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Some Bayes estimators for Exponentiated Rayleigh Distribution under Doubly Type II Censored Data

Muslim A. Qasim [1], Awatif R. Al-Dubaicy [2]

[1-2] Department of Mathematics, Education College, Mustansiriyah University,

Baghdad, Iraq ([1]muslim.abed@yahoo.com [2] draraldubaicy@yahoo.com)

ABSTRACT

Bayesian analysis for the shape parameter (β) of Exponentiated Rayleigh distribution or (Burr type x) has been considered in this paper, under two priors distributions [**Inverted Levy** and **Quasi**] using doubly type II censored data, with two loss functions [**Weighted** and **Precautionary**], we conducted a simulation study that evaluate the effectiveness of four estimators by six presumption experiments for varied values of distribution parameters and three sample sizes using mean squared error [MSE]. In general the best estimator of the shape parameter (β) of the (ERD) using Doubly type II censored data is under **Inverted Levy** prior under **Precautionary** loss function in all sample sizes

Keywords: *Bayesian analysis, Exponentiated Rayleigh distribution, Inverted Levy prior, doul type II censored data.*

1. Introduction

Company owners in markets ensure that their items are of high quality when showcasing them, excellent efficiency and long-lasting. This necessitates access to improving security, which necessitate the collecting of product information through so-called Causal research experiments, and also that the market demands these attributes to be present in industrial products **[1].** In the vast majority of cases life tests, it is possible that the researcher will not always be able to obtain all of the information here on particular experiment. As an example, in any industrial experiment, the modules may abruptly halt. Censored data (type I Censored and type II Censored Sample) refers to the information we obtain from these

*Corresponding author: Muslim A. Qasim
E-mail address: muslim.abed@yahoo.com

experiments. The two types are further separated into single and the double types. Our research will focus on double type II data[10]**.** As a result, the study concept necessitates the use of a statistical distribution as a failure model, and what has been used the distribution Exponentiated Rayleigh, this is distribution has received a lot of attention in recent times. An distribution of the set of parameters contains all of this distribution's attributes, for this useful statistical distribution, measurement may be found. In this context prior information was provided in the form of priors distributions [**Inverted Levy** and **Quasi**], under two loss functions[**Weighted** and **Precautionary**]. The purpose of the simulation study to determine the best estimator for the shape parameter (β), using mean square error.

The CDF for the Exponentiated Rayleigh distribution as:

$F(x;\alpha,\beta) = \left(1 - e^{-(\alpha x)^2}\right)^{\beta}, x > 0, \alpha, \beta > 0$ [5]

Where: (β) is the shape parameter and (α) is the scale parameter.

In this paper assume the scale parameter $(\alpha = 1)$ then the CDF and PDF becomes:

$$F(x;\beta) = \left(1 - e^{-x^2}\right)^{\beta}, x > 0, \beta > 0 \tag{1}$$

$$f(x;\beta) = 2\beta x e^{-x^2 + \ln\left(1-e^{-x^2}\right)^{-1}} e^{-\beta \ln\left(1-e^{-x^2}\right)^{-1}} \tag{2}$$

2. Censored Data: [4]

In rare cases, due to test costs or a tight timeline, we may not be able to screen all of the samples subjected to life testing (a waste of time, cost and effort). As a result, we either filter the sample for a specific period of time or a specific number of them.

***2.1 Doubly type II Censored Data*:**

We have (n) units that are being tested and we wish to censor the work of m units, we have this type of data, where $(m = (s - r) + 1)$ and $(r < s < n)$ for all censoring data $\left(x_{(r)}, \ldots, x_{(s)}\right)$ as a result. In this situation, the time is a random variable cannot be determined, the test is stopped to get (m) units of censored data, and once in the (s).

The likelihood function for this type of data as:

$$L(\beta|\underline{x}) = \left(\frac{n!}{(r-1)!\,(n-s)!}\right)\left(F(x_r)\right)^{r-1}\left(1 - F(x_s)\right)^{n-s} \prod_{i=r}^{s} f(x_i, \beta) \tag{3}$$

Such that:$x_1 < x_2 \ldots < x_{r-1} < x_r < x_{r+1} < \cdots < x_s < x_{s+1} < \cdots < x_n$.

3. Bayesian estimation: let x_1, $x_2, \ldots$, x_n, iid. from $f(x,\beta)$. We can update our ideas about the parameter by looking at each data observation density as a conditional density and determining the posterior density **[3]:**

$$\Phi(\beta|\underline{x}) = \frac{L\left(\underline{x}|\beta\right) g(\beta)}{\int_0^{\infty} L\left(\underline{x}|\beta\right) g(\beta)\, d\beta} \tag{4}$$

Where: $L(\underline{x}|\beta)$ is the likelihood density function.
$g(\beta)$ is the prior function for the shape parameter (β).
The loss functions using here as:

a) Weighted loss function[7]:

This is from the symmetric type given by: $L(\hat{\beta},\beta)=\frac{(\hat{\beta}-\beta)^2}{\beta}$

Then the Bayes estimator for the parameter (β) as:

$$\hat{\beta}_{W}=\frac{1}{E(\beta^{-1}|\underline{x})} \tag{5}$$

b) Precautionary loss function: [6]

This is from the asymmetric type given by:

$$L(\hat{\beta},\beta)=\frac{(\beta-\hat{\beta})^2}{\hat{\beta}}$$

Then Bayes estimator for the parameter (β) as:

$$\hat{\beta}_P=\sqrt{E(\beta^2|\underline{x})} \tag{6}$$

4. Bayesian analysis:

In this section Bayesian estimators for the shape parameter (β) found by substitute the equations (1) and (2) in (3), the likelihood function as:

$$L(\beta|\underline{x})=\left(\frac{n!}{(r-1)!\,(n-s)!}\right)\prod_{i=r}^{s} f(x_i,\beta)\left(F(x_r)\right)^{r-1}\left(1-F(x_s)\right)^{n-s}$$

$$L(\beta|\underline{x})=\left(\frac{n!}{(r-1)!\,(n-s)!}\right)\psi_1(x)\beta^m e^{-\beta\psi_2(x)}e^{-\beta(r-1)\ln\left(1-e^{-x_r^2}\right)^{-1}}\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}e^{-\beta j\ln\left(1-e^{-x_s^2}\right)^{-1}}$$

Such that: $\psi_1(x)=2^m e^{(\sum_{i=r}^{s}\ln x_i-\sum_{i=r}^{s}x_i^2+\sum_{i=1}^{s}\ln(1-e^{-x_i^2})^{-1}}$, $m=s-r+1$

And $\psi_2(x)=\sum_{i=r}^{s} ln(1-e^{-x_i^2})^{-1}$

$$=\left(\frac{n!\,\psi_1(x)}{(r-1)!\,(n-s)!}\right)\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\beta^m\, e^{-\beta[\psi_2(x)+(r-1)\ln\left(1-e^{-x_r^2}\right)^{-1}+jln\left(1-e^{-x_s^2}\right)^{-1}]}$$

$$\therefore L(\beta|\underline{x})=\psi_3(x)\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j\,\beta^m\, e^{-\beta(\psi_4(x))} \tag{7}$$

Such that: $\psi_3(x)=\frac{n!\psi_1(x)}{(r-1)!(n-s)!}$, $\psi_4(x)=\psi_2(x)+(r-1)\ln\left(1-e^{-x_r^2}\right)^{-1}+j\ln\left(1-e^{-x_s^2}\right)^{-1}$

4.1 posterior distribution under Inverted Levy prior:

The Inverted Levy prior is defined as:[8]

$$g(\beta) = \sqrt{\frac{b}{2\pi}}\,\beta^{\frac{-1}{2}}\, e^{\frac{-b\beta}{2}}, \beta > 0, b > 0 \tag{8}$$

The posterior distribution under Inverted Levy prior for this data we substitute equations (7) and (8) in (4) we get:

$$\Phi_I(\beta|\underline{x}) = \frac{\sqrt{\frac{b}{2\pi}}\,\psi_3(x)\sum_{j=0}^{n-S}\binom{n-s}{j}(-1)^j\ \beta^{m-\frac{1}{2}}\, e^{-\beta(\psi_4(x)+\frac{b}{2})}}{\sqrt{\frac{b}{2\pi}}\,\psi_3(x)\sum_{j=0}^{n-S}\binom{n-s}{j}(-1)^j\ \int_0^\infty \beta^{m-\frac{1}{2}}\, e^{-\beta(\psi_4(x)+\frac{b}{2})}\, d\beta}$$

$$\therefore\ \Phi_I(\beta|\underline{x}) = \frac{\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j\beta^{m-\frac{1}{2}}\, e^{-\beta(\psi_4(x)+\frac{b}{2})}}{\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j\,\frac{\Gamma\left(m+\frac{1}{2}\right)}{\left(\psi_4(x)+\frac{b}{2}\right)^{m+\frac{1}{2}}}} \tag{9}$$

4.2 The posterior distribution under Quasi prior:

The Quasi prior defined as: **[9]**

$$g(\beta) = \frac{1}{\beta^k}, \beta > 0, k > 0 \tag{10}$$

To find the posterior distribution under Quasi prior we substitute the equations (7) and equation (10) in (4) we get:

$$\Phi_Q(\beta|\underline{x}) = \frac{\psi_3(x)\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j\ \beta^{m-k}\, e^{-\beta\psi_4(x)}}{\psi_3(x)\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j\ \int_0^\infty \beta^{m-k}\, e^{-\beta\psi_4(x)}\, d\beta}$$

$$\Phi_Q(\beta|\underline{x}) = \frac{\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j\beta^{m-k}\, e^{-\beta\psi_4(x)}}{\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j\,\frac{\Gamma(m-k+1)}{(\psi_4(x))^{m-k+1}}} \tag{11}$$

4.3 Bayes estimator:

The Bayes estimator for the shape parameter (β) under Inverted Levy prior using two loss functions as:

a) Weighted loss function:

By equation (5) and (9), the Bayes estimator for (β) given by:

$$\therefore\ \hat{\beta}_{WI} = \frac{1}{E\left(\beta^{-1}\,|\underline{x}\,\right)} = \frac{\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j\,\frac{\Gamma\left(m+\frac{1}{2}\right)}{\left(\psi_4(x)+\frac{b}{2}\right)^{m+\frac{1}{2}}}}{\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j\,\frac{\Gamma\left(m-\frac{1}{2}\right)}{\left(\psi_4(x)+\frac{b}{2}\right)^{m-\frac{1}{2}}}} \tag{12}$$

b) Precautionary loss function:

By equations (6) and (9) the Bayes estimator for (β) given by:

$$\therefore \hat{\beta}_{PI} = \sqrt{E\left(\beta^2|\underline{x}\right)} = \left[\frac{\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j \frac{\Gamma\left(m+\frac{5}{2}\right)}{\left(\psi_4(x)+\frac{b}{2}\right)^{m+\frac{5}{2}}}}{\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j \frac{\Gamma\left(m+\frac{1}{2}\right)}{\left(\psi_4(x)+\frac{b}{2}\right)^{m+\frac{1}{2}}}}\right]^{\frac{1}{2}} \tag{13}$$

The Bayes estimator for the shape parameter (γ) under Quasi prior using two loss functions as:

a) Weighted loss function:

By equations (5) and (11), the Bayes estimator for (γ) given by:

$$\therefore \hat{\beta}_{WQ} = \frac{1}{E\left(\beta^{-1}|\underline{x}\right)} = \frac{\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j \frac{\Gamma(m-k+1)}{(\psi_4(x))^{m-k+1}}}{\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j \frac{\Gamma(m-k)}{(\psi_4(x))^{m-k}}} \tag{14}$$

b) Precautionary loss function:

By equations (6) and (11) the Bayes estimator for (β) given by:

$$\therefore \hat{\beta}_{PQ} = \sqrt{E\left(\beta^2|\underline{x}\right)} = \left[\frac{\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j \frac{\Gamma(m-k+3)}{(\psi_4(x))^{m-k+3}}}{\sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j \frac{\Gamma(m-k+1)}{(\psi_4(x))^{m-k+1}}}\right]^{\frac{1}{2}} \tag{15}$$

5. Study of Simulation:

Identify the best estimate for shape parameter (β) of the (ERD). A simulation technique is used, a variety of sample sizes (n=10, 50 and 100) in six experiments, by replicating each experiment (G= 1000), equation (1) is to generating different values of (x), by: R=F(x), where R is a random variable on interval (0,1) then:

$$x = \left(-\ln\left(1 - R^{\frac{1}{\beta}}\right)\right)^{\frac{1}{2}} \tag{16}$$

It is Monte _Carlo ś method [2], used to format of data generating (16) according to the values of (r=4, 35 and 80), (s=6, 42 and 93), (k=1.5), the default values parameters used as in table (1).

Table 1. The default Values of parameters(β, b)

Experiment	$\boldsymbol{\beta}$	$\boldsymbol{b}$
1	0.5	3
2	0.5	5
3	1.5	3

4	1.5	5
5	2.5	3
6	2.5	5

The best estimate for the shape parameter (β) is then compared using (MSE) as follows:

$$\text{MSE}\ (\hat{\beta}) = \frac{1}{G}\sum_{i=1}^{n}\left(\hat{\beta}-\beta\right)^2 \tag{17}$$

For G=(1000) the (MSE) for estimators by equations (12), (13), (14) and (15) as in tables (2, 3, 4, 5, 6 and 7).

Table 2. The (MSE) for estimate(β)/Ex.1, for (ERD)

n	**Inverted Levy Prior**		**Best**
	W_I	P_I	
10	0.1282	0.1054	P_I
50	0.1163	0.1041	P_I
100	0.0110	0.0123	W_I
	Quasi Prior		
	W_Q	P_Q	
10	0.1393	0.1143	P_Q
50	0.1207	0.1100	P_Q
100	0.0126	0.0148	W_Q

Table 3. The (MSE) for estimate (β) /Ex.2, for (ERD)

n	**Inverted Levy Prior**		**Best**
	W_I	P_I	
10	0.0292	0.0298	W_I
50	0.0153	0.0158	W_I
100	0.0112	0.0112	W_I, P_I
	Quasi Prior		
	W_Q	P_Q	
10	0.0385	0.0662	W_Q
50	0.0169	0.0209	W_Q
100	0.0120	0.0145	W_Q

Table 4. The (MSE) for estimate (β) /Ex.3, for (ERD)

n	Inverted Levy Prior		Best
	W_I	P_I	
10	1.9012	1.8229	P_I
50	1.5217	1.4589	P_I
100	0.0983	0.0936	P_I
	Quasi Prior		
	W_Q	P_Q	
10	1.9459	1.8650	P_Q
50	1.5479	1.4829	P_Q
100	0.1173	0.1375	W_Q

Table 5. The (MSE) for estimate (β) /Ex.4, for (ERD)

n	Inverted Levy Prior		Best
	W_I	P_I	
10	14034	1.2332	P_I
50	1.0907	1.0010	P_I
100	0.0990	0.0769	P_I
	Quasi Prior		
	W_Q	P_Q	
10	1.4314	1.23621	P_Q
50	1.2808	1.1846	P_Q
100	0.1074	0.1301	W_Q

Table 6. The (MSE) for estimate (β) /Ex.5, for (ERD)

n	Inverted Levy Prior		Best
	W_I	P_I	
10	4.7747	4.4553	P_I
50	3.6946	3.4834	P_I
100	0.2907	0.2269	P_I
	Quasi Prior		
	W_Q	P_Q	
10	4.8987	4.5512	P_Q
50	3.7143	3.4901	P_Q
100	0.3085	0.3663	W_Q

Table 7. The (MSE) for estimate (β) /Ex.6, for (ERD)

n	Inverted Levy Prior		Best
	W_I	P_I	
10	3.9637	3.2848	P_I
50	3.4963	3.1527	P_I
100	0.4185	0.2952	P_I
	Quasi Prior		
	W_Q	P_Q	
10	3.8537	3.2911	P_Q
50	3.3978	3.1529	P_Q
100	0.2997	0.3606	W_Q

Where:

$\boldsymbol{W_I}$: The **Weighted** loss function under **Inverted Levy** prior.

$\boldsymbol{P_I}$: The **Precautionary** loss function under **Inverted Levy** prior.

$\boldsymbol{W_Q}$: The **Weighted** loss function under **Quasi** prior.

$\boldsymbol{P_Q}$: The **Precautionary** loss function under **Quasi** prior.

6. Conclusions

From table (2) belong experiment (1) which contains the simulation results of MSE for Bayes estimator of the shape parameter (β), the best estimator under **Inverted Levy** prior using **Weighted** loss function in (n = 10,50), where in large sample size (100), the best estimator under the same prior using two loss functions (**Weighted** and **Precautionary**), but from table (3) belong experiment (2) the best estimator under **Inverted Levy** prior using **Precautionary** loss function in (n=10,50), where in large sample size (100),the best estimator under the same prior using **Weighted** loss function. From tables (4, 5, 6 and 7) belong experiments (3, 4, 5 and 6), in general the best estimator of the shape parameter (β) of the (ERD) using Doubly type II censored data is under **Inverted Levy** prior under **Precautionary** loss function in all sample sizes. Where the effectiveness of the default values of the parameter is that the smaller its value, the less the MES amount, and this means that the estimator approaches the exact value.

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