \emptyset$
Thus A and B are two disjoint closed set in Ӽ, by proposition (2.3)
Therefor X is normal space

Hence, there exists are two disjoint open set U and V in X

Such that, $A \subseteq U$ and $B \subseteq V$

Hence X is OR-normal space.

Remark 3.4 The convers of the above theorem is not necessarily true always as below.

Example 3.5

From example (3.2), clear , X is OR-normal space
To proof X is not normal space
Take $A = \{a, b\}$ and $B = \{d\}$ are two disjoint closed set in X
But, is not there exists are two disjoint open set counting $\{a, b\}$ and $\{d\}$
Thus X is not normal space .
The following theorem justify if Ӽ is OR- normal space and R- space then it will be normal space

Theorem 3.6 Let Ӽ is OR-normal space and R-space, then Ӽ be normal space .

Proof : Let Ӽ is OR-normal space, R-space
And let A and B are two closed set in Ӽ
Such that $A \cap B = \emptyset$
Since Ӽ is R-space
Hence, A and B are two disjoint regular closed set in Ӽ
Therefore, Ӽ is OR-normal space
There exists are two disjoint open set U and V in Ӽ
Such that $A \subseteq U$ and $B \subseteq V$
Thus, Ӽ is normal space.
Now we are introducing the following Lemma which is exist in **[4]** without proof. we will give the detail of the proof for it, because we will need it to proof the following theorem

Lemma 3.7 Let Ӽ a topological space and $A \subseteq X$, then A be rg-open set if and only if $F \subseteq A^{\circ}$, whenever $F \subseteq A$, where F is regular closed set in Ӽ.

Proof: The first direction ($\Longrightarrow$)
assume that A is rg-open set in Ҳ

Let $F \subseteq A$ where F regular closed set in Ҳ
To proof $F \subseteq A^{\circ}$
Therefore $F \subseteq A$,
Thus $A^c \subseteq F^c$
Therefore A^c is rg-closed set and F^c is regular open set
Hence $\overline{(A^c)} \subseteq F^c$
But $\overline{(A^c)} \subseteq (A^{\circ})^c$ by proposition (2.4)
Thus $(A^{\circ})^c \subseteq F^c$
Hence $F \subseteq A^{\circ}$
Second direction ($\Longleftarrow$)
Let $F \subseteq A^{\circ}$every regular closed set $F \subseteq A$
To proof A is rg-open set in Ҳ (to proof A^c is rg-closed set in Ҳ)
And let $A^c \subseteq U$ where U is regular open set in Ҳ
Thus $U^c \subseteq A$
Therefore U^c is regular closed set in Ҳ
Hence $U^c \subseteq A^{\circ}$ as per the assumption
Hence $(A^{\circ})^c \subseteq U$
But $(A^{\circ})^c = \overline{(A^c)}$ by proposition (2.4)
Thus $\overline{(A^c)} \subseteq U$
Thus A^c is rg-closed set in Ҳ
Hence A is rg-open set in Ҳ
Now we are introducing the most important theorem which gives a lot of character istics OR-normal space by using the two sets g-open and rg-open.

Theorem 3.8 For a topological space (Ҳ,Ҭ) the following are equivalent
1-Ҳ is OR-normal space
2-for every are two disjoint regular closed set $A \text{ and } B$ in Ҳ, there exists are two disjoint rg-open set $U \text{ and } V$ in Ҳ
Such that $A \subseteq U \text{ and } B \subseteq V$
3-for every are two disjoint regular closed set $A \text{ and } B$ in Ҳ, there exists are two disjoint rg-open set $U \text{ and } V$ in Ҳ such that
$A \subseteq U \text{ and } B \subseteq V$
4- for every A is regular closed set in Ҳ and for each V is regular open set in such that $A \subseteq U \subseteq \overline{U} \subseteq V$
proof(1⟹2)
let $A \text{ and } B$ are regular closed set in Ҳ, $A \cap B = \emptyset$
so, there are two disjoint open set $U \text{ and } V$ in Ҳ ,
according to (1)
such that $A \subseteq U \text{ and } B \subseteq V$,
U and V are two disjoint g-open set in Ҳ.

Proof(2⟹1)
Let $A\ and\ B$ are two regular closed set in Ҳ, such that $A \cap B = \emptyset$
Hence, there exists are two disjoint g-open set $U\ and\ V$ in Ҳ,
According to (2).
Such that $A \subseteq U\ and\ B \subseteq V$
Since for every g-open set is rg-open set,
Thus $U\ and\ V$ are two disjoint rg-open set in Ҳ
Now, let $U_1 = U^{\circ}$ and $V_1 = V^{\circ}$
Thus $U_1\ and\ V_1$ are two disjoint open set in Ҳ,
By lemma (3.7)
$A \subseteq U_1\ and\ B \subseteq U_1$
Hence Ҳ is OR-normal space
Proof (2⟹3)
See (b), by lemma (3.7)

Proof (3⟹4)
let A regular closed set in Ҳ,

and let V regular open set in Ҳ, such that $A \subseteq V$.
Thus $A\ and\ V^c$ are two regular closed set and $A \cap V^c = \emptyset$
Hence, there exists are two disjoint rg-open set $U\ and\ W$ in Ҳ
According two phrase (3)
Such that $A \subseteq U\ and\ V^c \subseteq W$,
By lemma (2.7), $V^c \subseteq W^{\circ}$
Since $U \cap W = \emptyset$
Thus $U \cap W^{\circ} = \emptyset$
Hence $U \subseteq (W^{\circ})^c$ and $\overline{U} \subseteq \overline{(W^{\circ})^c} \subseteq (W^{\circ})^c$,
Since $V^c \subseteq W^{\circ}$, thus $(W^{\circ})^c \subseteq V$
Hence $A \subseteq U \subseteq \overline{U} \subseteq (W^{\circ})^c \subseteq V$.
Proof (4⟹1)
Let $A\ and\ B$ are two regular closed set in Ҳ, such that $A \cap B = \emptyset$
Thus $A \subseteq B^c$ where B^c is regular open set in Ҳ,
Such that $A \subseteq G \subseteq \overline{G} \subseteq B^c$
Let $U = G^{\circ}$ where U is open set in Ҳ
Thus by lemma (3.7), $A \subseteq U$
Let $V = \left(\overline{G}\right)^c$ where V is open set in Ҳ,
Thus $A \subseteq U\ and\ B \subseteq V$
Since $V \cap U = \emptyset$
Hence Ҳ is OR-normal space

Now we are introducing og-normal space.

Definition 3.9 Let Ҳ a topological space, Ҳ is said to be Og-normal space, for each two disjoint
g-closed set, $A\ and\ B$ in Ҳ, there exists are two disjoint open set $U\ and\ V$ in Ҳ,
such that $A \subseteq U\ and\ B \subseteq V$

The following theorem justify the relation between og-normal space and
normal space.

Theorem 3.10 Every Og-normal space is normal space.

Proof: Clear, by proposition (2.5)

Remark 3.11 The convers of the above theorem is not true always as below.

Example 3.12 let $Ҳ = \{a, b, c\} and\ \Upsilon_x = \{Ҳ, \emptyset, \{a\}, \{b, c\}\}$, a topological space in Ҳ
g-closed set in $Ҳ = \{Ҳ, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$
To proof Ҳ is normal space
$A = \{a\} and\ B = \{b, c\} \longrightarrow A \cap B = \emptyset$ (closed set)
$U = \{a\} and\ V = \{b, c\} \longrightarrow U \cap V = \emptyset$ (open set)
Such that $A \subseteq U\ and\ B \subseteq V$
Hence Ҳ is normal space
To proof Ҳ is og-normal space
$A = \{c\} and\ B = \{b\} \longrightarrow A \cap B = \emptyset$ (g-closed set)
Clear, not there exists are two disjoint open set counting {c} and {b}
Thus, Ҳ is not og-normal space.
Now we are introducing the following theorem which justify if Ҳ is normal
space and $T_{\frac{1}{2}}$ –space then og-normal space

Theorem 3.13[5]: Let Ҳ is normal space, if Ҳ is $T_{\frac{1}{2}}$-space, then Ҳ is og-normal space.
Let Ҳ is normal space, $T_{\frac{1}{2}}$-space
And let $A\ and\ B$ are two g-closed set in Ҳ
Such that $A \cap B = \emptyset$
Since Ҳ is $T_{\frac{1}{2}}$-space
Thus $A\ and\ B$ are two disjoint closed set in Ҳ
Since Ҳ is normal space.
Hence, there exists are two disjoint open set $U\ and\ V$ in Ҳ
Such that $A \subseteq U\ and\ B \subseteq V$
Thus, Ҳ is og-normal space
The following theorem justify the relation between Og-normal space and
OR-normal space

Theorem 3.14 Every Og-normal space is OR-normal space.

Proof:
Let Ჯ is og-normal space
And let $A\ and\ B$ are two regular closed set in Ჯ
Such that $A \cap B \cap = \emptyset$
Hence $A\ and\ B$ are two disjoint g-closed set in Ჯ, by corollary (2.6)
Since Ჯ is og-normal space,
Thus, there exists are two disjoint open set, $U\ and\ V$ in Ჯ,
Such that $A \subseteq U\ and\ B \subseteq V$.
Thus, Ჯ is OR-normal space.

Remark 3.15 The converse is not necessarily true from example.

Example 3.16 From example (3.12), clear Ჯ is not og-normal space
To proof Ჯ is OR-normal space.
$Ჯ = \{a, b, c\}\ and\ T_X = \{Ჯ, \emptyset, \{a\}, \{b, c\}\}$
Regular closed set in Ჯ=$\{Ჯ, \emptyset, \{a\}, \{b, c\}\}$
Let $A = \{a\} and\ B = \{b, c\} \longrightarrow A \cap B = \emptyset$ (regular closed)
Let $U = \{a\}$ and $V = \{b, c\} \longrightarrow U \cap V = \emptyset$ (open set)
Such that $A \subseteq U\ and\ B \subseteq V$
Thus, Ჯ is OR-normal space
Now we will introduce the following important theorem which gives definition equivalent to the definition og-normal space

Theorem 3.17 For a topological space (Ჯ,Ţ) the following are equivalent.
1-Ჯis og-normal space.
2-for every are two disjoint g-closed set $A\ and\ B$ in Ჯ, there exists are two open set $U\ and\ V$ in Ჯ,
such that $A \subseteq U\ and\ B \subseteq V$,
$\overline{U} \cap \overline{V} = \emptyset$.

Proof (1⟶2)
Let $A\ and\ B$ are two g-closed set in Ჯ, such that $A \cap B = \emptyset$.
Since Ჯ is og-normal space
Thus, there exists are two disjoint open set $U^{\backslash}\ and\ V$ in Ჯ
Where $A \subseteq U^{\backslash}\ and\ B \subseteq V$
Since $U^{\backslash} \cap V = \emptyset$
Hence $V \subseteq \left(U^{\backslash}\right)^c\ and\ \overline{V} = \overline{((U^{\backslash})^c)} = \left(U^{\backslash}\right)^c$
Thus $\overline{V} \cap U^{\backslash} = \emptyset$,
Now $A\ and\ \overline{V}$ are two disjoint g-closed set in Ჯ
Since Ჯ is og-normal
There exists are two disjoint open set $H\ and\ G$ in Ჯ
Such that $A \subseteq G\ and\ \overline{V} \subseteq H$
Since $H \cap G = \emptyset$
Thus $G \subseteq H^c\ and\ \overline{G} \subseteq \overline{(H^c)} = H^c$

Hence $\overline{G} \cap H = \emptyset$
Therefore $\overline{V} \cap \overline{G} = \emptyset$
Now, let $U = U^{\setminus} \cap G$ where U is open set in Ҳ
Such that $A \subseteq U \; and \; B \subseteq V$
Now, to proof $\overline{V} \cap \overline{U} = \emptyset$
Since $\overline{U} = \overline{(U^{\setminus} \cap G)} \subseteq \overline{U^{\setminus}} \cap \overline{G}$
Thus $\overline{U} \cap \overline{V} \subseteq \overline{U^{\setminus}} \cap \overline{G} \cap \overline{V}$
But $\overline{G} \cap \overline{V} = \emptyset$
Hence $\overline{U} \cap \overline{V} \subseteq \overline{U^{\setminus}} \cap \emptyset = \emptyset$
Hence $\overline{U} \cap \overline{V} = \emptyset$.
Proof (2⟶1)
let $A \; and \; B$ are two g-closed set in Ҳ, such that $A \cap B = \emptyset$
there exists are two open set $U \; and \; V$ in Ҳ, according to phrase (2).
Such that $A \subseteq U \; and \; B \subseteq V$
Since $A \subseteq U \subseteq \overline{U} \; and \; b \subseteq V \subseteq \overline{V}$
Hence $A \cap B \subseteq U \cap V \subseteq \overline{U} \cap \overline{V}$
But $\overline{U} \cap \overline{V} = \emptyset$, according to phrase (2).
Thus $A \cap B \subseteq U \cap V \subseteq \emptyset$
Hence $U \cap V = \emptyset$,
Thus Ҳ is og-normal space
Now we are introducing Og-regular space and we will show in a later theorem
that og-normal space cannot be Og-regular space only if it was symmetric space.

Definition 3.18 Let Ҳ be a topological space, Ҳ is said to be Og-regular space, then for every g-closed set (A) in Ҳ and for every point (a) in Ҳ such that a $\notin A$, there exists are two disjoint open set $U \; and \; V \; in$ Ҳ, such that $A \subseteq U \; and \; a \in V$.

Remark 3.19 The convers of the above theorem is not necessarily true. From example (3.12) where Ҳ is regular space and not og-regular space. Because $\{b\}$ is g-closed set and $c \notin \{b\}$,
Clear, not there exists are two disjoint open set counting {b} and point $\{c\}$ both
of them in row.

Remark 3.20 Every Og-normal space is not necessary Og-regular space.
As shown in the following example,

Example 3.21 Let Ҳ $= \{a, b\} and \; T_x = \{X, \emptyset, \{a\}\}$ a topological space in Ҳ g-closed set in Ҳ=$\{\emptyset,$ Ҳ$, \{b\}\}$
Now, {b} is g-closed set in Ҳ where $a \notin \{b\}$,
There is not exists two disjoint open set in Ҳ counting a and {b},
Both of them in row
Hence Ҳ is not og-regular space,
Clear, Ҳ is og-normal space.

Now we will introducing symmetric space which will give us important condition in a later theorem

Definition 3.22 [6] Recall that a space (X,T) is said to be symmetric space if for any points x and y of X, If $x \in \overline{\{y\}}$ then $y \in \overline{\{x\}}$. See example (3.12).

Lemma 3.23 [6-9] for a topological space (X,T) the following are equivalent.
1-Xis symmetric space.
2-every singleton space {x} be g-closed set every $x \in$ X.

Theorem 3.24 Every og-normal space, symmetric space a topological space (X,T) is og-regular space.

Proof:
Let X is og-normal space, symmetric
And let p any point in X such that $p \notin A$,by lemma (3.23)
$\{p\}$ is g-closed set in X
Since X is Og-normal space and $\{p\} \cap A = \emptyset$,
There exists are two disjoint open set $U\ and\ V$ in X
Such that $A \subseteq U\ and\ \{p\} \subseteq V$
Thus $A \subseteq U\ and\ p \in V,$
Hence X is Og-regular space.

4 Conclusion

At the end of this paper, we are introducing the following chart which justify the relation between OR-normal space and og-normal space from one hand and its relation with normal space on other hand under certain conditions. Like that it justifies the relation between og-normal space and og-regular space under certain condition

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