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# New Convex Combination (H-version) method for unconstrained optimization problem

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A B S T R A C T

In this essay, we propose a new family for inverse of Hessian matrix which based on definition of convex combination class and on modified symmetric rank one update (MSR1) with ($\alpha$ −DFP) update and upgraded BFGS (MBFGS) revision for the inverse of the Hessian matrix and the properties of symmetric and positive definite will be shown and proved as well as the theoretical and numerical global converge is proposed , numerical experiment on eight functions are presented in a table.

**Keywords**: *MSR1 Update, $\alpha$ −DFP Update, MBFGS Update,Quasi-Newton, Positive Definite.*

## 1. Introduction

Different techniques have been combined to obtain the (H-version) of the Hessian matrix, including the Broyden family [1], Huang family [2], Fletcher switch technique [2], Update to the modified Broyden Convex class using the Hoshino parameter [3] and Hoshino method [2]. The concept of convex combination, the symmetric update with adjusted rank one (MSR1), the David Fletcher Powell (DFP) edit, also the altered BFGS (MBFGS) edit will all be utilised to introduce a new family in this paper. These three updates are positive definite and symmetric [1]. Major inverse Hessian matrix modifications have also been presented by certain researchers; for examples, see [4], [5], and [6]. There is a convex combination for inherently MSR1, $\alpha$ −DFP and MBFGS updates and the by definition of convex combination in [1] where the (H - variant) of this approach

$$H_{k+1}^{\lambda} = \lambda_1 H_{k+1}^{MSR1} + \lambda_2 H_{k+1}^{MBFGS} + \lambda_3 H_{k+1}^{\alpha-DFP} \tag{1}$$

In Eq.(1) , $\lambda_1 = 0.5$ , the equation for$\lambda_2$ This study includes the Hoshino parameter, which :

$$\lambda_2=\frac{1}{1+\left(y_k^T H_k y_k / s_k^T y_k\right)} \tag{2}$$

And $\lambda_3 = 1-(\lambda_1+\lambda_2)$ and $\sum_{i=1}^{3}\lambda_i = 1$

For the unconstrained optimization problem [7]

$$min_{x\in R^n} f: R^n \longrightarrow R \tag{3}$$

introduce the positive definite Upgrade for Modified Symmetric Rank One (MSR1) in [8]. Upgrade for Modified Symmetric Rank One (MSR1):

$$H_{k+1}^{MSR1} = H_k + \frac{(\alpha_k s_k - H_k y_k)(\alpha_k s_k - H_k y_k)^T}{(\alpha_k s_k - H_k y_k)^T y_k} \tag{4}$$

Where

$$\alpha_k = 2\frac{y_k^T H_k y_k}{y_k^T s_k} \tag{5}$$

in [9] the (H – version) of $\alpha$ −Daviden Fletcher Powell ($\alpha$ −DFP) update is :

$$H_{k+1}^{\alpha-DFP} = H_k + \frac{\tilde{s}_k \tilde{s}_k^T}{\tilde{s}_k^T y_k} - \frac{H_k y_k y_k^T H_k}{y_k^T H_k y_k} \tag{6}$$

Where $\tilde{s}_k = \tilde{\alpha} s_k$

And $\tilde{\alpha}= \frac{y_k^T H_k [Det(B_k)]^2}{s_k^T B_k s_k}$

And as in [10], introduce the (H- version) of the Modified Broyden Flecher Goldfrob -Shanno (MBFGS) edit , which is strongly positive definition and symmetric

$$H_{k+1}^{MBFGS} = \left[I - \frac{s_k y_k^T}{y_k^T s_k}\right] H_k \left[I - \frac{y_k s_k^T}{y_k^T s_k}\right] + \frac{\mu_k s_k s_k^T}{y_k^T s_k} \tag{7}$$

Where

$$\mu_k = \frac{s_k^T y_k}{s_k^T B_k s_k} \tag{8}$$

The fundamental concept behind this research is based on MSR1 [1], $\alpha$-DFP [11], and MBFGS [6], all of which are symmetric and positive definite . The (H- version) for convexity combinations also shares this property, and we will prove it in lemma(1) and lemma(2).

**Preliminaries**:

**Definition 1:**[1] A convex combination of the points $x_1, x_2, \dots, x_n$ is a point of the form $\sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ , i=1,…,n and $\sum_{i=1}^{n} \lambda_i = 1$.

**Definition 2:** [1] The matrix $A \in R^{n\times n}$ is called positive definite if $x^T A x > 0$ , for all vector $x \neq 0$ and $x \in R^n$**.**

**Definition 3:**[1] Given $f: R^n \to R$ , , if $\nabla f$ is differentiable we say that f is twice differetiable, and write the second derivative of f as

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}.$$

The matrix $\nabla^2 f(x)$ is called Hessian matrix of (f) at x.

**Main results:**

**Theorem (1)**

The (H-version) in Eq. (1), fulfills the modified Quasi-Newton-condition:

$H_{k+1}^{\lambda} y_k = \widetilde{U}_k$

where $\widetilde{U}_k = \lambda_1 \alpha_k + \lambda_2 \mu_k + \lambda_3 \tilde{\alpha}$

proof:

Multiplying Eq. (1), by $y_k$ we get:

$$H_{k+1}^{\lambda} y_k = \lambda_1 H_{k+1}^{MSR1} y_k + \lambda_2 H_{k+1}^{MBFGS} y_k + \lambda_3 H_{k+1}^{\alpha-DFP} y_k$$

$$= \lambda_1 \alpha_k s_k + \lambda_2 \mu_k y_k + \lambda_3 \tilde{\alpha} s_k$$

$$= (\lambda_1 \alpha_k + \lambda_2 \mu_k + \lambda_3 \tilde{\alpha})\, s_k$$

$$= \widetilde{U}_k s_k \qquad (9)$$

Where $\widetilde{U}_k = \lambda_1 \alpha_k + \lambda_2 \mu_k + \lambda_3 \tilde{\alpha}$

$$\widetilde{U}_k = \frac{y_k^T H_k y_k}{y_k^T s_k} + \frac{s_k^T y_k}{\left(1+\left(y_k^T H_k y_k / s_k^T y_k\right)\right) s_k^T B_k s_k} + \left(1 - (0.5 + \frac{1}{1+\left(y_k^T H_k y_k / s_k^T y_k\right)})\right)\left(\frac{y_k^T H_k [Det(B_k)]^2}{s_k^T B_k s_k}\right) \quad (10)$$

**Modified convex combination family** :

Within this section, we'll introduce the formula for our adapted class, which we'll use to compute the determinant and show the global convergence of Eq. (1) in a new form that matches the Sherman-Morrison-Woodburg formula. The new formula will now be used as follows.: -

$$H_{k+1}^{\lambda} = \lambda_1 H_{k+1}^{MSR1} + \left[\lambda_2 H_{k+1}^{MBFGS} + \lambda_3 H_{k+1}^{\alpha-DFP}\right]$$

Since both ($\alpha -$ DFP) and (MBFGS) are symmetric and positive definite and by definition of cholesky factorization we can write the above equation as follow

$$H_{k+1}^{\lambda} = \lambda_1 H_{k+1}^{MSR1} + \omega_k^* \omega_k^{*T} \quad (11)$$

Now to find the formula of $\omega_k^*$ multiplying Eq. (11), by $y_k$ we get

$$\lambda_1 H_{k+1}^{MSR1} y_k + \omega_k^* \omega_k^{*T} y_k = \widetilde{U}_k s_k$$

$$\lambda_1 \alpha_k s_k + \omega_k^* \omega_k^{*T} y_k = \widetilde{U}_k s_k$$

The result of multiplying the following equation by $y_k^T$ is

$$\lambda_1 \alpha_k y_k^T s_k + y_k^T \omega_k^* \omega_k^{*T} y_k = \widetilde{U}_k y_k^T s_k$$

$$\left(y_k^T \omega_k^*\right)^2 = \widetilde{U}_k y_k^T s_k - \lambda_1 \alpha_k y_k^T s_k \quad (12)$$

$$y_k^T \omega_k^* = \frac{\widetilde{U}_k y_k^T s_k - \lambda_1 \alpha_k y_k^T s_k}{y_k^T \omega_k^*}$$

$$y_k^T \omega_k^* = y_k^T \left(\frac{\widetilde{U}_k s_k - \lambda_1 \alpha_k s_k}{y_k^T \omega_k^*}\right)$$

By comparing the both sides of the above equation we have

$$\omega_k^* = \frac{\widetilde{U}_k s_k - \lambda_1 \alpha_k s_k}{y_k^T \omega_k^*} \quad (13)$$

And we conclude from Eq. (12),

$$y_k^T \omega_k^* = \left(\widetilde{U}_k y_k^T s_k - \lambda_1 \alpha_k y_k^T s_k\right)^{\frac{1}{2}} \tag{14}$$

By substitution Eq.(14) , in Eq. (13) , we get

$$\omega_k^* = \frac{\widetilde{U}_k s_k - \lambda_1 \alpha_k s_k}{(\widetilde{U}_k y_k^T s_k - \lambda_1 \alpha_k y_k^T s_k)^{\frac{1}{2}}}$$

So Eq. (11) can be rewriting as follow

$$H_{k+1}^{\lambda} = \lambda_1 H_{k+1}^{MSR1} + \frac{(\widetilde{U}_k s_k - \lambda_1 \alpha_k s_k)(\widetilde{U}_k s_k - \lambda_1 \alpha_k s_k)^T}{(\widetilde{U}_k y_k^T s_k - \lambda_1 \alpha_k y_k^T s_k)}$$

$$= \lambda_1 H_k + \frac{\lambda_1 (\alpha_k s_k - H_k y_k)(\alpha_k s_k - H_k y_k)^T}{(\alpha_k s_k - H_k y_k)^T y_k} + \frac{(\widetilde{U}_k s_k - \lambda_1 \alpha_k s_k)(\widetilde{U}_k s_k - \lambda_1 \alpha_k s_k)^T}{(\widetilde{U}_k y_k^T s_k - \lambda_1 \alpha_k y_k^T s_k)}$$

$$= \lambda_1 H_k + \frac{\lambda_1 (\alpha_k s_k - H_k y_k)(\alpha_k s_k - H_k y_k)^T}{(\alpha_k s_k - H_k y_k)^T y_k} + \frac{(\widetilde{U}_k - \lambda_1 \alpha_k) s_k s_k^T}{y_k^T s_k}$$

$$= \lambda_1 H_k + \frac{\lambda_1 (\alpha_k s_k - H_k y_k)(\alpha_k s_k - H_k y_k)^T}{(\alpha_k s_k - H_k y_k)^T y_k} + \frac{(\lambda_2 \mu_k + \lambda_3 \widetilde{\alpha}) s_k s_k^T}{y_k^T s_k} \tag{15}$$

**Lemma(1)**

The redesigned convex class in equation (1) is symmetric..

*Proof:*

$$\left(H_{k+1}^{\lambda}\right)^T = \left(\lambda_1 H_{k+1}^{MSR1} + \lambda_2 H_{k+1}^{MBFGS} + \lambda_3 H_{k+1}^{\alpha-DFP}\right)^T$$

$$= \lambda_1 \left(H_{k+1}^{MSR1}\right)^T + \lambda_2 \left(H_{k+1}^{MBFGS}\right)^T + \lambda_3 \left(H_{k+1}^{\alpha-DFP}\right)^T$$

$$= \lambda_1 H_{k+1}^{MSR1} + \lambda_2 H_{k+1}^{MBFGS} + \lambda_3 H_{k+1}^{\alpha-DFP}$$

$$= H_{k+1}^{\lambda}$$

This shows that the modified convex class is symmetric. ■

**Lemma (2)**

The modified- convex -class in Eq. (1), is positive definite if and only if $y_k^T \tilde{s}_k > 0$ .

*Proof:*

For $R^n \ni z \neq 0$ , as defined by the term "positive definite property" [1]

$z^T H_{k+1}^{\lambda} z = \lambda_1 z^T H_{k+1}^{MSR1} z + \lambda_2 z^T H_{k+1}^{MBFGS} z + \lambda_3 z^T H_{k+1}^{\alpha-DFP} z$

Since $z^T H_{k+1}^{MSR1} z > 0$ so that

$\lambda_1 z^T H_{k+1}^{MSR1} z > 0$

and since $z^T H_{k+1}^{MBFGS} z > 0$ this implies to

$\lambda_2 z^T H_{k+1}^{MBFGS} z > 0$

and since $y_k^T \tilde{s}_k > 0$ $z^T H_{k+1}^{\alpha-DFP} z > 0$ this implies to

$\lambda_3 z^T H_{k+1}^{\alpha-DFP} z > 0$

we receive from above :

$z^T H_{k+1}^{\lambda} z > 0$, The proof is therefore complete. ■

***Lemma (3)***

The next estimate of the inverse of the Hessian matrix for the modified convex class in Eq. (1) has the determinant given by:

$$|H_{k+1}^{\lambda}| = |H_k| \left(\frac{b_k^T r_k}{b_k^T y_k}\right) (\tilde{t}) \qquad (16)$$

here $b_k = (H_k y_k - \alpha_k s_k)$ , $r_k = y_k - B_k b_k$ and

$$\tilde{t} = \frac{\left(\frac{\tilde{U}_k{}^2}{\alpha_k} - \lambda_1 \tilde{U}_k\right)}{(\lambda_1 \lambda_2 \mu_k + \lambda_1 \lambda_3 \tilde{\alpha})}$$

Proof:

since $H_{k+1}^{MSR1}$ is definite in the affirmative then a triangular matrix exists $L_k \in R^{n \times n}$ $\ni$ $H_{k+1}^{MSR1} = L_k L_k{}^T$

$$\left|H_{k+1}^{\lambda}\right| = \left|\lambda_1 H_{k+1}^{MSR1} + \omega_k^* \omega_k^{*T}\right|$$

$$\left|H_{k+1}^{\lambda}\right| = \left|\lambda_1 H_{k+1}^{MSR1}\right| \left|I + \frac{1}{\lambda_1} L_k^{-1} \lambda \omega_k^* \omega_k^{*T} L_k^{T^{-1}}\right|$$

$$= \lambda_1 \left|H_{k+1}^{MSR1}\right| \left|I + \frac{1}{\lambda_1} L_k^{-1} \omega_k^* \left(L_k^{-1} \omega_k^*\right)^T\right|$$

$$= \lambda_1 \left|H_{k+1}^{MSR1}\right| |I + u\, v^T|$$

Where $u = \frac{1}{\lambda_1} L_k^{-1} \omega_k^*$ and $v^T = \left(L_k^{-1} \omega_k^*\right)^T$

By Sherman – Morrison– formula [1] follows that :

$$\left|H_{k+1}^{\lambda}\right| = \lambda_1 \left|H_{k+1}^{MSR1}\right| (1 + u^T v)$$

$$= \lambda_1 \left|H_{k+1}^{MSR1}\right| \left(1 + \left(\frac{1}{\lambda_1} L_k^{-1} \omega_k^*\right)^T \left(L_k^{-1} \omega_k^*\right)\right)$$

$$= \lambda_1 \left|H_{k+1}^{MSR1}\right| \left(1 + \frac{1}{\lambda_1} \omega_k^{*T} B_{k+1}^{MSR1} \omega_k^*\right)$$

$$=\lambda_1 \left|H_{k+1}^{MSR1}\right| \left(1 + \frac{1}{\lambda_1} \frac{(\widetilde{U}_k s_k - \lambda_1 \alpha_k s_k)^T B_{k+1}^{MSR1} (\widetilde{U}_k s_k - \lambda_1 \alpha_k s_k)}{(\widetilde{U}_k y_k^T s_k - \lambda_1 \alpha_k y_k^T s_k)}\right)$$

$$= \lambda_1 \left|H_{k+1}^{MSR1}\right| \left(\frac{\left(\frac{\widetilde{U}_k{}^2}{\alpha_k} - \lambda_1 \widetilde{U}_k\right)}{(\lambda_1 \widetilde{U}_k - \lambda_1^2 \alpha_k)}\right)$$

$$= \left|H_{k+1}^{MSR1}\right| \left(\frac{\left(\frac{\widetilde{U}_k{}^2}{\alpha_k} - \lambda_1 \widetilde{U}_k\right)}{(\lambda_1 \lambda_2 \mu_k + \lambda_1 \lambda_3 \widetilde{\alpha})}\right)$$

And in [8] shows that $\left|H_{k+1}^{MSR1}\right| = |H_k| \left(\frac{b_k^T r_k}{b_k^T y_k}\right)$

so we get:

$$\left|H_{k+1}^{\lambda}\right| = |H_k| \left(\frac{b_k^T r_k}{b_k^T y_k}\right) \left(\frac{\left(\frac{\widetilde{U}_k{}^2}{\alpha_k} - \lambda_1 \widetilde{U}_k\right)}{(\lambda_1 \lambda_2 \mu_k + \lambda_1 \lambda_3 \widetilde{\alpha})}\right)$$

$$\left|H_{k+1}^{\lambda}\right| = |H_k| \left(\frac{b_k^T r_k}{b_k^T y_k}\right) (\tilde{t})$$

where $\tilde{t} = \frac{\left(\frac{\widetilde{U}_k{}^2}{\alpha_k} - \lambda_1 \widetilde{U}_k\right)}{(\lambda_1 \lambda_2 \mu_k + \lambda_1 \lambda_3 \widetilde{\alpha})}$ . ■

**Modified Broydn Convex Class Algorithm**

1. given a starting point $x_0 \in R^n$ with a starting matrix that is positive definite and symmetric. $H_0 \in R^{n\times n}$ , $\epsilon > 0$ , *let k = 0.*
2. Evaluate $\nabla f(x_k)$
3. if $\|\nabla f(x_{k+1})\| < \epsilon$ stop, and the optimal solution is $x_k$ if not, move on to the next step.
4. Compute $\tilde{p}_k = -H_k \nabla f(x_k)$.
5. Do line search to fined $\tilde{\alpha}_k > 0 \ni f(x_k + \tilde{\alpha}_k \tilde{p}_k) < f(x_k)$.
6. Set $x_{k+1} = x_k + \tilde{\alpha}_k \tilde{p}_k$
7. Set $s_k = x_{k+1} - x_k$ , $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$
8. Compute $H_{k+1}$ from Eq. (1).
9. Set k = k+1 then go to step 2.

**Convergence of the method**:

This section provides information relating to the H- version convergence of the modified convex combination class update. The lemma and assumption listed below are required.

**Assumption 1**, [1]

(A) $f: D \subseteq R^n \longrightarrow R$ is assumed to be continuously and differentiable twice on convex set $D \subseteq R^n$.

(B) f(x) is uniformly convex. In another meaning there is a positive constants c and C $\ni$ for all $x \in L(x) = \{x: f(x) \le f(x_0)\}$ , where $x_0$ is the starting point , we have $c\|u\|^2 \le u^T \nabla^2 f(x) u \le C\|u\|^2$ , $\forall u \in R^n$.

**Lemma(4)** , [9]

Let $f: R^n \longrightarrow R$ satisfy Assumption 1, then $\frac{\|s_k\|}{\|y_k\|}, \frac{\|y_k\|}{\|s_k\|}, \frac{s_k^T y_k}{\|s_k\|^2}, \frac{s_k^T y_k}{\|y_k\|^2}$ and $\frac{\|y_k\|^2}{s_k^T y_k}$ are bounded .

Note that from Assumption 1 and by lemma then $\frac{w_k^T y_k}{\|y_k\|^2}$ , $\frac{(p_k - \alpha_k)\|s_k\|}{y_k^T s_k}$ and $\left(\frac{b_k^T r_k}{b_k^T y_k}\right)\left(a_k y_k^T s_k\right)$ are bounded.

**Theorem (2)**

Suppose that f(x) satisfies Assumption one then the sequence $\{x_k\}$ generated by modified convex class in Eq. (11), is converges to the minimizer $x^*$ of the function f.

*Proof:*

writing Eq. (11) ,as follow :

$$H_{k+1}^{\lambda} = \lambda_1 H_k + \frac{\lambda_1(\alpha_k s_k - H_k y_k)(\alpha_k s_k - H_k y_k)^T}{(\alpha_k s_k - H_k y_k)^T y_k} + \frac{(\lambda_2 \mu_k + \lambda_3 \tilde{\alpha}) s_k s_k^T}{y_k^T s_k}$$

$$H_{k+1}^{\lambda} = \lambda_1 H_k - \frac{\lambda_1 w^*_k w^{*T}_k}{w^{*T}_k y_k} + \frac{(\lambda_2 \mu_k + \lambda_3 \tilde{\alpha}) s_k s_k^T}{y_k^T s_k}$$

Where $w^*_k = H_k y_k - \alpha_k s_k$

Define $\tilde{\varphi}(H_k) = tr(\lambda_1 H_k) - \ln(|H_k|) > 0$

By replacing $H_k$ by $H_{k+1}$ in above equation then :

$$0 < \tilde{\varphi}(H_{k+1}) = tr(\lambda_1 H_{k+1}) - \ln(|H_{k+1}|)$$

$$= \lambda_1 tr(H_k) - \frac{\lambda_1 w^*_k w^{*T}_k}{w^{*T}_k y_k} + \frac{(\lambda_2 \mu_k + \lambda_3 \tilde{\alpha}) \|s_k\|}{y_k^T s_k} - \ln(|H_{k+1}|)$$

From Eq.(16), we get :

$$0 < \tilde{\varphi}(H_{k+1}) = tr(\lambda_1 H_k) - \frac{\lambda_1 w^*_k w^{*T}_k}{w^{*T}_k y_k} + \frac{(\lambda_2 \mu_k + \lambda_3 \tilde{\alpha}) \|s_k\|}{y_k^T s_k}$$

$$- \ln\left( |H_k| \left(\frac{b_k^T r_k}{b_k^T y_k}\right) (\tilde{N}) \right)$$

$$0 < \tilde{\varphi}(H_{k+1}) = tr(\lambda_1 H_k) - \frac{\lambda_1 w^*_k w^{*T}_k}{w^{*T}_k y_k} + \frac{(\lambda_2 \mu_k + \lambda_3 \tilde{\alpha}) \|s_k\|}{y_k^T s_k}$$

$$- \ln(|H_k|) - \ln\left( \left(\frac{b_k^T r_k}{b_k^T y_k}\right) (\tilde{N}) \right)$$

$$0 < \tilde{\varphi}(H_{k+1}) = \tilde{\varphi}(H_k) - \lambda_1 \frac{q}{cos^2 \vartheta_k} + \frac{(\lambda_2 \mu_k + \lambda_3 \tilde{\alpha}) \|s_k\|}{y_k^T s_k}$$

$$- \ln\left( \left(\frac{b_k^T r_k}{b_k^T y_k}\right) (\tilde{\alpha}) \right)$$

(17)

Where $\cos(\vartheta_k) = \frac{w^{*T}_k y_k}{\|w^*_k\| \|y_k\|}$ and $q = \frac{w^{*T}_k y_k}{\|y_k\|^2}$ so that $\frac{q}{cos^2 \vartheta_k} = \frac{w^{*T}_k w^*_k}{w^{*T}_k\ y_k}$

By adding one to the right side of Eq. (17), we get

$$0 < \tilde{\varphi}(H_{k+1}) < \tilde{\varphi}(H_k) - \lambda_1 \frac{q}{cos^2 \vartheta_k} + \frac{(\lambda_2 \mu_k + \lambda_3 \tilde{\alpha}) \|s_k\|}{y_k^T s_k}$$

$$- \ln\left( \left(\frac{b_k^T r_k}{b_k^T y_k}\right) (\tilde{N}) \right) + 1$$

By adding and subtracting $\ln\left(\frac{\lambda_1 q}{cos^2\vartheta_k}\right)$ for the above inequality

$$0 < \tilde{\varphi}(H_{k+1}) < \tilde{\varphi}(H_k) + \frac{(\lambda_2 \mu_k + \lambda_3 \tilde{\alpha})\|s_k\|}{y_k^T s_k} - \ln\left(\frac{\lambda_1 q}{cos^2\vartheta_k}\right)$$

$$-\ln\left(\left(\frac{b_k^T r_k}{b_k^T y_k}\right)(\tilde{N})\right) + \left(1 - \lambda_1 \frac{q}{cos^2\vartheta_k} + \ln\left(\frac{\lambda_1 q}{cos^2\vartheta_k}\right)\right)$$

$$< \tilde{\varphi}(H_k) + \frac{(\lambda_2 \mu_k + \lambda_3 \tilde{\alpha})\|s_k\|}{y_k^T s_k}$$

$$-\ln\left(\frac{\lambda_1 q}{cos^2\vartheta_k}\right) - \ln\left(\left(\frac{b_k^T r_k}{b_k^T y_k}\right)(\tilde{N})\right)$$

Since the maximum value of $(1 - \omega + \ln(\omega))$ is zero that is by letting $F(\omega)=1 - \omega + \ln(\omega)$ then $\frac{dF(\omega)}{d\omega} = -1 + \frac{1}{\omega} = 0$

$\Longrightarrow \omega = 1$ so the maximum value of $F(1) = 0$ then we get :

Where $\omega = \frac{\lambda_1 q}{cos^2\vartheta_k}$ then we get :

$$0 < \tilde{\varphi}(H_{k+1}) < \tilde{\varphi}(H_k) + \frac{(\lambda_2 \mu_k + \lambda_3 \tilde{\alpha})\|s_k\|}{y_k^T s_k} - \ln(\lambda_1 q) + \ln(cos^2\vartheta_k) -$$

$$\ln\left(\left(\frac{b_k^T r_k}{b_k^T y_k}\right)(\tilde{N})\right)$$

$$0 < \tilde{\varphi}(H_{k+1}) < \tilde{\varphi}(H_k) + C + \ln(cos^2\vartheta_k)$$

Where $C = -\ln(\lambda_1 q) + \frac{(\lambda_2 \mu_k + \lambda_3 \tilde{\alpha})\|s_k\|}{y_k^T s_k} - \ln\left(\left(\frac{b_k^T r_k}{b_k^T y_k}\right)(\tilde{N})\right)$

By assuming that the terms: $\lambda_1 q$ , $\frac{(\lambda_2 \mu_k + \lambda_3 \tilde{\alpha})\|s_k\|}{y_k^T s_k}$ and $\left(\frac{b_k^T r_k}{b_k^T y_k}\right)(\tilde{N})$ are bounded and by summing from j=0 up to k we get

$$0 < \sum_{j=0}^{k} \tilde{\varphi}(H_{j+1}) < \sum_{j=0}^{k} \tilde{\varphi}(H_j) + \sum_{j=0}^{k} C + \sum_{j=0}^{k} \ln(cos^2\vartheta_j)$$

$$0 < \tilde{\varphi}(H_1) + \tilde{\varphi}(H_2) + \cdots + \tilde{\varphi}(H_{k+1}) < \tilde{\varphi}(H_0) + \tilde{\varphi}(H_1) + \tilde{\varphi}(H_2) + \cdots + \tilde{\varphi}(H_{k+1}) + C\,k + \sum_{j=0}^{k} \ln(cos^2\vartheta_j)$$

$$0 < \tilde{\varphi}(H_{k+1}) < \tilde{\varphi}(H_0) + C\,k + \sum_{j=0}^{k} \ln(cos^2\vartheta_j) \quad (18)$$

Where the constant C is assumed to be positive without loss of the generality , from Zoutendijk condition (1) (if f satisfy assumption 1, then $\sum cos^2\vartheta_k \|\nabla f\|^2 < \infty$ )

$\lim_{k\to\infty}\|\nabla f\|.\cos\vartheta_k = 0$ . if $\vartheta_k$ is bounded away from 90° , ∃ M ∈ $R^+$ ∋ $\cos\vartheta_k$ > M > 0 , for k sufficient large and hence $\|\nabla f\| \to 0$ and by the first order necessary condition the prove is complete .

Now assume by contradiction that $\cos\vartheta_k \to 0$ then ∃ $k_1 > 0$ ∋ ∀ j > $k_1$ ,

$$\ln\left(cos^2\vartheta_j\right) < -2C \quad (19)$$

Since $\sum_{j=0}^{k}\ln\left(cos^2\vartheta_j\right) = \sum_{j=0}^{k_1}\ln\left(cos^2\vartheta_j\right) + \sum_{j=k_1+1}^{k}\ln\left(cos^2\vartheta_j\right)$

Then Eq.(18), become

$$0 < \tilde{\varphi}(H_{k+1}) < \tilde{\varphi}(H_0) + C\, k + \sum_{j=0}^{k_1}\ln\left(cos^2\vartheta_j\right) + \sum_{j=k_1+1}^{k}\ln\left(cos^2\vartheta_j\right) \quad (20)$$

By substitution Eq. (19) in Eq. (20) we have

$$0 < \tilde{\varphi}(H_{k+1}) < \tilde{\varphi}(H_0) + C\, k + \sum_{j=0}^{k_1}\ln\left(cos^2\vartheta_j\right) - \sum_{j=k_1+1}^{k} 2C$$

$$0 < \tilde{\varphi}(H_{k+1}) < \tilde{\varphi}(H_0) + C\, k + \sum_{j=0}^{k_1}\ln\left(cos^2\vartheta_j\right) - 2C(k - k_1)$$

$$0 < \tilde{\varphi}(H_{k+1}) < \tilde{\varphi}(H_0) + C\, k + \sum_{j=0}^{k_1}\ln\left(cos^2\vartheta_j\right) - 2Ck + 2Ck_1$$

$$0 < \tilde{\varphi}(H_{k+1}) < \tilde{\varphi}(H_0) + \sum_{j=0}^{k_1}\ln\left(cos^2\vartheta_j\right) - Ck + 2Ck_1 < 0$$

For k sufficiently large which is contradiction .

Then $cos^2\vartheta_j \to 0$ is not true and $\lim_{k\to\infty}\inf\|\nabla f\| \to 0$ and by the first order necessary condition(FONC) the prove is complete . ■

**Numerical Experiments and figures** :

Within this section, we'll introduce a number of eight functions were the subject of numerical experiments utilizing the modified convex combination class. from Eq. (1), where the numerical outcomes are supported by the table, and the parameter is the Hoshino parameter, which has the formula from Eq. (2). The following list of test functions is chosen:

1- Least square equation for two dimension[9]
   $f(x) = (1 - x_1)^2 + (1 - x_2)^2$.

2- Cubic function[7]

$f(x) = 100(x_2 - x_1^3)^2 + (1 - x_1)^2$

3- A quadratic- function: [10]

$$f(x) = \sum_{i=1}^{4}\left(10^{i-1}x_i^4 + x_i^3 + 10^{1-i}x_i^2\right)$$

4- Rosenbroc'k -cliff –function: [11]

$f(x) = 10^{-4}(x_1 - 3)^2 - (x_1 - x_2) + e^{20(x_1 - x_2)}$

5- Generalized Edger function[10]

$$f(x) = \sum_{i=1}^{n/2}[(x_{2i-1} - 2)^4 + (x_{2i-1} - 2)^2 x_{2i}^2 + (x_{2i} + 1)^2]$$

6- Extended –Himmelbla- function : [10]

$$f(x) = \sum_{i=1}^{n/2}[(x_{2i-1}^2 + x_{2i} - 11)^2 + (x_{2i-1} + x_{2i}^2 - 2)^2].$$

7- Rosen -rock's –function: [7]

$$f(x) = \sum_{i=1}^{n/2}[100(x_i - x_i^3)^2 + (1 - x_i)^2]$$

8- Watson function: [10]

$$F(x) = \sum_{i=1}^{j} f_i^2(x)$$

$$f_i(x) = \sum_{j=2}^{3}(j-1)x_j t_j^{j-2} - \left(\sum_{j=1}^{3} x_j t_j^{j-1}\right)^2 - 1$$

**Table 1. Numerical Results for Modified Convex Combination family**

| *Fun.* | *Stating points* | *Dim* | *Feval* | *Iter.* |
|---|---|---|---|---|
| 1a | $[0; 0]^T$ | 2 | 8.2008e−016 | 2 |
| 1b | $[1; 2]^T$ | 2 | 6.4758e-019 | 2 |
| 1c | $[-1; -2]^T$ | 2 | 3.3087e-019 | 2 |
| 2a | $[0; 0]^T$ | 2 | 0.1262 | 34 |

| | | | | |
|---|---|---|---|---|
| 2b | $[-1; -2]^{T}$ | 2 | 4.6912 | 35 |
| 2c | $[1; 5]^{T}$ | 2 | 0.4599 | 34 |
| 3a | $[0; 0; 0; 0]^{T}$ | 4 | $-5.6449e - 014$ | 1 |
| 3b | $[1; 2; 3; 4]^{T}$ | 4 | $-3.5987$ | 66 |
| 3c | $[-1; -2; -3; -4]^{T}$ | 4 | $-3.4168$ | 54 |
| 4a | $[0; 0]^{T}$ | 2 | 0.2007 | 3 |
| 4b | $[1; 2]^{T}$ | 2 | 0.9245 | 30 |
| 4c | $[-1; -2]^{T}$ | 2 | 5.9617e+005 | 27 |
| 5a | $[0; 0]^{T}$ | 2 | 2.6165e-008 | 5 |
| 5b | $[1; 2]^{T}$ | 2 | 1.7470e-010 | 6 |
| 5c | $[-1; -2]^{T}$ | 2 | 1.7867e-00 | 5 |
| 6a | $[0; 0]^{T}$ | 2 | 7.6789e-010 | 48 |
| 6b | $[1; 2]^{T}$ | 2 | 6.2341e-012 | 43 |
| 6c | $[-1; -2]^{T}$ | 2 | 3.3773e-011 | 49 |
| 7a | $[0; 0]^{T}$ | 2 | 1.7728e−010 | 2 |
| 7b | $[1; 2]^{T}$ | 2 | 5.3891e-012 | 4 |
| 7c | $[-1; -2]^{T}$ | 2 | 7.9797 | 50 |

| | | | | |
|---|---|---|---|---|
| 8a | $[0;0;0;0]^T$ | 4 | 6.2296e−010 | 2 |
| 8b | $[1;2;3;4]^T$ | 4 | 5.1579e-017 | 3 |
| 8c | $[-1;-2;-3;-4]^T$ | 4 | 2.4288e-016 | 4 |

It is clear from the above Table 1 that our new method gave a minimization of the tested functions. We conducted the experiment on several starting points, starting with zero and at positive values and others with negative values for each function, and we clarified the number of iterations in the last column for each test of the function to reach the best minimizing to the tested functions.

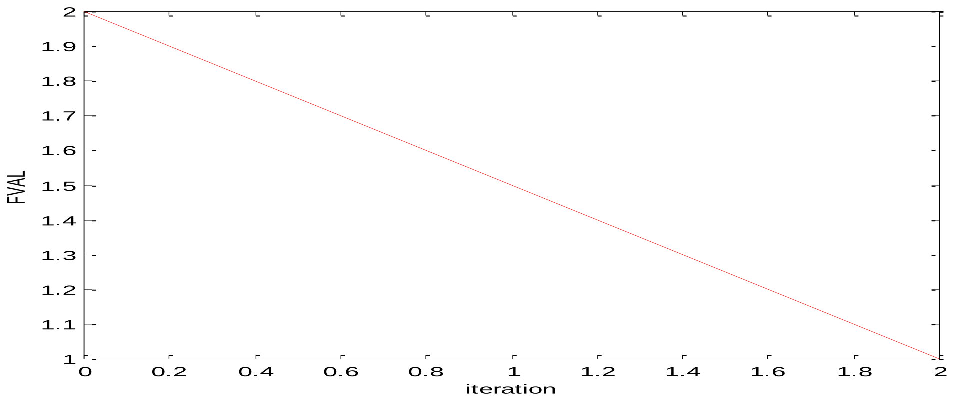


Figure (1)

Shows Function (1) Least square, with Dimension 2, and starting point [0;0]

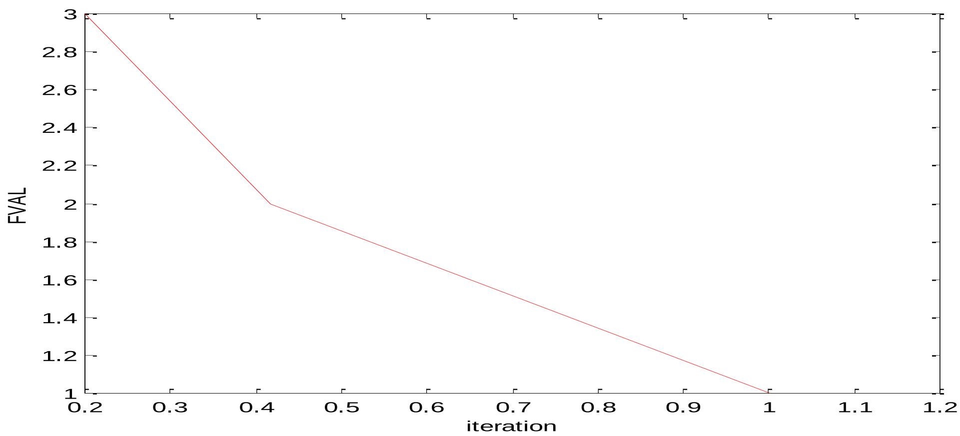


Figure (2)

Shows Function Rosen broc'k cliff, with Dimension 2, and starting point [0;0]

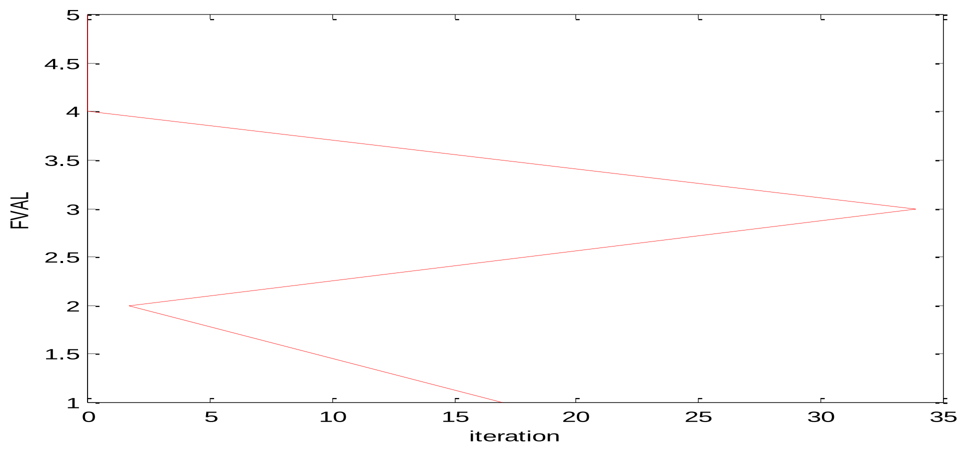


Figure (3)

Shows Function Generalized Edeger, with Dimension 2, and starting point [0;0]

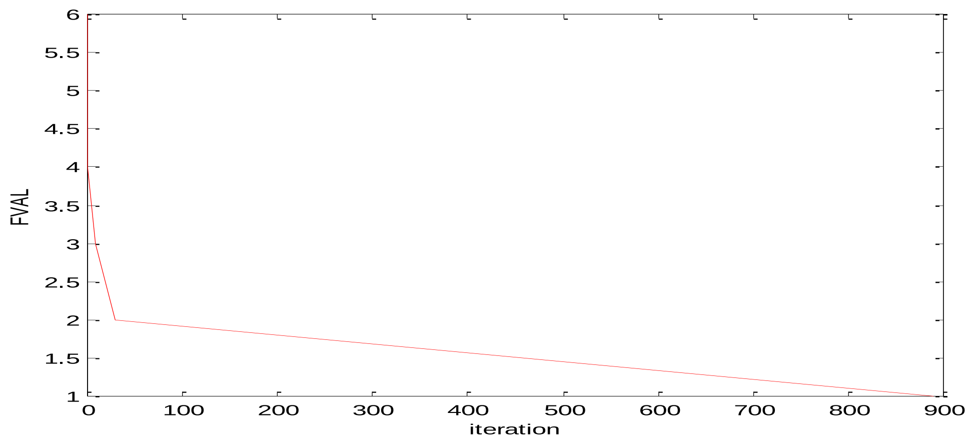


Figure (4)

Shows Function Extended Himmelblau function, with Dimension 2, and starting point [1;2]

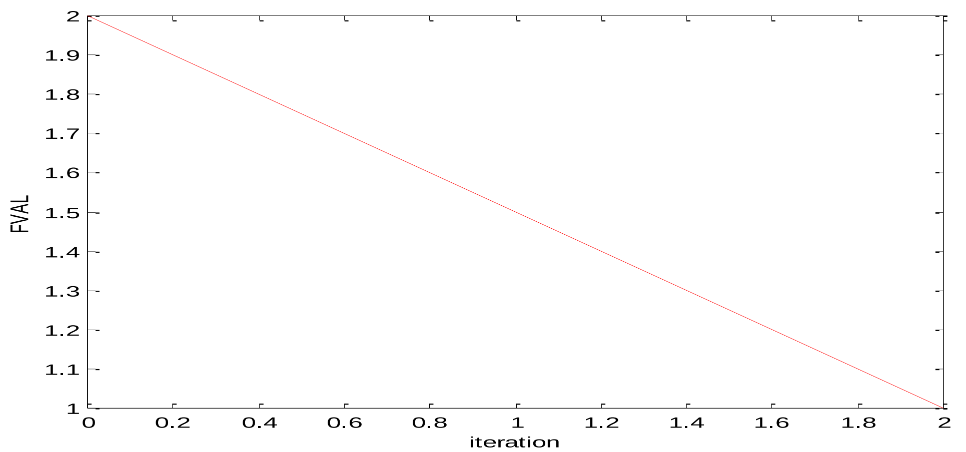

Figure (5)

Shows Function Rosen rock's function, with Dimension 2, and starting point [0;0]

## *Conclusion:*

In this paper the Broyden convex class (H-Version) in[1] was updated to modify convex combination class which is based on rank1 update with two formulas of rank 2 updates together , As we can see, the latest update is symmetric , positive definite ,the determent was found and the converge of this update was found as we see in Theorem(2).

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