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Linear Formulas for Estimating the Reliability of Generalized Inverse Weibull Distributi

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A B S T R A C T

Estimation methods that depend on linear formulas to estimate the reliability of the three parameters (α, β, θ) generalized inverse Weibull distribution (GIW) were used in this research, including each of Least Square method (LS), Weight Least Square method (WLS), White method (W), Modified White method (MW) and Linear Regression method (REG). The parameters and reliability of the distribution were estimated in four experiments using simulation to generating the required samples and the results were compared using the mean square error criterion. The results showed: In general the Modified White estimators are better than estimators of other methods.

Keywords: *Generalized Inverse Weibull distribution, estimation methods, simulation and mean square error.*

1. Introduction

In life testing, the Weibull distribution is one of the important continuous distributions. It can be used in many scientific fields, including: failure times, minimum and maximum temperatures, and many other scientific fields, such as medical fields are closely related to application of the Weibull distribution. It has been studied by many researches like Khan and King (2013) they developed a transmuted modified Weibull distribution with three parameters [1]. In (2017), Loganathan and Uma estimated the scale and shape parameters of inverse Weibull distribution [2]. In (2018), Hassan and Nassr they introduced the inverse Weibull generated family [3]. In (2019), Aldahlan introduced a new model named the inverse Weibull inverse exponential distribution [4].

The generalized inverse Weibull (GIW) distribution of three parameters was introduced and studied by de Gusmao, Ortega and Cordeiro in (2011) [5]. The probability density function (pdf), cumulative distribution function (cdf) and reliability function for (GIW) distribution are as follows, respectively [5]:

$$f(x;\alpha,\beta.\theta) = \theta\beta\alpha^{\beta}x^{-(\beta+1)}exp\left(-\theta\left(\frac{\alpha}{x}\right)^{\beta}\right), x > 0 \tag{1}$$

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$$F(x;\alpha,\beta.\theta) = exp\left(-\theta\left(\frac{\alpha}{x}\right)^{\beta}\right), x > 0 \tag{2}$$

$$R(x;\alpha,\beta.\theta) = 1 - exp\left(-\theta\left(\frac{\alpha}{x}\right)^{\beta}\right), x > 0 \tag{3}$$

Whereas α , θ and β are scale and shape parameters, respectively and $\alpha, \beta, \theta > 0$.
The quintile function can be given from equation (2), as follows:

$$x_F = \alpha\left(\frac{-lnF}{\theta}\right)^{-\frac{1}{\beta}} \tag{4}$$

2. Estimation Methods

This section includes a presentation of the estimation method used in this research

2.1 Estimating the initial values of the parameters

The method of finding the initial values of the parameters was suggested, based on the equality of the distribution median with the sample median, as follows [2]:
Median of (GIW) distribution is [1]

$$x_{med} = \alpha\left(\frac{-\,ln(0.5)}{\theta}\right)^{\frac{-1}{\beta}}, \quad -\,ln(0.5) = 0.693147 \tag{5}$$

Using equation (5), we get the initial estimates of the parameters and as follows:

$$\hat{\alpha}_0 = \; x_{med}\left(\frac{0.693147}{\theta}\right)^{\frac{1}{\beta}}, \tag{6}$$

Now,

$$\left(\frac{x_{med}}{\alpha}\right)^{-\beta} = \frac{0.693147}{\theta} \tag{7}$$

By taking the natural logarithm for equation (7), getting:

$$-\beta ln\left(\frac{x_{med}}{\alpha}\right) = ln\left(\frac{0.693147}{\theta}\right)$$

$$\hat{\beta}_0 = -\frac{ln\left(\frac{0.693147}{\theta}\right)}{ln\left(\frac{x_{med}}{\alpha}\right)} \tag{8}$$

And
From equation (7), getting:

$$\hat{\theta}_0 = (0.693147)\left(\frac{x_{med}}{\alpha}\right)^{\beta} \tag{9}$$

$\hat{\alpha}_0, \hat{\beta}_0$ and $\hat{\theta}_0$ in equations (6), (8) and (9) are the initial values for parameters and x_{med} it's possible to get it from the generating sample.

2.2 Least Square Method (LS)

This method depends on converting the extracted formula from the inverse of the distribution to the linear regression formula, as follows [6]:
The plotting position formula is

$$p_i = \frac{i}{n+1}, i = 1,2, \dots, n \tag{10}$$

By substituting (cdf) for the distribution by equation (10) into equation (4), getting:

$$x_{(p_i)} = \alpha \left(\frac{-ln(p_i)}{\theta}\right)^{-\frac{1}{\beta}} \tag{11}$$

Let $z_i = -ln(p_i)$, equation (11) become:

$$x_{(p_i)} = \alpha \left(\frac{z_i}{\theta}\right)^{-\frac{1}{\beta}} \qquad (12$$

Taking the natural logarithm for equation (12), getting:

$$ln\left(x_{(p_i)}\right) = ln(\alpha) - \frac{1}{\beta} ln\left(\frac{z_i}{\theta}\right) \qquad (13$$

The formula for linear regression is:

$$Y_i = a + b\emptyset_i + \epsilon_i \qquad (14$$

Comparing the equation (13) with the equation (14), getting:

$$Y_i = ln\left(x_{(p_i)}\right)$$

$$a = ln(\alpha) \Rightarrow \alpha = \exp(a) \qquad (15$$

$$b = \frac{-1}{\beta} \Rightarrow \beta = \frac{-1}{b} \tag{16}$$

$$\emptyset_i = ln\left(\frac{z_i}{\theta}\right)$$

From equation (14), getting:

$$\epsilon_i = Y_i - a - b\emptyset_i \tag{17}$$

Squaring equation (17), and taking the sum of both sides, getting:

$$\sum_{i=1}^{n} \epsilon_i^{\,2} = \sum_{i=1}^{n} (Y_i - a - b\emptyset_i)^2$$

Let

$$\delta = \sum_{i=1}^{n} \epsilon_i^{\,2}$$

$$\delta = \sum_{i=1}^{n} (Y_i - a - b\emptyset_i)^2 \tag{18}$$

Now, taking partial derivative for equation (18) with respect to (a) and (b), and equality to zero, getting two equations are respectively:

$$\frac{\partial\delta}{\partial a} = 2\sum_{i=1}^{n} (Y_i - a - b\emptyset_i)\,(-1) = 0$$

$$\sum_{i=1}^{n} Y_i = na + b\sum_{i=1}^{n} \emptyset_i \tag{19}$$

$$\frac{\partial\delta}{\partial b} = 2\sum_{i=1}^{n}(Y_i - a - b\emptyset_i)\,(-\emptyset_i) = 0$$

$$\sum_{i=1}^{n} Y_i\,\emptyset_i = a\sum_{i=1}^{n} \emptyset_i + b \tag{20}$$

By multiplication equation (19) by $\sum_{i=1}^{n} \emptyset_i$ and equation (20) by (n), getting:

$$\sum_{i=1}^{n} Y_i \sum_{i=1}^{n} \emptyset_i = na\sum_{i=1}^{n} \emptyset_i + b\left(\sum_{i=1}^{n} \emptyset_i\right)^2 \tag{21}$$

$$n\sum_{i=1}^{n} Y_i\emptyset_i = na\sum_{i=1}^{n} \emptyset_i + nb\sum_{i=1}^{n} {\emptyset_i}^2 \tag{22}$$

Subtracting equation (22) from equation (21), getting:

$$\sum_{i=1}^{n} Y_i \sum_{i=1}^{n} \emptyset_i - n\sum_{i=1}^{n} Y_i\,\emptyset_i = b\left((\sum_{i=1}^{n} \emptyset_i)^2 - n\sum_{i=1}^{n} {\emptyset_i}^2\right)$$

$$\hat{b}_{LS} = \frac{\sum_{i=1}^{n} Y_i \sum_{i=1}^{n} \emptyset_i - n\sum_{i=1}^{n} Y_i\,\emptyset_i}{(\sum_{i=1}^{n} \emptyset_i)^2 - n\sum_{i=1}^{n} {\emptyset_i}^2} \tag{23}$$

By substituting value of $(\hat{b}_{LS})$ from equation (23) into equation (16), getting:

$$\hat{\beta}_{LS} = \frac{-1}{\hat{b}_{LS}} \tag{24}$$

From equations (19) and (23), getting:

$$\hat{a}_{LS} = \frac{\sum_{i=1}^{n} Y_i - \hat{b}_{LS}\sum_{i=1}^{n} \emptyset_i}{n} \tag{25}$$

By substituting value of $(\hat{a}_{LS})$ from equation (25) into equation (15), getting:

$$\hat{\alpha}_{LS} = \ exp\left(\frac{\sum_{i=1}^{n} Y_i + \frac{\sum_{i=1}^{n} \emptyset_i}{\hat{\beta}_{LS}}}{n}\right) \tag{26}$$

Now, by taking the natural logarithm for equation (2), we get:

$$ln\big(F(x)\big) = -\theta\left(\frac{\alpha}{x}\right)^{\beta}$$

$$\theta = \frac{-ln\big(F(x)\big)}{\left(\frac{\alpha}{x}\right)^{\beta}} \tag{27}$$

Replacing $F(x)$ in equation (27) by plotting position in equation (10), getting:

$$\hat{\theta}_{LS} = \frac{-\sum_{i=1}^{n} ln(p_i)}{\sum_{i=1}^{n}\left(\frac{\hat{\alpha}_{LS}}{x_{(i)}}\right)^{\hat{\beta}_{LS}}} \tag{28}$$

By substituting equations (24), (26) and (28) in to equation (3), getting:

$$\hat{R}_{LS} = 1 - \exp\left(-\hat{\theta}_{LS}\left(\frac{\hat{\alpha}_{LS}}{x}\right)^{\hat{\beta}_{LS}}\right) \tag{29}$$

3.2 Weighted Least Square Method (WLS)

This method suggested by Swain, Venkatraman, and Wilson (1988) [7]. When the conventional least squares assumption of constant variance in the errors is violated, the weighted least squares approach can be utilized. This method is based on equation (17) by divide it (Y_i), squaring it and taking the sum of both sides, getting [8]:

$$\sum_{i=1}^{n}\left(\frac{\epsilon_i}{Y_i}\right)^2 = \sum_{i=1}^{n}\left(1 - \frac{1}{Y_i}a - \frac{\emptyset_i}{Y_i}b\right)^2 \tag{30}$$

Let

$$\delta = \sum_{i=1}^{n}\left(\frac{\epsilon_i}{Y_i}\right)^2, \sigma_i = \frac{1}{Y_i}, \varphi_i = \frac{\emptyset_i}{Y_i}$$

Substituted in equation (30), getting:

$$\delta = \sum_{i=1}^{n}(1 - \sigma_i a - \varphi_i b)^2 \tag{31}$$

By taking the partial derivative for equation (31) with respect to (a) and (b), and equaling to zero, getting two equations are respectively:

$$\frac{\partial\delta}{\partial a} = 2\sum_{i=1}^{n}(1 - a\sigma_i - b\varphi_i)(-\sigma_i) = 0$$

$$a\sum_{i=1}^{n}\sigma_i^{\,2} = \sum_{i=1}^{n}\sigma_i - b\sum_{i=1}^{n}\sigma_i\varphi_i$$

$$a = \frac{\sum_{i=1}^{n}\sigma_i - b\sum_{i=1}^{n}\sigma_i\varphi_i}{\sum_{i=1}^{n}\sigma_i^{\,2}} \tag{32}$$

Now,

$$\frac{\partial\delta}{\partial b} = 2\sum_{i=1}^{n}(1 - a\sigma_i - b\varphi_i)(-\varphi_i) = 0$$

$$a\sum_{i=1}^{n}\sigma_i\varphi_i = \sum_{i=1}^{n}\varphi_i - b\sum_{i=1}^{n}{\varphi_i}^2$$

$$a = \frac{\sum_{i=1}^{n}\varphi_i - b\sum_{i=1}^{n}{\varphi_i}^2}{\sum_{i=1}^{n}\sigma_i\varphi_i} \tag{33}$$

By equaling equation (32) with equation (33), getting:

$$\sum_{i=1}^{n}\sigma_i\sum_{i=1}^{n}\sigma_i\varphi_i - b\left(\sum_{i=1}^{n}\sigma_i\varphi_i\right)^2 = \sum_{i=1}^{n}\varphi_i\sum_{i=1}^{n}{\sigma_i}^2 - b\sum_{i=1}^{n}{\varphi_i}^2\sum_{i=1}^{n}{\sigma_i}^2$$

$$b\sum_{i=1}^{n}{\varphi_i}^2\sum_{i=1}^{n}{\sigma_i}^2 - b\left(\sum_{i=1}^{n}\sigma_i\varphi_i\right)^2 = \sum_{i=1}^{n}\varphi_i\sum_{i=1}^{n}{\sigma_i}^2 - \sum_{i=1}^{n}\sigma_i\sum_{i=1}^{n}\sigma_i\varphi_i$$

$$\hat{b}_{WLS} = \frac{\sum_{i=1}^{n}\varphi_i\sum_{i=1}^{n}{\sigma_i}^2 - \sum_{i=1}^{n}\sigma_i\sum_{i=1}^{n}\sigma_i\varphi_i}{\sum_{i=1}^{n}{\varphi_i}^2\sum_{i=1}^{n}{\sigma_i}^2 - \left(\sum_{i=1}^{n}\sigma_i\varphi_i\right)^2} \tag{34}$$

From equation (16) and equation (34), getting:

$$\hat{\beta}_{WLS} = \frac{\left(\sum_{i=1}^{n}\sigma_i\varphi_i\right)^2 - \sum_{i=1}^{n}{\varphi_i}^2\sum_{i=1}^{n}{\sigma_i}^2}{\sum_{i=1}^{n}\varphi_i\sum_{i=1}^{n}{\sigma_i}^2 - \sum_{i=1}^{n}\sigma_i\sum_{i=1}^{n}\sigma_i\varphi_i} \tag{35}$$

From equation (15) and equations (32) and (35), getting:

$$\hat{\alpha}_{WLS} = exp\left(\frac{\sum_{i=1}^{n}\sigma_i + \frac{1}{\hat{\beta}_{WLS}}\sum_{i=1}^{n}\sigma_i\varphi_i}{\sum_{i=1}^{n}{\sigma_i}^2}\right) \tag{36}$$

Now, from equation (27), getting:

$$\hat{\theta}_{WLS} = \frac{-\sum_{i=1}^{n}ln(p_i)}{\sum_{i=1}^{n}\left(\frac{\hat{\alpha}_{WLS}}{x_{(i)}}\right)^{\hat{\beta}_{WLS}}} \tag{37}$$

By substituting equations (35), (36) and (37) into equation (3), getting:

$$\hat{R}_{WLS} = 1 - exp\left(-\hat{\theta}_{WLS}\left(\frac{\hat{\alpha}_{WLS}}{x}\right)^{\hat{\beta}_{WLS}}\right) \tag{38}$$

4.2 White Method (W)

White's method depends on converting the reliability function into an equation form similar to the linear regression equation and then estimating the parameters and as follows [9]:
From equation (3), getting:

$$1 - R(x) = exp\left(-\theta\left(\frac{\alpha}{x}\right)^{\beta}\right) \tag{39}$$

By taking the natural logarithm for equation (39), getting:

$$-ln\big(1 - R(x)\big) = \theta\left(\frac{\alpha}{x}\right)^{\beta} \tag{40}$$

Now taking the natural logarithm for equation (40), getting:

$$ln\Big(-ln\big(1 - R(x)\big)\Big) = ln(\theta) + \beta ln\left(\frac{\alpha}{x}\right) \tag{41}$$

Equating equation (2) with equation (10), getting:

$$F(x_i) = \frac{i}{n+1}\ , i = 1,2, \dots, n$$

Since

$$R(x_i) = 1 - F(x_i)$$

Then:

$$R(x_i) = 1 - \frac{i}{n+1}\ , i = 1,2, \dots, n \tag{42}$$

Comparing equation (41) with equation (14), getting:

$$Y_i = ln\Big(-ln\Big(1 - R\big(x_{(i)}\big)\Big)\Big), \bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n} \quad i = 1,2, \dots, n \tag{43}$$

$$a = ln(\theta) \Rightarrow \theta = exp(a) \tag{44}$$

$$b = \beta \tag{45}$$

$$\emptyset_i = ln\left(\frac{\alpha}{x_{(i)}}\right), i = 1,2, \dots, n \tag{46}$$

$$\widehat{\emptyset}_i = ln\left(\frac{\alpha_0}{x_{(i)}}\right), \overline{\emptyset} = \frac{\sum_{i=1}^{n} \widehat{\emptyset}_i}{n} \quad , i = 1,2, \dots, n \tag{47}$$

Formula for extracting the value of (b) by White's method is

$$\hat{b}_W = \frac{\sum_{i=1}^{n}\big(\widehat{\emptyset}_i - \overline{\emptyset}\big)(Y_i - \bar{Y})}{\sum_{i=1}^{n}\big(\widehat{\emptyset}_i - \overline{\emptyset}\big)^2} \tag{48}$$

Substituting equations (43) and (47) into equation (48), getting:

$$\hat{\beta}_W = \hat{b}_W \tag{49}$$

From equation (14), getting:

$$\hat{a}_W = \bar{Y} - \hat{\beta}_W \overline{\emptyset} \tag{50}$$

Substitute equation (50) in equation (44), getting:

$$\hat{\theta}_W = exp(\bar{Y} - \hat{\beta}_W \bar{\emptyset}) \tag{51}$$

From equation (46), getting:

$$\sum_{i=1}^{n} exp(\emptyset_i) = \alpha \sum_{i=1}^{n} \left(\frac{1}{x_{(i)}}\right)$$

$$\hat{\alpha}_W = \frac{\sum_{i=1}^{n} exp(\hat{\emptyset}_i)}{\sum_{i=1}^{n} \left(\frac{1}{x_{(i)}}\right)} \tag{52}$$

By substituting equations (49), (51) and (52) into equation (3), getting:

$$\hat{R}_W = 1 - exp\left(-\hat{\theta}_W \left(\frac{\hat{\alpha}_W}{x}\right)^{\hat{\beta}_W}\right) \tag{53}$$

5.2 Modification White Method (MW)

This method differs from White's method in that it depends on converting the risk function into an equation similar to the linear regression formula and not on the reliability function. [9] The hazard function for (GIW) distribution is [1]

$$h(x) = \frac{\theta\beta\alpha^{\beta} x^{-(\beta+1)} exp\left(-\theta \left(\frac{\alpha}{x}\right)^{\beta}\right)}{1 - exp\left(-\theta \left(\frac{\alpha}{x}\right)^{\beta}\right)} \tag{54}$$

$$\frac{1}{h(x)} = \frac{1 - exp\left(-\theta \left(\frac{\alpha}{x}\right)^{\beta}\right)}{\theta\beta\alpha^{\beta} x^{-(\beta+1)} exp\left(-\theta \left(\frac{\alpha}{x}\right)^{\beta}\right)}$$

$$\frac{1}{h(x)} = \frac{1}{\theta\beta\alpha^{\beta} x^{-(\beta+1)} exp\left(-\theta \left(\frac{\alpha}{x}\right)^{\beta}\right)} - \frac{1}{\theta\beta\alpha^{\beta} x^{-(\beta+1)}}$$

$$\frac{1}{\frac{1}{h(x)} + \frac{1}{\theta\beta\alpha^{\beta} x^{-(\beta+1)}}} = \theta\beta\alpha^{\beta} x^{-(\beta+1)} exp\left(-\theta \left(\frac{\alpha}{x}\right)^{\beta}\right) \tag{55}$$

By taking the natural logarithm for equation (55), getting:

$$ln\left(\frac{1}{\frac{1}{h(x)} + \frac{1}{\theta\beta\alpha^{\beta} x^{-(\beta+1)}}}\right) = ln(\theta) + ln(\beta) + \beta ln(\alpha) - (\beta+1) ln(x) - \theta \left(\frac{\alpha}{x}\right)^{\beta}$$

$$ln\left(\frac{1}{\frac{1}{h(x)}+\frac{1}{\theta\beta\alpha^{\beta}x^{-(\beta+1)}}}\right)-ln(\theta)-\beta ln(\alpha)+(\beta+1)ln(x)=ln(\beta)-\theta\left(\frac{\alpha}{x}\right)^{\beta} \quad (56)$$

By comparison equation (56) with equation (14), getting:

$$Y_i=ln\left(\frac{1}{\frac{1}{h_0\left(x_{(i)}\right)}+\frac{1}{\theta_0\beta_0\alpha_0^{\beta_0}x_{(i)}{}^{-(\beta_0+1)}}}\right)-ln(\theta_0)-\beta_0 ln(\alpha_0)+(\beta_0+1)ln\left(x_{(i)}\right),i=1,2\ldots n \quad (57)$$

Whereas; $h_0\left(x_{(i)}\right)=\frac{\theta_0\beta_0\alpha_0^{\beta_0}x_{(i)}{}^{-(\beta_0+1)}exp\left(-\theta_0\left(\frac{\alpha_0}{x_{(i)}}\right)^{\beta_0}\right)}{1-exp\left(-\theta_0\left(\frac{\alpha_0}{x_{(i)}}\right)^{\beta_0}\right)}$ and $\bar{Y}=\frac{\sum_{i=1}^{n}Y_I}{n}$

$$a=ln(\beta)\Rightarrow\beta=exp(a) \quad (58)$$

$$b=\theta \quad (59)$$

$$\emptyset_i=-\left(\frac{\alpha}{x_{(i)}}\right)^{\beta} \quad (60)$$

$$\widehat{\emptyset}_i=-\left(\frac{\alpha_0}{x_{(i)}}\right)^{\beta_0},\overline{\emptyset}=\frac{\sum_{i=1}^{n}\widehat{\emptyset}_i}{n}\ ,i=1,2,\ldots,n \quad (61)$$

Substituting equations (57) and (61) into equation (48), getting:

$$\hat{\theta}_{MW}=\hat{b}_{\ MW} \quad (62)$$

By using equation (14), getting:

$$\hat{a}_{MW}=\bar{Y}-\hat{\theta}_{MW}\overline{\emptyset} \quad (63)$$

Substituting equation (63) into equation (58), getting:

$$\hat{\beta}_{MW}=exp\left(\bar{Y}-\hat{\theta}_{MW}\overline{\emptyset}\ \right) \quad (64)$$

Now, taking the natural logarithm for equation (60), getting:

$$ln(-\emptyset_i)=\beta ln\left(\frac{\alpha}{x_{(i)}}\right)$$

$$\frac{\alpha}{x_{(i)}}=exp\left(\frac{ln(-\emptyset_i)}{\beta}\right)$$

$$\sum_{i=1}^{n}\alpha=\sum_{i=1}^{n}x_{(i)}exp\left(\frac{ln(-\emptyset_i)}{\beta}\right)$$

$$\hat{\alpha}_{MW}=\frac{\sum_{i=1}^{n}x_{(i)}exp\left(\frac{ln(-\widehat{\emptyset}_i)}{\hat{\beta}_{MW}}\right)}{n} \quad (65)$$

By substituting equations (62), (64) and (65) in equation (3), getting:

$$\hat{R}_{MW}=1-exp\left(-\hat{\theta}_{MW}\left(\frac{\hat{\alpha}_{MW}}{x}\right)^{\hat{\beta}_{MW}}\right) \quad (66)$$

6.2 Linear Regression Method (REG)

This method is based on finding estimates for the parameters of the distribution by converting the (cdf) into a linear regression equation as follows [10]:
By taking the natural logarithm for equation (2), getting:

$$ln\left(F\left(x_{(i)}\right)\right) = -\theta\left(\frac{\alpha}{x_{(i)}}\right)^{\beta}$$

$$-ln\left(F\left(x_{(i)}\right)\right) = \theta\left(\frac{\alpha}{x_{(i)}}\right)^{\beta} \tag{67}$$

Now, taking the natural logarithm for equation (67), getting:

$$ln\left(-ln\left(F\left(x_{(i)}\right)\right)\right) = ln(\theta) + \beta ln\left(\frac{\alpha}{x_{(i)}}\right) \tag{68}$$

Comparing equation (68) with equation (14), getting:

$$Y_i = ln\left(-ln\left(F\left(x_{(i)}\right)\right)\right) \to \bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$$

$$a = ln(\theta) \Rightarrow \theta = exp(a) \tag{69}$$

$$b = \beta \tag{70}$$

$$\emptyset = ln\left(\frac{\alpha}{x}\right) \tag{71}$$

$$\widehat{\emptyset}_i = ln\left(\frac{\alpha_0}{x_{(i)}}\right) \to \overline{\emptyset} = \frac{\sum_{i=1}^{n} \widehat{\emptyset}_i}{n} \tag{72}$$

And formula for extracting parameter (b) by Linear Regression method is:

$$\hat{b}_{REG} = \frac{\sum_{i=1}^{n}\left(\widehat{\emptyset}_i - \overline{\emptyset}\right)Y_i}{\sum_{i=1}^{n}\left(\widehat{\emptyset}_i - \overline{\emptyset}\right)^2} \tag{73}$$

Substituting equation (73) in equation (70), getting:

$$\hat{\beta}_{REG} = \hat{b}_{REG} \tag{74}$$

From equations (14) and (74), getting:

$$na = \sum_{i=1}^{n} Y_i - b\sum_{i=1}^{n} \emptyset_i$$

$$\hat{a} = \bar{Y} - \hat{\beta}_{REG}\overline{\emptyset} \tag{75}$$

Substituting equation (75) into equation (69), getting:

$$\hat{\theta}_{REG} = exp\left(\bar{Y} - \hat{\beta}_{REG}\overline{\emptyset}\right) \tag{76}$$

From equation (71), getting:

$$\sum_{i=1}^{n}\left(exp(\emptyset_i)\right) = \alpha\sum_{i=1}^{n}\left(\frac{1}{x_{(i)}}\right) \tag{77}$$

Substituting equation (72) in equation (77), getting:

$$\hat{\alpha}_{REG} = \frac{\sum_{i=1}^{n}\left(exp\left(\widehat{\emptyset}_i\right)\right)}{\sum_{i=1}^{n}\left(\frac{1}{x_{(i)}}\right)} \tag{78}$$

By substituting equations (74), (76) and (78) in equation (3), getting:

$$\hat{R}_{REG} = 1 - exp\left(-\hat{\theta}_{REG}\left(\frac{\hat{\alpha}_{REG}}{x}\right)^{\hat{\beta}_{REG}}\right) \quad (79)$$

3. Experiment of Simulation

The simulation technique was applied to generate the samples required to estimate the three parameters of (GIW) distribution. Using MATLAB language version R2015b to find all results. The following steps are for the stages of parameter estimation and the reliability function:

1. Four experiments were identified for the real parameter values, which are shown in Table 1.

Table 1. The real values of the parameters

Experiments→ Parameters↓	E_1	E_2	E_3	E_4
α	0.5	2	5	0.5
β	3.5	2.5	3	0.5
θ	1.5	3	2	0.5

2. Choosing a small, medium and large sample size $(n = 10,40,70,100)$ and replicated sample (N=1000).
3. Parameters estimated through equations 24,26,28,35,36,37,49,51,52,62,64,65,74,76 and 78 , and then estimate the reliability of the distribution through equations 29,38,53,66 and 79.
4. After finding the estimates, they are compared by the squares error MSE criterion: $MSE(\hat{\omega}) = \frac{\sum_{i=1}^{n}(\hat{\omega}_i - \omega)^2}{N}$, where $(\hat{\omega})$ is estimator for parameter (ω).

4. Simulation Results

The following tables 2, 3, 4 and 5 show MSE values for parameter estimates as well as for the reliability function

Table 2. The MSE values for (α, β, θ) reliability using (E_1)

Estimators	n	LS	WLS	W	MW	REG	Best
$\hat{\alpha}$	10	0.00147	0.01037	0.00046	0.000092	0.00047	MW
$\hat{\beta}$		1.23476	21.79118	1.05849	0.085454	0.34059	MW
$\hat{\theta}$		0.09910	0.10220	0.10457	0.010596	0.06045	MW
$\hat{R}$		0.00149	0.00154	0.00108	0.000122	0.00263	MW
$\hat{\alpha}$	40	0.00042	0.00646	0.00037	0.000089	0.00037	MW
$\hat{\beta}$		0.30175	4.08804	0.34626	0.083518	0.28912	MW
$\hat{\theta}$		0.03874	0.03673	0.03663	0.010249	0.04623	MW
$\hat{R}$		0.00053	0.00144	0.00023	0.000106	0.00077	MW
$\hat{\alpha}$	70	0.00025	0.00414	0.00030	0.000084	0.00032	MW
$\hat{\beta}$		0.18465	1.78025	0.20204	0.079102	0.26122	MW

$\hat{\theta}$		0.02409	0.02426	0.02529	0.009545	0.03909	MW
$\hat{R}$		0.00024	0.00122	0.00017	0.000069	0.00043	MW
$\hat{\alpha}$	100	0.00018	0.00350	0.00028	0.000083	0.00028	MW
$\hat{\beta}$		0.12911	1.58215	0.15128	0.077981	0.23877	MW
$\hat{\theta}$		0.01785	0.01969	0.02065	0.009434	0.03452	MW
$\hat{R}$		0.00019	0.00095	0.00010	0.000067	0.00027	MW

Table 3. The MSE values for $(\alpha, \beta, \theta)$and (R) using (E_2)

Estimators	n	LS	WLS	W	MW	REG	Best
$\hat{\alpha}$	10	1.10204	0.09941	0.02364	0.02131	0.02407	MW
$\hat{\beta}$		0.69257	0.50418	0.61789	0.06945	0.15916	MW
$\hat{\theta}$		0.93781	1.09937	2.46154	0.34632	0.31718	REG
$\hat{R}$		0.00037	0.00068	0.00127	0.00375	0.00240	LS
$\hat{\alpha}$	40	0.03375	0.03000	0.01727	0.01449	0.01760	MW
$\hat{\beta}$		0.16141	0.10217	0.17574	0.05112	0.10534	MW
$\hat{\theta}$		0.47721	0.46377	0.40902	0.22904	0.23860	MW
$\hat{R}$		0.00022	0.00023	0.00036	0.00134	0.00090	LS
$\hat{\alpha}$	70	0.02056	0.01799	0.01341	0.01245	0.01372	MW
$\hat{\beta}$		0.09074	0.05679	0.10805	0.04590	0.07668	MW
$\hat{\theta}$		0.33468	0.28735	0.26816	0.19378	0.18929	REG
$\hat{R}$		0.00018	0.00008	0.00011	0.00081	0.00030	WLS
$\hat{\alpha}$	100	0.01374	0.01370	0.01119	0.00958	0.01110	MW
$\hat{\beta}$		0.06134	0.04179	0.08231	0.03745	0.05536	MW
$\hat{\theta}$		0.20707	0.21945	0.20486	0.14610	0.15809	MW
$\hat{R}$		0.00010	0.00007	0.00010	0.00027	0.00009	WLS

Table 4. The MSE values for $(\alpha, \beta, \theta)$and (R) using (E_3)

Estimators	n	LS	WLS	W	MW	REG	Best
$\hat{\alpha}$	10	0.33930	0.29340	0.11288	0.03280	0.11201	MW
$\hat{\beta}$		0.91539	0.67838	0.90597	0.08237	0.25630	MW
$\hat{\theta}$		0.24526	0.22913	0.44247	0.05039	0.18633	MW
$\hat{R}$		0.00193	0.00202	0.00110	0.00040	0.00154	MW
$\hat{\alpha}$	40	0.09161	0.08432	0.08267	0.03040	0.07931	MW
$\hat{\beta}$		0.21676	0.15234	0.25110	0.07899	0.20410	MW
$\hat{\theta}$		0.11288	0.11194	0.10922	0.04622	0.12679	MW
$\hat{R}$		0.00044	0.00053	0.00031	0.00025	0.00042	MW
$\hat{\alpha}$	70	0.05588	0.05358	0.06246	0.02787	0.06197	MW

$\hat{\beta}$		0.12886	0.10178	0.15274	0.07280	0.16365	MW
$\hat{\theta}$		0.08366	0.08305	0.06894	0.04229	0.09742	MW
$\hat{R}$		0.00017	0.00014	0.00015	0.00021	0.00030	WLS
$\hat{\alpha}$	100	0.03950	0.03866	0.05051	0.02594	0.05108	MW
$\hat{\beta}$		0.09698	0.06053	0.11032	0.06808	0.14107	WLS
$\hat{\theta}$		0.06402	0.06456	0.05177	0.03928	0.07887	MW
$\hat{R}$		0.00016	0.00013	0.00011	0.00017	0.00014	W

Table 5. The MSE values for $(\alpha, \beta, \theta)$and reliability using (E_4)

Estimators	n	LS	WLS	W	MW	REG	Best
$\hat{\alpha}$	10	2.29988	1.26854	0.04797	0.04444	0.04527	MW
$\hat{\beta}$		0.02468	0.47279	0.02410	0.05476	0.09020	W
$\hat{\theta}$		0.06500	0.06547	0.01593	0.01687	0.01632	W
$\hat{R}$		0.00337	0.00169	0.00413	0.03450	0.02140	WLS
$\hat{\alpha}$	40	0.06212	0.03964	0.02348	0.01821	0.02161	MW
$\hat{\beta}$		0.00624	0.41202	0.00694	0.03375	0.07316	LS
$\hat{\theta}$		0.06228	0.06043	0.00569	0.00556	0.00622	MW
$\hat{R}$		0.00205	0.00066	0.00038	0.00458	0.00237	W
$\hat{\alpha}$	70	0.02564	0.03109	0.01727	0.01321	0.01604	MW
$\hat{\beta}$		0.00323	0.23612	0.00446	0.02796	0.05849	LS
$\hat{\theta}$		0.05051	0.04933	0.00341	0.00382	0.00437	W
$\hat{R}$		0.00196	0.00060	0.00020	0.00270	0.00091	W
$\hat{\alpha}$	100	0.01713	0.01699	0.01389	0.01104	0.01288	MW
$\hat{\beta}$		0.00257	0.05393	0.00331	0.02696	0.05277	LS
$\hat{\theta}$		0.04535	0.04572	0.00259	0.00304	0.00332	W
$\hat{R}$		0.00191	0.00058	0.00010	0.00124	0.00062	W

From the above tables we conclude the following:

- In experiment (E_1) ,$\hat{\alpha}_{MW}, \hat{\beta}_{MW}$ and $\hat{\theta}_{MW}$ are better than other estimators. $(\hat{R}_{MW})$ is better than other estimators.
- In experiment(E_2), $\hat{\alpha}_{MW}, \hat{\beta}_{MW}$ is best in all samples, but $\hat{\theta}_{MW}$ is the best just in (n=40,100) and $(\hat{\theta}_{REG})$ is better in$(n = 10{,}70)$. $(\hat{R}_{LS})$ is better in $(n = 10{,}40)$, but $(\hat{R}_{WLS})$ is better in $(n = 70{,}100)$.
- In experiment (E_3), $(\hat{\alpha}_{MW})$ is better than other estimators and $(\hat{\beta}_{MW})$ is better in$(n = 10{,}40{,}70)$, but $(\hat{\beta}_{WLS})$ is better in$(n = 100)$. $(\hat{\theta}_{MW})$ is better than other estimators. $(\hat{R}_{MW})$ is better in $(n = 10{,}40)$, but $(\hat{\beta}_{WLS})$ is better in $(n = 70)$and $(\hat{R}_W)$ is better in $(n = 100)$.

- In experiment(E_4), $(\hat{\alpha}_{MW})$ is better than other estimators in all samples. $(\hat{\beta}_W)$ is better in $(n = 10)$, but $(\hat{\beta}_{LS})$ is better in $(n = 40,70,100)$. $(\hat{\theta}_W)$ is better in $(n = 10,70,100)$, but $(\hat{\theta}_{MW})$ is better in $(n = 40)$. $(\hat{R}_{WLS})$ is better in $(n = 10)$, but $(\hat{R}_W)$ is better in $(n = 40,70,100)$

5. Conclusion

From the results in Tables (2, 3, 4 and 5) we note that $\hat{\alpha}_{MW}$,$\hat{\beta}_{MW}$, $\hat{\theta}_{MW}$ and $\hat{R}_{MW}$ are better than estimators of other methods.

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