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Two Doubly Truncated Semicircular Distributions: Some Important Properties

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A B S T R A C T

In this paper, two doubly truncated semicircular distributions, doubly truncated Semicircular Exponentiated Weibull ($\boldsymbol{TSEW}$) and doubly truncated Semicircular Generalized Gompertz ($\boldsymbol{TSGGo}$) are presented. The most important statistical properties, including moments, characteristic function, trigonometric moments, quantile function, simulated data, reliability stress strength model, Shannon entropy, and relative entropy, are obtained.

Keywords: *Semicircular distributions; Exponentiated Weibull; Generalized Gompertz; Entropy; Stress-Strength reliability*
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1. Introduction

In most cases in real life, the study and analysis of truncated probabilistic models makes more sense. This issue takes on a greater dimension when studying the angular and circular data. Below are some studies on both topics.

Moments of doubly truncated Logistic distribution are considered by Balakrishnan and Kocherlakota (1986). Mittal and Dahiya (1987) discussed some methods of estimation for doubly truncated normal distribution. Khurana and Jha (1987) derived an expression for rth moment function of order statistic of doubly truncated Pareto distribution. Wingo (1988) presented the doubly truncated Weibull distribution with some estimation issues. Mohie El-Din et al. (1997) studied the moments of order statistics from doubly truncated linear exponential distribution. Ismail and Abu-Youssef (2014) studied the recurrence relations between single and product moments of order statistics from doubly truncated modified Makeham distribution. Xin et al. (2020) presented an accelerated Life Test Method for the Doubly Truncated Burr Type XII Distribution. Abid and Jani (2021) presented two doubly truncated generalized distributions, doubly truncated generalized Gompertz distribution and doubly truncated Marshal-Olkin extended Uniform distribution. Toshihiro Abe et al (2010) applied Inverse Stereographic Projection to develop symmetric circular models. Dattatreya Rao et al (2011) generated Cauchy type models by inducing Stereographic Projection. Phani et al. (2013) constructed some semicircular distributions by applying Inverse Stereographic

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projection. Girija et al (2013) presented a new circular model called Stereographic Lognormal distribution on the lines of Minh and Farnum (2003). Dattatreya Rao et al. (2016) developed a circular logistic distribution by applying inverse stereographic projection. Goodness of fit is conducted for a real data. Yedlapalli et al (2017) derived the trigonometric Moments of the Stereographic Semicircular Gamma Distribution. Yedlapalli et al (2020) presented an arc tan-Exponential Type Distributions.

Suppose that $G(\theta)$ and $g(\theta)$ are the cdf and pdf of a semicircular distribution on the interval $[0,\pi)$, then the truncated cdf and pdf of that distribution on the interval $[a,b]$ are given respectively by

$$F(\theta)_T = \frac{G(\theta) - G(a)}{G(b) - G(a)} \; ; a < \theta < b \tag{1}$$

$$f(\theta)_T = \frac{g(\theta)}{G(b) - G(a)}; \; a < \theta < b \tag{2}$$

Where $0 < a < b < \pi$. The following two subsections are interested in using the cdf and pdf mentioned above to introduce new truncated semicircular distributions that are useful for studying truncated semicircular data.

2. Truncated Semicircular Exponentiated Weibull ($TSEW$) Distribution

Suppose $G(.)$ and $g(.)$ in (1) and (2) represent the cdf and pdf of the semicircular exponentiated Weibull distribution that are given respectively,

$$G(\theta)_{SEW} = \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha} \; ; \; \theta \in [0,\pi) \, , \lambda, \alpha, \beta > 0 \tag{3}$$

$$g(\theta)_{SEW} = \frac{\lambda\alpha}{2\beta^{\lambda}} \tan^{\lambda-1}\left(\frac{\theta}{2}\right) e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}} \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha-1} \sec^2\left(\frac{\theta}{2}\right) \tag{4}$$

Then the cdf and pdf of $TSEW$ are given respectively by

$$F(\theta)_{TSEW} = \frac{\left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha} - \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}{\left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha} - \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}} \tag{5}$$

$$f(\theta)_{TSEW} = \frac{\frac{\lambda\alpha}{2\beta^{\lambda}}\tan^{\lambda-1}\left(\frac{\theta}{2}\right)e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha-1}\sec^{2}\left(\frac{\theta}{2}\right)}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}, 0 < a < \theta < b < \pi ; \lambda, \alpha, \beta > 0 \quad (6)$$

The $TSEW$ distribution reliability measures include reliability function $\tau_1(\theta)_{TSEW}$, hazard function $\tau_2(\theta)_{TSEW}$, cumulative hazard function $\tau_3(\theta)_{TSEW}$, and reverse hazard function $\tau_4(\theta)_{TSEW}$ can easily be written respectively as

$$\tau_1(\theta)_{TSEW} = 1 - F(\theta)_{TSEW}$$

$$= 1 - \frac{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}} \quad (7)$$

$$\tau_2(\theta)_{TSEW} = \frac{f(\theta)_{TSEW}}{1 - F(\theta)_{TSEW}}$$

$$= \frac{\frac{\lambda\alpha}{2\beta^{\lambda}}\tan^{\lambda-1}\left(\frac{\theta}{2}\right)e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha-1}\sec^{2}\left(\frac{\theta}{2}\right)}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha}} \quad (8)$$

$$\tau_3(\theta)_{TSEW} = -\ln(1 - F(\theta)_{TSEW})$$

$$= -\ln\left(1 - \frac{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}\right) \quad (9)$$

$$\tau_4(\theta)_{TSEW} = \frac{f(\theta)_{TSEW}}{F(\theta)_{TSEW}}$$

$$= \frac{\frac{\lambda\alpha}{2\beta^{\lambda}}\tan^{\lambda-1}\left(\frac{\theta}{2}\right)e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha-1}\sec^{2}\left(\frac{\theta}{2}\right)}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}} \quad (10)$$

The r^{th} non-central moment of the $TSEW$ distribution, $E(\theta^r)_{TSEW}$, can be obtained as follows, where, $\sec^2\left(\frac{\theta}{2}\right) = 1 + \tan^2\left(\frac{\theta}{2}\right)$, and

$$A = \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha} - \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}$$

$$E(\theta^r)_{TSEW} = \int_a^b \theta^r f(\theta)_{TSEW}\, d\theta$$

$$= \int_a^b \theta^r \frac{\lambda\alpha}{2A\beta^{\lambda}} \tan^{\lambda-1}\left(\frac{\theta}{2}\right) e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}} \left[1 - e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha-1} \left(1 + \tan^2\left(\frac{\theta}{2}\right)\right) d\theta \quad (10)$$

Using the transformation $x = v\tan\left(\frac{\theta}{2}\right), \theta = 2\tan^{-1}\left(\frac{x}{v}\right)$, and $d\theta = \frac{2}{v+\frac{x^2}{v}}dx$ where $x \in [a^*, b^*]$, $a^* = v\tan\left(\frac{a}{2}\right)$ and $b^* = v\tan\left(\frac{b}{2}\right)$, then (10) will be

$$E(\theta^r)_{TSEW} = \frac{\lambda\alpha}{2A\beta^{\lambda}} \int_{a^*}^{b^*} \left(2\tan^{-1}\left(\frac{x}{v}\right)\right)^r \left(\frac{x}{v}\right)^{\lambda-1} e^{-\left(\frac{x}{\beta v}\right)^{\lambda}} \left[1 - e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{\alpha-1} \left(1 + \left(\frac{x}{v}\right)^2\right) \frac{2}{v+\frac{x^2}{v}} dx$$

$$= 2^r \frac{\lambda\alpha}{A(\beta v)^{\lambda}} \int_{a^*}^{b^*} \left(\tan^{-1}\left(\frac{x}{v}\right)\right)^r x^{\lambda-1} e^{-\left(\frac{x}{\beta v}\right)^{\lambda}} \left[1 - e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{\alpha-1} dx$$

Based on , $\tan^{-1}(x) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}\,(k!)^2\,(2k+1)} \left(\frac{x^2}{x^2+1}\right)^{k+\frac{1}{2}}$; $x^2 < \infty$, , we get

$$E(\theta^r)_{TSEW} = 2^r \frac{\lambda\alpha}{A(\beta v)^{\lambda}} \int_{a^*}^{b^*} \left(\sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} \left(\frac{\left(\frac{x}{v}\right)^2}{\left(\frac{x}{v}\right)^2+1}\right)^{k+\frac{1}{2}}\right)^r x^{\lambda-1}$$

$$e^{-\left(\frac{x}{\beta v}\right)^{\lambda}} \left[1 - e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{\alpha-1} dx$$

Let $u = \frac{\left(\frac{x}{v}\right)^2}{\left(\frac{x}{v}\right)^2+1} \Rightarrow x = v\left(\frac{1}{u} - 1\right)^{-1/2} \Rightarrow dx = \frac{v}{2u^2}\left(\frac{1}{u} - 1\right)^{-3/2} du$, where $u \in [a^{**}, b^{**}]$ with $a^{**} = \frac{\left(\frac{a^*}{v}\right)^2}{\left(\frac{a^*}{v}\right)^2+1}$, and $b^{**} = \frac{\left(\frac{b^*}{v}\right)^2}{\left(\frac{b^*}{v}\right)^2+1}$. Now

$$E(\theta^r)_{TSEW} = 2^r \frac{\lambda\alpha}{A(\beta v)^{\lambda}} \int_{a^{**}}^{b^{**}} \left(\sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} u^{k+\frac{1}{2}}\right)^r \left(v\left(\frac{1}{u} - 1\right)^{-1/2}\right)^{\lambda-1}$$

$$e^{-\frac{1}{(\beta v)^{\lambda}}\left(v\left(\frac{1}{u}-1\right)^{-1/2}\right)^{\lambda}} \left[1 - e^{-\frac{1}{(\beta v)^{\lambda}}\left(v\left(\frac{1}{u}-1\right)^{-1/2}\right)^{\lambda}}\right]^{\alpha-1} \frac{v}{2u^2}\left(\frac{1}{u} - 1\right)^{-3/2} du$$

$$= 2^{r-1} \frac{\lambda\alpha v^{\lambda}}{A(\beta v)^{\lambda}} \int_{a^{**}}^{b^{**}} \left(\sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} u^k\right)^r u^{\frac{r}{2}-2} \left(\frac{1}{u} - 1\right)^{-\frac{\lambda}{2}-1}$$

$$e^{-\frac{v^{\lambda}}{(\beta v)^{\lambda}}\left(\frac{1}{u}-1\right)^{-\lambda/2}} \left[1 - e^{-\frac{v^{\lambda}}{(\beta v)^{\lambda}}\left(\frac{1}{u}-1\right)^{-\lambda/2}}\right]^{\alpha-1} du$$

$$= 2^{r-1} \frac{\lambda\alpha}{A\beta^{\lambda}} \int_{a^{**}}^{b^{**}} \left(\sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} u^k\right)^r u^{\frac{r}{2}-2} \left(\frac{1}{u} - 1\right)^{-\frac{\lambda}{2}-1}$$

$$e^{-\frac{1}{\beta^{\lambda}}\left(\frac{1}{u}-1\right)^{-\lambda/2}} \left[1 - e^{-\frac{1}{\beta^{\lambda}}\left(\frac{1}{u}-1\right)^{-\lambda/2}}\right]^{\alpha-1} du$$

Based on $(1-z)^n=\sum_{k=0}^{\infty}(-1)^k\binom{n}{k}z^k$; $|z|<1$ and $n>0$, $\left[1-e^{-\frac{1}{\beta^\lambda}\left(\frac{1}{u}-1\right)^{-\lambda/2}}\right]^{\alpha-1}=\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}e^{-\frac{i}{\beta^\lambda}\left(\frac{1}{u}-1\right)^{-\lambda/2}}$. Now

$$E(\theta^r)_{TSEW}=2^{r-1}\frac{\lambda\alpha}{A\beta^\lambda}\int_{a^{**}}^{b^{**}}\left(\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}u^k\right)^r u^{\frac{r}{2}-2}\left(\frac{1}{u}-1\right)^{-\frac{\lambda}{2}-1}\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}e^{-\frac{(i+1)}{\beta^\lambda}\left(\frac{1}{u}-1\right)^{-\lambda/2}}du$$

and based on $e^{-z}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}z^k$, $E(\theta^r)_{TSEW}$ will be

$$E(\theta^r)_{TSEW}=2^{r-1}\frac{\lambda\alpha}{A\beta^\lambda}\int_{a^{**}}^{b^{**}}\left(\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}u^k\right)^r u^{\frac{r}{2}-2}\left(\frac{1}{u}-1\right)^{-\frac{\lambda}{2}-1}\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(\frac{(i+1)}{\beta^\lambda}\left(\frac{1}{u}-1\right)^{-\frac{\lambda}{2}}\right)^j du$$

$$=2^{r-1}\frac{\lambda\alpha}{A}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j}{j!\,\beta^{\lambda(j+1)}}\binom{\alpha-1}{i}\int_{a^{**}}^{b^{**}}\left(\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}u^k\right)^r u^{\frac{r}{2}-2}\left(\frac{1}{u}-1\right)^{-\frac{\lambda}{2}(j+1)-1}du$$

Using Newton Binomial series, $(a+b)^n=\sum_{k=0}^{\infty}\binom{n}{k}a^{n-k}b^k=\sum_{k=0}^{\infty}\binom{n}{k}a^k b^{n-k}$; $n\geq 0$, then, $\left(\frac{1}{u}-1\right)^{-\frac{\lambda}{2}(j+1)-1}=\sum_{\ell=0}^{\infty}\binom{-\frac{\lambda}{2}(j+1)-1}{\ell}(-1)^\ell\left(\frac{1}{u}\right)^{-\frac{\lambda}{2}(j+1)-\ell-1}$, we get

$$E(\theta^r)_{TSEW}=2^{r-1}\frac{\lambda\alpha}{A}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j}{j!\,\beta^{\lambda(j+1)}}\binom{\alpha-1}{i}\int_{a^{**}}^{b^{**}}\left(\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}u^k\right)^r u^{\frac{r}{2}-2}\sum_{\ell=0}^{\infty}\binom{-\frac{\lambda}{2}(j+1)-1}{\ell}(-1)^\ell\left(\frac{1}{u}\right)^{-\frac{\lambda}{2}(j+1)-\ell-1}du$$

$$=2^{r-1}\frac{\lambda\alpha}{A}\sum_{i,j,\ell=0}^{\infty}\frac{(-1)^{i+j+\ell}(i+1)^j}{j!\,\beta^{\lambda(j+1)}}\binom{\alpha-1}{i}\binom{-\frac{\lambda}{2}(j+1)-1}{\ell}\int_{a^{**}}^{b^{**}}\left(\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}u^k\right)^r u^{\frac{1}{2}(r+\lambda(j+1))+\ell-1}du$$

According to $\left(\sum_{k=0}^{\infty}a_k u^k\right)^r=\sum_{k=0}^{\infty}b_k u^k$; r is a natural number, $b_0=a_0^r$ and $b_m=\frac{1}{a_0\,m}\sum_{k=1}^{m}(kr-m+k)\,a_k b_{m-k}$; $m\geq 1$, $E(\theta^r)_{TSEW}$ with $a_k=\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}$ will be

$$E(\theta^r)_{TSEW}=2^{r-1}\frac{\lambda\alpha}{A}\sum_{i,j,\ell,k=0}^{\infty}\frac{(-1)^{i+j+\ell}(i+1)^j b_k}{j!\,\beta^{\lambda(j+1)}}\binom{\alpha-1}{i}\binom{-\frac{\lambda}{2}(j+1)-1}{\ell}\int_{a^{**}}^{b^{**}}u^{k+\ell+\frac{1}{2}(r+\lambda(j+1))-1}du$$

$$=2^{r-1}\frac{\lambda\alpha}{A}\sum_{i,j,\ell,k=0}^{\infty}\frac{(-1)^{i+j+\ell}(i+1)^j\,b_k}{j!\,\beta^{\lambda(j+1)}}\binom{\alpha-1}{i}\binom{-\frac{\lambda}{2}(j+1)-1}{\ell}\frac{{b^{**}}^{k+\ell+\frac{1}{2}(r+\lambda(j+1))}-{a^{**}}^{k+\ell+\frac{1}{2}(r+\lambda(j+1))}}{k+\ell+\frac{1}{2}(r+\lambda(j+1))}$$

Thus, the r^{th} non-central moment of the $TSEW$ distribution is given by

$$E(\theta^r)_{TSEW} = 2^{r-1} \frac{\lambda\alpha}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha} - \left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}$$

$$\sum_{i,j,\ell,k=0}^{\infty} \frac{(-1)^{i+j+\ell}\,(i+1)^j\, b_k}{j!\,\beta^{\lambda(j+1)}} \binom{\alpha-1}{i}$$

$$\binom{-\frac{\lambda}{2}(j+1)-1}{\ell} \frac{b^{**k+\ell+\frac{1}{2}(r+\lambda(j+1))} - a^{**k+\ell+\frac{1}{2}(r+\lambda(j+1))}}{k+\ell+\frac{1}{2}(r+\lambda(j+1))} \qquad (11)$$

So, the characteristic function of the $TSEW$ distribution can be obtained as

$$\varphi_p(\theta)_{TSEW} = E\left(e^{ip\theta}\right)_{TSEW} = \sum_{r=0}^{\infty} \frac{(ip)^r}{r!} E(\theta^r)_{TSEW}$$

$$\varphi_p(\theta)_{TSEW} = \frac{\lambda\alpha}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha} - \left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}$$

$$\sum_{i,j,\ell,k,r=0}^{\infty} \frac{(-1)^{i+j+\ell}\,(i+1)^j\,(ip)^r 2^{r-1}\, b_k}{j!\,r!\,\beta^{\lambda(j+1)}} \binom{\alpha-1}{i}$$

$$\binom{-\frac{\lambda}{2}(j+1)-1}{\ell} \frac{b^{**k+\ell+\frac{1}{2}(r+\lambda(j+1))} - a^{**k+\ell+\frac{1}{2}(r+\lambda(j+1))}}{k+\ell+\frac{1}{2}(r+\lambda(j+1))} \qquad (12)$$

Furthermore, from $\varphi_p(\theta)_{TSEW}$, the p^{th}; $p = 0, \pm 1, \pm 2, \ldots$ non-central trigonometric moments can be obtained as

$\varphi_p(\theta)_{TSEW} = \nabla_p + i\,\Delta_p = E\left(\cos(p\theta)\right)_{TSEW} + i\,E\left(\sin(p\theta)\right)_{TSEW}$, where

$$\Delta_p = E(\sin(p\theta))_{TSEW} = \sum_{l=0}^{\infty} \frac{(-1)^l p^{2l+1}}{(2l+1)!} E(\theta^{2l+1})_{TSEW}$$

$$\nabla_p = E(\cos(p\theta))_{TSEW} = \sum_{l=0}^{\infty} \frac{(-1)^l p^{2l}}{(2l)!} E(\theta^{2l})_{TSEW}$$

where $E(\theta^{2l+1})_{TSEW}$ and $E(\theta^{2l})_{TSEW}$ as in (11) respectively with $r = 2l+1$ and $r = 2l$.

2.1 Stress Strength Model of TSEW Distribution

Consider two independent random variables, say $Y{:}\,stress$ and Z: strength, that follow $TSEW$ distribution with different parameters. The reliability stress strength model of the $TSEW$ distribution can be obtained by

$$SS_{TSEW} = P(Y < Z)_{TSEW} = E(F_Y(\theta)_{TSEW}) = \int_a^b F_Y(\theta)_{TSEW} f_Z(\theta)_{TSEW}\, d\theta \qquad (13)$$

where $F_Y(\theta)_{TSEW}$ represents the cdf of the $TSEW$ distribution as in (5) with parameters $\lambda_1, \alpha_1, \beta_1$, i.e.

$$F_Y(\theta)_{TSEW} = \frac{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1} - \left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}}{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{b}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1} - \left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}}$$

and $f_Z(\theta)_{TSEW}$ represents the pdf of the $TSEW$ distribution with parameters λ, α, β as in (6). Since,

$$\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\alpha_1}{k} \left(\frac{k}{\beta_1^{\lambda_1}}\right)^m \tan^{m\lambda_1}\left(\frac{\theta}{2}\right)$$, we get

$$F_Y(\theta)_{TSEW} = \frac{\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!}\binom{\alpha_1}{k}\left(\frac{k}{\beta_1^{\lambda_1}}\right)^m \tan^{m\lambda_1}\left(\frac{\theta}{2}\right) - \left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}}{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{b}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1} - \left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}} \quad (14)$$

Substituting (14) in (13), we get

$$SS_{TSEW} = \int_a^b \frac{\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!}\binom{\alpha_1}{k}\left(\frac{k}{\beta_1^{\lambda_1}}\right)^m \tan^{m\lambda_1}\left(\frac{\theta}{2}\right) - \left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}}{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{b}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1} - \left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}} f_Z(\theta)_{TSEW}\, d\theta$$

$$= \int_a^b \frac{\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!}\binom{\alpha_1}{k}\left(\frac{k}{\beta_1^{\lambda_1}}\right)^m \tan^{m\lambda_1}\left(\frac{\theta}{2}\right)}{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{b}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1} - \left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}} f_Z(\theta)_{TSEW}\, d\theta$$

$$- \frac{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}}{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{b}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1} - \left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}}$$

$$= \frac{\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!}\binom{\alpha_1}{k}\left(\frac{k}{\beta_1^{\lambda_1}}\right)^m}{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{b}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1} - \left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}}$$

$$\int_a^b \frac{\frac{\lambda\alpha}{2\beta^{\lambda}}\tan^{m\lambda_1+\lambda-1}\left(\frac{\theta}{2}\right) e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha-1}\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha} - \left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}} d\theta$$

$$- \frac{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}}{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{b}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1} - \left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}}$$

Let, $I = \int_a^b \frac{\lambda\alpha}{2\beta^{\lambda}}\tan^{m\lambda_1+\lambda-1}\left(\frac{\theta}{2}\right) e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha-1}\left(1+\tan^2\left(\frac{\theta}{2}\right)\right) d\theta$

Using the transformation, $x = v\tan\left(\frac{\theta}{2}\right)$, $\theta = 2\tan^{-1}\left(\frac{x}{v}\right)$ and $d\theta = \frac{2}{v+\frac{x^2}{v}}dx$, then

$$I = \frac{\lambda\alpha}{2\beta^{\lambda}}\int_{a^*}^{b^*}\left(\frac{x}{v}\right)^{m\lambda_1+\lambda-1} e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{\alpha-1}\left(1+\left(\frac{x}{v}\right)^2\right)\frac{2}{v+\frac{x^2}{v}}dx$$

$$= \frac{\lambda\alpha}{v\beta^{\lambda}}\int_{a^*}^{b^*}\left(\frac{x}{v}\right)^{m\lambda_1+\lambda-1} e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{\alpha-1} dx$$

Since, $\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{\alpha-1} = \sum_{i=0}^{\infty}(-1)^{i}\binom{\alpha-1}{i}e^{-i\left(\frac{x}{\beta v}\right)^{\lambda}}$, then

$$I = \frac{\lambda\alpha}{v\beta^{\lambda}}\sum_{i=0}^{\infty}(-1)^{i}\binom{\alpha-1}{i}\int_{a^*}^{b^*}\left(\frac{x}{v}\right)^{m\lambda_1+\lambda-1} e^{-(i+1)\left(\frac{x}{\beta v}\right)^{\lambda}}\,dx$$

$$= \frac{\lambda\alpha}{v\beta^{\lambda}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{j!}(i+1)^{j}\binom{\alpha-1}{i}\int_{a^*}^{b^*}\left(\frac{x}{v}\right)^{m\lambda_1+\lambda-1}\left(\frac{x}{\beta v}\right)^{j\lambda}dx$$

$$= \lambda\alpha\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\,(i+1)^{j}}{j!\,v^{m\lambda_1+(j+1)\lambda}\beta^{(j+1)\lambda}}\binom{\alpha-1}{i}\int_{a^*}^{b^*}x^{m\lambda_1+(j+1)\lambda-1}\,dx$$

$$= \lambda\alpha\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\,(i+1)^{j}}{j!\,v^{m\lambda_1+(j+1)\lambda}\beta^{(j+1)\lambda}}\binom{\alpha-1}{i}\frac{{b^*}^{m\lambda_1+(j+1)\lambda}-{a^*}^{m\lambda_1+(j+1)\lambda}}{m\lambda_1+(j+1)\lambda}$$

$$= \frac{\lambda\alpha\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j+k+m}\,(i+1)^{j}}{j!m!v^{m\lambda_1+(j+1)\lambda}\beta^{(j+1)\lambda}}\binom{\alpha-1}{i}\binom{\alpha_1}{k}\left(\frac{k}{\beta_1^{\lambda_1}}\right)^{m}}{\left(\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{b}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}-\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}\right)}$$

$$\frac{{b^*}^{m\lambda_1+(j+1)\lambda}-{a^*}^{m\lambda_1+(j+1)\lambda}}{(m\lambda_1+(j+1)\lambda)\left(\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}\right)}$$

$$-\frac{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}}{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{b}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}-\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}} \qquad (15)$$

2.2 Shannon Entropy of $TSEW$ Distribution

The Shannon entropy SH_{TSEW} can be obtained as $E(-\ln(f(\theta)_{TSEW})$. Since

$$\ln(f(\theta)_{TSEW}) = \ln\left(\frac{\lambda\alpha}{2\beta^{\lambda}}\right)+(\lambda-1)\ln\left(\tan\left(\frac{\theta}{2}\right)\right)-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}$$
$$+(\alpha-1)\ln\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right)+2\ln\left(\sec\left(\frac{\theta}{2}\right)\right)$$
$$-\ln\left(\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}\right)$$

, then

$$SH_{TSEW} = \ln\left(\frac{2\beta^{\lambda}}{\lambda\alpha}\right)-(\lambda-1)E\left(\ln\left(\tan\left(\frac{\theta}{2}\right)\right)\right)+\frac{1}{\beta^{\lambda}}E\left(\tan^{\lambda}\left(\frac{\theta}{2}\right)\right)$$
$$-(\alpha-1)E\left(\ln\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]\right)-2E\left(\ln\left(\sec\left(\frac{\theta}{2}\right)\right)\right)$$
$$+\ln\left(\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}\right) \qquad (16)$$

Let $\ddot{I}_1 = E\left(\ln\left(\tan\left(\frac{\theta}{2}\right)\right)\right), \ddot{I}_2 = E\left(\tan^{\lambda}\left(\frac{\theta}{2}\right)\right)$ $\quad \ddot{I}_3 = E\left(\ln\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]\right)$ and $\ddot{I}_4 = E\left(\ln\left(\sec\left(\frac{\theta}{2}\right)\right)\right)$.

Now, for $\ddot{I}_1 = \int_a^b \ln\left(\tan\left(\frac{\theta}{2}\right)\right)f(\theta)_{TSEW}d\theta$, based on $\sec^2\left(\frac{\theta}{2}\right) = 1+\tan^2\left(\frac{\theta}{2}\right)$ and

$$A=\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}$$, we get

$$\ddot{I}_1=\frac{\lambda\alpha}{2A\beta^{\lambda}}\int_a^b \ln\left(\tan\left(\frac{\theta}{2}\right)\right)\tan^{\lambda-1}\left(\frac{\theta}{2}\right)\ e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha-1}\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)d\theta$$

Using the transformation, $x=v\tan\left(\frac{\theta}{2}\right),\theta=2\tan^{-1}\left(\frac{x}{v}\right)$, and $d\theta=\frac{2}{v+\frac{x^2}{v}}dx$, then

$$\ddot{I}_1=\frac{\lambda\alpha}{2A\beta^{\lambda}}\int_{a^*}^{b^*}\ln\left(\frac{x}{v}\right)\left(\frac{x}{v}\right)^{\lambda-1}e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{\alpha-1}\left(1+\left(\frac{x}{v}\right)^2\right)\frac{2}{v+\frac{x^2}{v}}dx$$

$$=\frac{\lambda\alpha}{A\beta^{\lambda}v^{\lambda}}\int_{a^*}^{b^*}\ln\left(\frac{x}{v}\right)x^{\lambda-1}e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{\alpha-1}dx$$

Based on previous formulas , we get

$$e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{\alpha-1}=e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}e^{-i\left(\frac{x}{\beta v}\right)^{\lambda}}$$

$$=\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}e^{-(i+1)\left(\frac{x}{\beta v}\right)^{\lambda}}$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j}{j!}\binom{\alpha-1}{i}\left(\frac{x}{\beta v}\right)^{j\lambda}$$, So

$$\ddot{I}_1=\frac{\lambda\alpha}{A\beta^{\lambda}v^{\lambda}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j}{j!}\binom{\alpha-1}{i}\int_{a^*}^{b^*}\ln\left(\frac{x}{v}\right)x^{\lambda-1}\left(\frac{x}{\beta v}\right)^{j\lambda}dx$$

$$=\frac{\lambda\alpha}{A}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j}{j!\beta^{(j+1)\lambda}v^{(j+1)\lambda}}\binom{\alpha-1}{i}\int_{a^*}^{b^*}\ln\left(\frac{x}{v}\right)x^{(j+1)\lambda-1}dx$$

Using the retail integration, $\int u dv=uv-\int v du$, with

$u=\ln\left(\frac{x}{v}\right)\Rightarrow du=\frac{1}{x}dx, dv=x^{(j+1)\lambda-1}dx\Rightarrow v=\frac{x^{(j+1)\lambda}}{(j+1)\lambda}$, then

$$\int_{a^*}^{b^*}\ln\left(\frac{x}{v}\right)x^{(j+1)\lambda-1}dx=\ln\left(\frac{x}{v}\right)\frac{x^{(j+1)\lambda}}{(j+1)\lambda}\Bigg|_{a^*}^{b^*}-\frac{1}{(j+1)\lambda}\int_{a^*}^{b^*}x^{(j+1)\lambda-1}dx$$

$$=\frac{b^{*(j+1)\lambda}\ln\left(\frac{b^*}{v}\right)-a^{*(j+1)\lambda}\ln\left(\frac{a^*}{v}\right)}{(j+1)\lambda}-\frac{b^{*(j+1)\lambda}-a^{*(j+1)\lambda}}{\left((j+1)\lambda\right)^2}$$

Therefore, $\ddot{I}_1$ will be

$$\ddot{I}_1=\frac{\lambda\alpha}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j}{j!\beta^{(j+1)\lambda}v^{(j+1)\lambda}}\binom{\alpha-1}{i}\left(\frac{b^{*(j+1)\lambda}\ln\left(\frac{b^*}{v}\right)-a^{*(j+1)\lambda}\ln\left(\frac{a^*}{v}\right)}{(j+1)\lambda}-\frac{b^{*(j+1)\lambda}-a^{*(j+1)\lambda}}{\left((j+1)\lambda\right)^2}\right)\quad(17)$$

For $\ddot{I}_2=\int_a^b\tan^{\lambda}\left(\frac{\theta}{2}\right)f(\theta)_{TSEW}\,d\theta$,

$\sec^2\left(\frac{\theta}{2}\right)=1+\tan^2\left(\frac{\theta}{2}\right)$ and $A=\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}$, we get

$$\ddot{I}_2=\frac{\lambda\alpha}{2A\beta^{\lambda}}\int_a^b\tan^{2\lambda-1}\left(\frac{\theta}{2}\right)e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha-1}\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)d\theta$$

Using the transformation, $x=v\tan\left(\frac{\theta}{2}\right),\theta=2\tan^{-1}\left(\frac{x}{v}\right)$, and $d\theta=\frac{2}{v+\frac{x^2}{v}}dx$, then

$$\ddot{I}_2 = \frac{\lambda\alpha}{2A\beta^{\lambda}}\int_{a^*}^{b^*}\left(\frac{x}{v}\right)^{2\lambda-1} e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{\alpha-1}\left(1+\left(\frac{x}{v}\right)^2\right)\frac{2}{v+\frac{x^2}{v}}dx$$

$$= \frac{\lambda\alpha}{A\beta^{\lambda}v^{2\lambda}}\int_{a^*}^{b^*} x^{2\lambda-1} e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{\alpha-1} dx$$

Since $e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\left[1-e^{-\left(\frac{x}{\beta v}\right)^{\lambda}}\right]^{\alpha-1} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j}{j!}\binom{\alpha-1}{i}\left(\frac{x}{\beta v}\right)^{j\lambda}$, then

$\ddot{I}_2 = \frac{\lambda\alpha}{A}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\binom{\alpha-1}{i}(i+1)^j}{j!\beta^{(j+1)\lambda}v^{(j+2)\lambda}}\int_{a^*}^{b^*} x^{(j+2)\lambda-1}dx$, Therefore, $\ddot{I}_2$ will be

$$\ddot{I}_2 = \frac{\lambda\alpha}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}\binom{\alpha-1}{i}(i+1)^j}{j!\beta^{(j+1)\lambda}v^{(j+2)\lambda}}\left(\frac{b^{*(j+2)\lambda}-a^{*(j+2)\lambda}}{(j+2)\lambda}\right) \quad (18)$$

For $\ddot{I}_3 = \int_a^b \ln\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right) f(\theta)_{TSEW}\, d\theta$, using some previous formulas, then

$\ln\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right) = \sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{r\, m!}\left(\frac{r}{\beta^{\lambda}}\right)^m \tan^{m\lambda}\left(\frac{\theta}{2}\right)$. Now

$$\ddot{I}_3 = \sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{r\, m!}\left(\frac{r}{\beta^{\lambda}}\right)^m\int_a^b \tan^{m\lambda}\left(\frac{\theta}{2}\right) f(\theta)_{TSEW}\, d\theta$$

$$= \sum_{r=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{r\, m!}\left(\frac{r}{\beta^{\lambda}}\right)^m E\left(\tan^{m\lambda}\left(\frac{\theta}{2}\right)\right)$$

where $E\left(\tan^{m\lambda}\left(\frac{\theta}{2}\right)\right)$ can be attained similarly to $\ddot{I}_2$ with $\lambda = m\lambda$. Therefore, $\ddot{I}_3$ will be

$$\ddot{I}_3 = \frac{\lambda\alpha}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}\sum_{r=1}^{\infty}\sum_{i,j,m=0}^{\infty}\frac{(-1)^{i+j+m+1}\binom{\alpha-1}{i}(i+1)^j}{r\, j!\, m!\beta^{(j+1)\lambda}v^{(j+m+1)\lambda}}$$
$$\left(\frac{r}{\beta^{\lambda}}\right)^m\left(\frac{b^{*(j+m+1)\lambda}-a^{*(j+m+1)\lambda}}{(j+m+1)\lambda}\right) \quad (19)$$

Finally for $\ddot{I}_4 = \int_a^b \ln\left(\sec\left(\frac{\theta}{2}\right)\right) f(\theta)_{TSEW}d\theta = \int_a^b -\ln\left(\cos\left(\frac{\theta}{2}\right)\right) f(\theta)_{TSEW}d\theta$ and based on formula A_6,

$-\ln\left(\cos\left(\frac{\theta}{2}\right)\right) = \sum_{m=1}^{\infty}\frac{2^{2m-1}(2^{2m}-1)}{m(2m)!}|B_{2t}|\left(\frac{\theta}{2}\right)^{2m}$; $|\theta| < \pi$, with the Bernoulli numbers B_{2m}. Now

$$\ddot{I}_4 = \sum_{m=1}^{\infty}\frac{2^{2m-1}(2^{2m}-1)}{m(2m)!}|B_{2m}|\int_a^b\left(\frac{\theta}{2}\right)^{2m} f(\theta)_{TSEW}\, d\theta = \sum_{m=1}^{\infty}\frac{2^{2m}-1}{2m(2m)!}|B_{2m}|\, E(\theta^{2m})_{TSEW}$$

Based on (11) with $r = 2m$, $\ddot{I}_4$ will be

$$\ddot{I}_4 = \frac{\lambda\alpha}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}$$
$$\sum_{m=1}^{\infty}\sum_{i,j,\ell,k=0}^{\infty}\frac{(-1)^{i+j+\ell}(i+1)^j\, 2^{2(m-1)}(2^{2m}-1)b_k|B_{2m}|}{m(2m)!j!\,\beta^{\lambda(j+1)}}\binom{\alpha-1}{i}$$
$$\binom{-\frac{\lambda}{2}(j+1)-1}{\ell}\frac{b^{**k+\ell+\frac{1}{2}(r+\lambda(j+1))}-a^{**k+\ell+\frac{1}{2}(r+\lambda(j+1))}}{k+\ell+\frac{1}{2}(2m+\lambda(j+1))} \quad (20)$$

Therefore, the Shannon entropy of the $TSEW$ distribution can be obtained as

$$SH_{TSEW} = \ln\left(\frac{2\beta^{\lambda}}{\lambda\alpha}\right) - (\lambda-1)\ddot{I}_1 + \frac{1}{\beta^{\lambda}}\ddot{I}_2 - (\alpha-1)\ddot{I}_3 - 2\ddot{I}_4 + \ln\left(\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha} - \left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}\right) \quad (21)$$

where $\ddot{I}_1$, $\ddot{I}_2$, $\ddot{I}_3$ and $\ddot{I}_4$ are respectively given in $(17), (18), (19)$ and (20).

2.3 Relative Entropy of TSEW Distribution

The relative entropy of the $TSEW$ distribution can be obtained through the following formula

$$RE_{TSEW} = E\left(\ln\left[\frac{f(\theta)_{TSEW}}{f_1(\theta)_{TSEW}}\right]\right) = \int_a^b \ln\left[\frac{f(\theta)_{TSEW}}{f_1(\theta)_{TSEW}}\right] f(\theta)_{TSEW}\, d\theta \quad (22)$$

Taking the natural logarithm of the $f(\theta)_{TSEW}$ with parameters λ, α, β relative of the $f_1(\theta)_{TSEW}$ with parameters $\lambda_1, \alpha_1, \beta_1$, then

$$\ln\left[\frac{f(\theta)_{TSEW}}{f_1(\theta)_{TSEW}}\right] = \ln\left[\frac{\dfrac{\frac{\lambda\alpha}{\beta^{\lambda}}\tan^{\lambda-1}\left(\frac{\theta}{2}\right)e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right]^{\alpha-1}}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}}{\dfrac{\frac{\lambda_1\alpha_1}{\beta_1^{\lambda_1}}\tan^{\lambda_1-1}\left(\frac{\theta}{2}\right)e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1}}\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1-1}}{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{b}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}-\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}}}\right]$$

$$= \ln\left(\frac{\lambda\alpha\beta_1^{\lambda_1}}{\lambda_1\alpha_1\beta^{\lambda}}\right) + (\lambda-1)\ln\left(\tan\left(\frac{\theta}{2}\right)\right) - \left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda} + (\alpha-1)\ln\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right)$$
$$- (\lambda_1-1)\ln\left(\tan\left(\frac{\theta}{2}\right)\right) + \left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1} - (\alpha_1-1)\ln\left(1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1}}\right)$$
$$+ \ln\left(\frac{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{b}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}-\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}\right)$$

Now, the relative entropy of the $TSEW$ distribution can be obtained as

$$RE_{TSEW} = \ln\left(\frac{\lambda\alpha\beta_1^{\lambda_1}}{\lambda_1\alpha_1\beta^{\lambda}}\right) + (\lambda-\lambda_1)E\left(\ln\left(\tan\left(\frac{\theta}{2}\right)\right)\right)$$
$$-\frac{1}{\beta^{\lambda}}E\left(\tan^{\lambda}\left(\frac{\theta}{2}\right)\right) + \frac{1}{\beta_1^{\lambda_1}}E\left(\tan^{\lambda_1}\left(\frac{\theta}{2}\right)\right)$$
$$+(\alpha-1)E\left(\ln\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right)\right) - (\alpha_1-1)E\left(\ln\left(1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1}}\right)\right)$$
$$+ \ln\left(\frac{\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{b}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}-\left[1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{a}{2}\right)\right)^{\lambda_1}}\right]^{\alpha_1}}{\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{b}{2}\right)\right)^{\lambda}}\right]^{\alpha}-\left[1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{a}{2}\right)\right)^{\lambda}}\right]^{\alpha}}\right) \quad (23)$$

where $E\left(\ln\left(\tan\left(\frac{\theta}{2}\right)\right)\right)$, $E\left(\tan^{\lambda}\left(\frac{\theta}{2}\right)\right)$, $E\left(\tan^{\lambda_1}\left(\frac{\theta}{2}\right)\right)$, $E\left(\ln\left(1-e^{-\left(\frac{1}{\beta}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda}}\right)\right)$ and $E\left(\ln\left(1-e^{-\left(\frac{1}{\beta_1}\tan\left(\frac{\theta}{2}\right)\right)^{\lambda_1}}\right)\right)$ as previously extracted with specified parameters.

3. Truncated Semicircular Generalized Gompertz ($TSGGo$) Distribution

Suppose $G(.)$ and $g(.)$ in (1) and (2) represent the cdf and pdf of the semicircular generalized Gompertz distribution that are given respectively,

$$G(\theta)_{SGGo} = \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta} \;;\; \theta \in [0,\pi), \lambda, \beta, \delta, \alpha > 0 \qquad (24)$$

$$g(\theta)_{SGGo} = \frac{\alpha\lambda\beta}{2\delta} e^{\alpha\tan\left(\frac{\theta}{2}\right)} e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)} \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta-1} \sec^2\left(\frac{\theta}{2}\right) \qquad (25)$$

then the cdf and pdf of the $TSGGo$ distribution with parameters λ, β, δ, and $\alpha = \delta v$ are respectively given by

$$F(\theta)_{TSGGo} = \frac{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta} - \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}}{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta} - \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}} \qquad (26)$$

$$f(\theta)_{TSGGo} = \frac{\frac{\alpha\lambda\beta}{2\delta} e^{\alpha\tan\left(\frac{\theta}{2}\right)} e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)} \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta-1} \sec^2\left(\frac{\theta}{2}\right)}{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta} - \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}}$$

$$;0 < a < \theta < b < \pi \,;\, \lambda, \beta, \delta, \alpha > 0 \qquad (27)$$

The $TSGGo$ reliability measures can be easily found respectively as

$\tau_1(\theta)_{TSGGo} = 1 - F(\theta)_{TSGGo}$

$$= 1 - \frac{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta} - \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}}{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta} - \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}} \qquad (28)$$

$$\tau_2(\theta)_{TSGGo} = \frac{f(\theta)_{TSGGo}}{1-F(\theta)_{TSGGo}}$$

$$= \frac{\frac{\alpha\lambda\beta}{2\delta} e^{\alpha\tan\left(\frac{\theta}{2}\right)} e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)} \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta-1} \sec^2\left(\frac{\theta}{2}\right)}{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta} - \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta}} \quad (29)$$

$$\tau_3(\theta)_{TSGGo} = -\ln(1-F(\theta)_{TSGGo})$$

$$= -\ln\left(1 - \frac{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta} - \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}}{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta} - \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}}\right) \quad (30)$$

$$\tau_4(\theta)_{TSGGo} = \frac{f(\theta)_{TSGGo}}{F(\theta)_{TSGGo}} = \frac{\frac{\alpha\lambda\beta}{2\delta} e^{\alpha\tan\left(\frac{\theta}{2}\right)} e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)} \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta-1} \sec^2\left(\frac{\theta}{2}\right)}{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta} - \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}} \quad (31)$$

The r^{th} non-central moment of the $TSGGo$ distribution, $E(\theta^r)_{TSGGo}$ can be obtained as follows

$$E(\theta^r)_{TSGGo} = \int_a^b \theta^r f(\theta)_{TSGGo} d\theta$$

$$= \int_a^b \theta^r \frac{\alpha\lambda\beta}{2\dot{A}\delta} e^{\alpha\tan\left(\frac{\theta}{2}\right)} e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)} \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta-1} \left(1+\tan^2\left(\frac{\theta}{2}\right)\right) d\theta \quad (32)$$

Using the transformation $x = v\tan\left(\frac{\theta}{2}\right)$, $\theta = 2\tan^{-1}\left(\frac{x}{v}\right)$, and $d\theta = \frac{2}{v+\frac{x^2}{v}}dx$ where $x \in [a^*, b^*]$, $a^* = v\tan\left(\frac{a}{2}\right)$ and $b^* = v\tan\left(\frac{b}{2}\right)$, then

$$E(\theta^r)_{TSGGo} = 2^r \frac{\alpha\lambda\beta}{\dot{A}\delta v} \int_{a^*}^{b^*} \left(\tan^{-1}\left(\frac{x}{v}\right)\right)^r e^{\alpha\left(\frac{x}{v}\right)} e^{-\frac{\lambda}{\delta}\left(e^{\alpha\left(\frac{x}{v}\right)}-1\right)} \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\left(\frac{x}{v}\right)}-1\right)}\right]^{\beta-1} dx$$

$$= 2^r \frac{\alpha\lambda\beta}{\dot{A}\delta v} \int_{a^*}^{b^*} \left(\sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} \left(\frac{\left(\frac{x}{v}\right)^2}{\left(\frac{x}{v}\right)^2+1}\right)^{k+\frac{1}{2}}\right)^r e^{\alpha\left(\frac{x}{v}\right)} e^{-\frac{\lambda}{\delta}\left(e^{\alpha\left(\frac{x}{v}\right)}-1\right)} \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\left(\frac{x}{v}\right)}-1\right)}\right]^{\beta-1} dx$$

Let $u=\frac{\left(\frac{x}{v}\right)^2}{\left(\frac{x}{v}\right)^2+1} \Rightarrow x=v\left(\frac{1}{u}-1\right)^{-1/2} \Rightarrow dx=\frac{v}{2u^2}\left(\frac{1}{u}-1\right)^{-3/2} du$, where $u\in[a^{**},b^{**}]$ with $a^{**}=\frac{\left(\frac{a^*}{v}\right)^2}{\left(\frac{a^*}{v}\right)^2+1}$, and $b^{**}=\frac{\left(\frac{b^*}{v}\right)^2}{\left(\frac{b^*}{v}\right)^2+1}$. Now

$$E(\theta^r)_{TSGGo}=2^r\frac{\alpha\lambda\beta}{\dot{A}\delta v}\int_{a^{**}}^{b^{**}}\left(\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}u^{k+\frac{1}{2}}\right)^r e^{\alpha\left(\frac{1}{u}-1\right)^{-1/2}}$$

$$e^{-\frac{\lambda}{\delta}\left(e^{\alpha\left(\frac{1}{u}-1\right)^{-1/2}}-1\right)}\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\left(\frac{1}{u}-1\right)^{-1/2}}-1\right)}\right]^{\beta-1}\frac{v}{2u^2}\left(\frac{1}{u}-1\right)^{-3/2}du$$

Since, $\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\left(\frac{1}{u}-1\right)^{-1/2}}-1\right)}\right]^{\beta-1}=\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}e^{-\frac{i\lambda}{\delta}\left(e^{\alpha\left(\frac{1}{u}-1\right)^{-1/2}}-1\right)}$. Now

$$E(\theta^r)_{TSGGo}=2^{r-1}\frac{\alpha\lambda\beta}{\dot{A}\delta}\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}\int_{a^{**}}^{b^{**}}\left(\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}u^{k+\frac{1}{2}}\right)^r$$

$$e^{\alpha\left(\frac{1}{u}-1\right)^{-1/2}}e^{(i+1)\frac{\lambda}{\delta}}e^{-(i+1)\frac{\lambda}{\delta}e^{\alpha\left(\frac{1}{u}-1\right)^{-1/2}}}\frac{1}{u^2}\left(\frac{1}{u}-1\right)^{-3/2}du$$

By using $e^{-(i+1)\frac{\lambda}{\delta}e^{\alpha\left(\frac{1}{u}-1\right)^{-1/2}}}=\sum_{j=0}^{\infty}\frac{(-1)^j(i+1)^j}{j!}\left(\frac{\lambda}{\delta}\right)^j e^{j\alpha\left(\frac{1}{u}-1\right)^{-1/2}}$, we have

$$E(\theta^r)_{TSGGo}=2^{r-1}\frac{\alpha\lambda\beta}{\dot{A}\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j}{j!}\left(\frac{\lambda}{\delta}\right)^j\binom{\beta-1}{i}e^{(i+1)\frac{\lambda}{\delta}}$$

$$\int_{a^{**}}^{b^{**}}\left(\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}u^{k+\frac{1}{2}}\right)^r e^{(j+1)\alpha\left(\frac{1}{u}-1\right)^{-1/2}}\frac{1}{u^2}\left(\frac{1}{u}-1\right)^{-3/2}du$$

By using $e^{(j+1)\alpha\left(\frac{1}{u}-1\right)^{-1/2}}=\sum_{m=0}^{\infty}\frac{(j+1)^m}{m!}\alpha^m\left(\frac{1}{u}-1\right)^{-m/2}$, then

$$E(\theta^r)_{TSGGo}=2^{r-1}\frac{\alpha\lambda\beta}{\dot{A}\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j(j+1)^m}{j!m!}\alpha^m\left(\frac{\lambda}{\delta}\right)^j$$

$$\binom{\beta-1}{i}e^{(i+1)\frac{\lambda}{\delta}}\int_{a^{**}}^{b^{**}}\left(\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}u^k\right)^r u^{\frac{r}{2}-2}\left(\frac{1}{u}-1\right)^{-\frac{1}{2}(m+3)}du$$

By using A_4, $\left(\frac{1}{u}-1\right)^{-\frac{1}{2}(m+3)}=\sum_{\ell=0}^{\infty}\binom{-\frac{1}{2}(m+3)}{\ell}(-1)^{\ell}\left(\frac{1}{u}\right)^{-\frac{1}{2}(m+3)-\ell}$. We have

$$E(\theta^r)_{TSGGo}=2^{r-1}\frac{\alpha\lambda\beta}{\dot{A}\delta}\sum_{i,j,m,\ell=0}^{\infty}\frac{(-1)^{i+j+\ell}(i+1)^j(j+1)^m}{j!\,m!}\alpha^m\left(\frac{\lambda}{\delta}\right)^j$$

$$\binom{\beta-1}{i}\binom{-\frac{1}{2}(m+3)}{\ell}e^{(i+1)\frac{\lambda}{\delta}}\int_{a^{**}}^{b^{**}}\left(\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}u^k\right)^r u^{\frac{r}{2}-2}\left(\frac{1}{u}\right)^{-\frac{1}{2}(m+3)-\ell}du$$

$$=2^{r-1}\frac{\alpha\lambda\beta}{\dot{A}\delta}\sum_{i,j,m,\ell=0}^{\infty}\frac{(-1)^{i+j+\ell}(i+1)^j(j+1)^m}{j!\,m!}\alpha^m\left(\frac{\lambda}{\delta}\right)^j$$

$$\binom{\beta-1}{i}\binom{-\frac{1}{2}(m+3)}{\ell}e^{(i+1)\frac{\lambda}{\delta}}\int_{a^{**}}^{b^{**}}\left(\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}u^k\right)^r u^{\frac{1}{2}(r+m-1)+\ell}du$$

According to $\left(\sum_{k=0}^{\infty}a_k u^k\right)^r=\sum_{k=0}^{\infty}b_k u^k$; r is a natural number, $b_0=a_0^r$ and $b_m=\frac{1}{a_0 m}\sum_{k=1}^{m}(kr-m+k)\,a_k b_{m-k}, m\geq 1$, the $E(\theta^r)_{TSGGo}$ with $a_k=\frac{(2k)!}{2^{2k}(k!)^2(2k+1)}$ we get

$$E(\theta^r)_{TSGGo} = 2^{r-1}\frac{\alpha\lambda\beta}{\dot{A}\delta}\sum_{i,j,m,\ell=0}^{\infty}\frac{(-1)^{i+j+\ell}(i+1)^j(j+1)^m b_k}{j!\,m!}\alpha^m\left(\frac{\lambda}{\delta}\right)^j \binom{\beta-1}{i}\binom{-\frac{1}{2}(m+3)}{l}e^{(i+1)\frac{\lambda}{\delta}}\int_{a^{**}}^{b^{**}}u^{k+\frac{1}{2}(r+m-1)+\ell}du$$

$$= 2^{r-1}\frac{\alpha\lambda\beta}{\dot{A}\delta}\sum_{i,j,m,\ell=0}^{\infty}\frac{(-1)^{i+j+\ell}(i+1)^j(j+1)^m b_k}{j!\,m!}\alpha^m\left(\frac{\lambda}{\delta}\right)^j \binom{\beta-1}{i}\binom{-\frac{1}{2}(m+3)}{\ell}e^{(i+1)\frac{\lambda}{\delta}}\frac{b^{**k+\frac{1}{2}(r+m+1)+\ell}-a^{**k+\frac{1}{2}(r+m+1)+\ell}}{k+\frac{1}{2}(r+m+1)+\ell}$$

Thus, the r^{th} non-central moment of the $TSGGo$ distribution is given by

$$E(\theta^r)_{TSGGo} = 2^{r-1}\frac{\alpha\lambda\beta}{\delta\left(\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta}-\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}\right)}\sum_{i,j,m,\ell=0}^{\infty}\frac{(-1)^{i+j+\ell}(i+1)^j(j+1)^m b_k}{j!\,m!}\alpha^m\left(\frac{\lambda}{\delta}\right)^j\binom{\beta-1}{i}\binom{-\frac{1}{2}(m+3)}{\ell}e^{(i+1)\frac{\lambda}{\delta}}\frac{b^{**k+\frac{1}{2}(r+m+1)+\ell}-a^{**k+\frac{1}{2}(r+m+1)+\ell}}{k+\frac{1}{2}(r+m+1)+\ell} \quad (33)$$

So, the characteristic function of the $TSGGo$ distribution can be obtained as

$$\varphi_p(\theta)_{TSGGo} = \int_a^b\sum_{r=0}^{\infty}\frac{(ip\theta)^r}{r!}f(\theta)_{TSGGo}\,d\theta = \sum_{r=0}^{\infty}\frac{(ip)^r}{r!}E(\theta^r)_{TSGGo}$$

Furthermore, from $\varphi_p(\theta)_{TSGGo}$, the p^{th}; $p = 0, \pm1, \pm2, \dots$ non-central trigonometric moments can be obtained as

$$\varphi_p(\theta)_{TSGGo} = \nabla_p + i\,\Delta_p = E\big(\cos(p\theta)\big)_{TSGGo} + i\,E\big(\sin(p\theta)\big)_{TSGGo}$$

where

$$\Delta_p = E(\sin(p\theta))_{TSGGo} = \sum_{l=0}^{\infty}\frac{(-1)^l p^{2l+1}}{(2l+1)!}E(\theta^{2l+1})_{TSGGo}$$

$$\nabla_p = E(\cos(p\theta))_{TSGGo} = \sum_{l=0}^{\infty}\frac{(-1)^l p^{2l}}{(2l)!}E(\theta^{2l})_{TSGGo}$$

where $E(\theta^{2l+1})_{TSGGo}$ and $E(\theta^{2l})_{TSGGo}$ as in (33) respectively with $r = 2l+1$ and $r = 2l$.

3.1 Stress Strength Model of $TSGGo$ Distribution

Consider two independent random variables, say Y: $stress$ and Z: strength, that follow $TSGGo$ distribution with different parameters. Let $F_Y(\theta)_{TSGGo}$ represents the cdf of the $TSGGo$ distribution with parameters $\lambda_1, \beta_1, \delta_1$, and α_1,

$$F_Y(\theta)_{TSGGo} = \frac{\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta_1}-\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta_1}}{\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta_1}-\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta_1}}$$

and $f_Z(\theta)_{TSGGo}$ represents the pdf of the $TSGGo$ distribution with parameters λ, β, δ, and α.

Using A_1, A_2 and A_3, then

$$\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta_1} = \sum_{z=0}^{\infty}(-1)^z\binom{\beta_1}{z}e^{-\frac{\lambda_1 z}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{\theta}{2}\right)}-1\right)}$$

$$= \sum_{z=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^{z+r}}{r!} \left(\frac{\lambda_1 z}{\delta_1}\right)^r \binom{\beta_1}{z} \left(e^{\alpha_1 \tan\left(\frac{\theta}{2}\right)} - 1\right)^r$$

$$= \sum_{z=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^{z+2r}}{r!} \left(\frac{\lambda_1 z}{\delta_1}\right)^r \binom{\beta_1}{z} \left(1 - e^{\alpha_1 \tan\left(\frac{\theta}{2}\right)}\right)^r$$

$$= \sum_{z=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{z+2r+m}}{r!} \left(\frac{z\lambda_1}{\delta_1}\right)^r \binom{\beta_1}{z} \binom{r}{m} e^{\mathrm{m}\alpha_1 \tan\left(\frac{\theta}{2}\right)}$$

$$= \sum_{z=0}^{\infty} \sum_{r=0}^{\infty} \sum_{m=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^{z+2r+m} (m\alpha_1)^t}{r!t!} \left(\frac{z\lambda_1}{\delta_1}\right)^r \binom{\beta_1}{z} \binom{r}{m} \tan^t\left(\frac{\theta}{2}\right)$$

Now, $F_Y(\theta)_{TSEW}$ will be

$$F_Y(\theta) = \frac{\sum_{z,r,m,t=0}^{\infty} \frac{(-1)^{z+2r+m}(m\alpha_1)^t}{r!t!} \left(\frac{z\lambda_1}{\delta_1}\right)^r \binom{\beta_1}{z}\binom{r}{m} \tan^t\left(\frac{\theta}{2}\right) - \left[1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1 \tan\left(\frac{a}{2}\right)} - 1\right)}\right]^{\beta_1}}{\left[1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1 \tan\left(\frac{b}{2}\right)} - 1\right)}\right]^{\beta_1} - \left[1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1 \tan\left(\frac{a}{2}\right)} - 1\right)}\right]^{\beta_1}} \quad (34)$$

The reliability stress strength model of the $TSGGo$ distribution can be obtained by

$$SS_{TSGGo} = P(Y < M)_{TSGGo} = E(F_Y(\theta)_{TSGGo}) = \int_a^b F_Y(\theta)_{TSGGo} f_M(\theta)_{TSGGo}\, d\theta \quad (35)$$

Substituting (34) in (35), we get

$$SS_{TSGGo} = \frac{\sum_{z,r,m,t=0}^{\infty} \frac{(-1)^{z+2r+m}(m\alpha_1)^t}{r!t!} \left(\frac{z\lambda_1}{\delta_1}\right)^r \binom{\beta_1}{z}\binom{r}{m} \int_a^b \tan^t\left(\frac{\theta}{2}\right) f_M(\theta)_{TSGGo}\, d\theta}{\left[1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1 \tan\left(\frac{b}{2}\right)} - 1\right)}\right]^{\beta_1} - \left[1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1 \tan\left(\frac{a}{2}\right)} - 1\right)}\right]^{\beta_1}} - \frac{\left[1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1 \tan\left(\frac{a}{2}\right)} - 1\right)}\right]^{\beta_1}}{\left[1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1 \tan\left(\frac{b}{2}\right)} - 1\right)}\right]^{\beta_1} - \left[1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1 \tan\left(\frac{a}{2}\right)} - 1\right)}\right]^{\beta_1}}$$

$$= \frac{\sum_{z,r,m,t=0}^{\infty} \frac{(-1)^{z+2r+m}(m\alpha_1)^t}{r!t!} \left(\frac{z\lambda_1}{\delta_1}\right)^r \binom{\beta_1}{z}\binom{r}{m}}{\left[1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1 \tan\left(\frac{b}{2}\right)} - 1\right)}\right]^{\beta_1} - \left[1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1 \tan\left(\frac{a}{2}\right)} - 1\right)}\right]^{\beta_1}} E\left(\tan^t\left(\frac{\theta}{2}\right)\right) - \frac{\left[1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1 \tan\left(\frac{a}{2}\right)} - 1\right)}\right]^{\beta_1}}{\left[1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1 \tan\left(\frac{b}{2}\right)} - 1\right)}\right]^{\beta_1} - \left[1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1 \tan\left(\frac{a}{2}\right)} - 1\right)}\right]^{\beta_1}}$$

Using the transformation $x = v\tan\left(\frac{\theta}{2}\right)$, $\theta = 2\tan^{-1}\left(\frac{x}{v}\right)$, and $d\theta = \frac{2}{v + \frac{x^2}{v}}dx$ where $x \in [a^*, b^*]$, $a^* = v\tan\left(\frac{a}{2}\right)$ and $b^* = v\tan\left(\frac{b}{2}\right)$, then

$$E\left(\tan^t\left(\frac{\theta}{2}\right)\right)=\frac{\alpha\lambda\beta}{2\dot{A}\delta}\int_{a^*}^{b^*}\left(\frac{x}{v}\right)^t e^{\alpha\left(\frac{x}{v}\right)}e^{-\frac{\lambda}{\delta}\left(e^{\alpha\left(\frac{x}{v}\right)}-1\right)}\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\left(\frac{x}{v}\right)}-1\right)}\right]^{\beta-1}\left(1+\left(\frac{x}{v}\right)^2\right)\frac{2}{v+\frac{x^2}{v}}dx$$

$$=\frac{\alpha\lambda\beta}{\dot{A}\delta v}\int_{a^*}^{b^*}\left(\frac{x}{v}\right)^t e^{\alpha\left(\frac{x}{v}\right)}e^{-\frac{\lambda}{\delta}\left(e^{\alpha\left(\frac{x}{v}\right)}-1\right)}\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\left(\frac{x}{v}\right)}-1\right)}\right]^{\beta-1}dx$$

Based on A_1, $\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\left(\frac{x}{v}\right)}-1\right)}\right]^{\beta-1}=\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}e^{-\frac{i\lambda}{\delta}\left(e^{\alpha\left(\frac{x}{v}\right)}-1\right)}$ and then based on some previous formulas, we get

$$E\left(\tan^t\left(\frac{\theta}{2}\right)\right)=\frac{\alpha\lambda\beta}{\dot{A}\delta v}\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}\int_{a^*}^{b^*}\left(\frac{x}{v}\right)^t e^{\alpha\left(\frac{x}{v}\right)}e^{-\frac{(i+1)\lambda}{\delta}\left(e^{\alpha\left(\frac{x}{v}\right)}-1\right)}dx$$

$$=\frac{\alpha\lambda\beta}{\dot{A}\delta v}\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}e^{\frac{(i+1)\lambda}{\delta}}\int_{a^*}^{b^*}\left(\frac{x}{v}\right)^t e^{\alpha\left(\frac{x}{v}\right)}e^{-\frac{(i+1)\lambda}{\delta}e^{\alpha\left(\frac{x}{v}\right)}}dx$$

$$=\frac{\alpha\lambda\beta}{\dot{A}\delta v}\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}e^{\frac{(i+1)\lambda}{\delta}}\int_{a^*}^{b^*}\left(\frac{x}{v}\right)^t e^{\alpha\left(\frac{x}{v}\right)}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left((i+1)\frac{\lambda}{\delta}\right)^j e^{j\alpha\left(\frac{x}{v}\right)}dx$$

$$=\frac{\alpha\lambda\beta}{\dot{A}\delta v}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{j!}\left((i+1)\frac{\lambda}{\delta}\right)^j\binom{\beta-1}{i}e^{\frac{(i+1)\lambda}{\delta}}\int_{a^*}^{b^*}\left(\frac{x}{v}\right)^t e^{\alpha(j+1)\left(\frac{x}{v}\right)}dx$$

$$=\frac{\alpha\lambda\beta}{\dot{A}\delta v}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}}{j!}\left((i+1)\frac{\lambda}{\delta}\right)^j\binom{\beta-1}{i}e^{\frac{(i+1)\lambda}{\delta}}\int_{a^*}^{b^*}\left(\frac{x}{v}\right)^t\sum_{k=0}^{\infty}\frac{(j+1)^k\alpha^k}{k!}\left(\frac{x}{v}\right)^k dx$$

$$=\frac{\beta}{\dot{A}v^{t+k+1}}\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j(j+1)^k\alpha^{k+1}}{j!k!}\left(\frac{\lambda}{\delta}\right)^{j+1}\binom{\beta-1}{i}e^{\frac{(i+1)\lambda}{\delta}}\int_{a^*}^{b^*}x^{t+k}dx$$

Then , $E\left(\tan^t\left(\frac{\theta}{2}\right)\right)$ will be

$$E\left(\tan^t\left(\frac{\theta}{2}\right)\right)=\frac{\beta\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j(j+1)^k\alpha^{k+1}}{j!k!v^{t+k+1}}\left(\frac{\lambda}{\delta}\right)^{j+1}}{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta}-\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}}\binom{\beta-1}{i}e^{\frac{(i+1)\lambda}{\delta}}\left(\frac{{b^*}^{t+k+1}-{a^*}^{t+k+1}}{t+k+1}\right)\quad(36)$$

By inserting $E\left(\tan^t\left(\frac{\theta}{2}\right)\right)$ in SS_{TSGGo}, the stress strength of the $TSGGo$ distribution can be obtained as follows

$$SS_{TSGGo}=\frac{\beta\sum_{i,j,k,z,r,m,t=0}^{\infty}\frac{(-1)^{i+j+z+2r+m}(i+1)^{j}(j+1)^{k}\alpha^{k+1}(m\alpha_1)^{t}}{j!k!r!t!\,v^{t+k+1}}}{\left(\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta_1}-\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta_1}\right)}$$

$$\frac{\left(\frac{\lambda}{\delta}\right)^{j+1}\left(\frac{z\lambda_1}{\delta_1}\right)^{r}\binom{\beta_1}{z}\binom{r}{m}\binom{\beta-1}{i}e^{\frac{(i+1)\lambda}{\delta}}\left(\frac{b^{*t+k+1}-a^{*t+k+1}}{t+k+1}\right)}{\left(\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta}-\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}\right)}$$

$$-\frac{\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta_1}}{\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta_1}-\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta_1}} \qquad (37)$$

3.2 Shannon Entropy of $TSGGo$ Distribution

Recall $f(\theta)_{TSGGo}$ in (27) and take the natural logarithm, yields

$$\ln(f(\theta)_{TSGGo})=\ln\left(\frac{\alpha\lambda\beta}{2\delta}\right)+\alpha\tan\left(\frac{\theta}{2}\right)-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)$$

$$+(\beta-1)\ln\left(1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right)+2\ln\left(sec\left(\frac{\theta}{2}\right)\right)$$

$$-\ln\left(\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta}-\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}\right)$$

Now, the Shannon entropy will be

$$SH_{TSGGo}=\ln\left(\frac{2\delta}{\alpha\lambda\beta}\right)-\alpha E\left(\tan\left(\frac{\theta}{2}\right)\right)+\frac{\lambda}{\delta}E\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}\right)-\frac{\lambda}{\delta}$$

$$-(\beta-1)E\left(\ln\left(1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right)\right)-2E\left(\ln\left(\sec\left(\frac{\theta}{2}\right)\right)\right)$$

$$+\ln\left(\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta}-\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}\right) \qquad (38)$$

Let $\ddot{J}_1=E\left(\tan\left(\frac{\theta}{2}\right)\right), \ddot{J}_2=E\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}\right), \ddot{J}_3=E\left(\ln\left(1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right)\right)$ and

$\ddot{J}_4=E\left(\ln\left(\sec\left(\frac{\theta}{2}\right)\right)\right)$.

Now,

$$\ddot{J}_1=\int_a^b\tan\left(\frac{\theta}{2}\right)f(\theta)_{TSGGo}d\theta=\sum_{t=1}^{\infty}\frac{(-1)^{t+1}2(2^{2t}-1)B_{2t}}{(2t)!}\int_a^b\theta^{2t-1}f(\theta)_{TSGGo}d\theta$$

$$=\sum_{t=1}^{\infty}\frac{(-1)^{t+1}2(2^{2t}-1)B_{2t}}{(2t)!}E(\theta^{2t-1})_{TSGGo}$$

Based on (33) with $r = 2t - 1$, $\ddot{J}_1$ will be

$$\ddot{J}_1 = \frac{\alpha\lambda\beta}{\delta\left(\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta}-\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}\right)}$$

$$\sum_{i,j,m,\ell=0}^{\infty}\sum_{t=1}^{\infty}\frac{(-1)^{i+j+\ell+t+1}(i+1)^j(j+1)^m 2^{2t-1}(2^{2t}-1)b_k B_{2t}}{j!\,m!(2t)!}\alpha^m\left(\frac{\lambda}{\delta}\right)^j$$

$$\binom{\beta-1}{i}\binom{-\frac{1}{2}(m+3)}{\ell}e^{(i+1)\frac{\lambda}{\delta}}\frac{b^{**k+\frac{1}{2}(2t+m)+\ell}-a^{**k+\frac{1}{2}(2t+m)+\ell}}{k+\frac{1}{2}(2t+m)+\ell} \quad (39)$$

By using the same argument, $\ddot{J}_2 = \int_a^b e^{\alpha\tan\left(\frac{\theta}{2}\right)} f(\theta)_{TSGGo}d\theta$ will be

$$\ddot{J}_2 = \frac{\beta\sum_{i,j,k,t=0}^{\infty}\frac{(-1)^{i+j}(i+1)^j(j+1)^k\alpha^{t+k+1}}{j!k!t!v^{t+k+1}}\left(\frac{\lambda}{\delta}\right)^{j+1}}{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta}-\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}}\binom{\beta-1}{i}e^{\frac{(i+1)\lambda}{\delta}}\left(\frac{b^{*t+k+1}-a^{*t+k+1}}{t+k+1}\right) \quad (40)$$

For $\ddot{J}_3 = \int_a^b \ln\left(1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right)f(\theta)_{TSGGo}d\theta$ based on some previous formulas, we get $\ln\left(1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right) = -\sum_{z=1}^{\infty}\frac{1}{z}e^{-\frac{\lambda z}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}$

$$= \sum_{z=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{z\,m!}\left(\frac{\lambda z}{\delta}\right)^m\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)^m$$

$$= \sum_{z=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{2m+1}}{z\,m!}\left(\frac{\lambda z}{\delta}\right)^m\left(1-e^{\alpha\tan\left(\frac{\theta}{2}\right)}\right)^m$$

$$= \sum_{z=1}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^{2m+r+1}}{z\,m!}\left(\frac{\lambda z}{\delta}\right)^m\binom{m}{r}e^{r\alpha\tan\left(\frac{\theta}{2}\right)}$$

$$= \sum_{z=1}^{\infty}\sum_{m,r,t=0}^{\infty}\frac{(-1)^{2m+r+1}(r\alpha)^t}{z\,m!t!}\left(\frac{\lambda z}{\delta}\right)^m\binom{m}{r}\tan^t\left(\frac{\theta}{2}\right)$$

Now

$$\ddot{J}_3 = \sum_{z=1}^{\infty}\sum_{m,r,t=0}^{\infty}\frac{(-1)^{2m+r+1}(r\alpha)^t}{z\,m!t!}\left(\frac{\lambda z}{\delta}\right)^m\binom{m}{r}E\left(\tan^t\left(\frac{\theta}{2}\right)\right)$$

$$= \frac{\beta\sum_{i,j,k,m,r,t=0}^{\infty}\sum_{z=1}^{\infty}\frac{(-1)^{i+j+2m+r+1}(i+1)^j(j+1)^k\alpha^{k+1}(r\alpha)^t z^{m-1}}{z\,m!t!j!k!v^{t+k+1}}}{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta}-\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}}$$

$$\left(\frac{\lambda}{\delta}\right)^{m+j+1}\binom{m}{r}\binom{\beta-1}{i}e^{\frac{(i+1)\lambda}{\delta}}\left(\frac{b^{*t+k+1}-a^{*t+k+1}}{t+k+1}\right) \quad (41)$$

Finally for $\ddot{J}_4 = \int_a^b \ln\left(\sec\left(\frac{\theta}{2}\right)\right)f(\theta)_{TSGGo}d\theta = \int_a^b -\ln\left(\cos\left(\frac{\theta}{2}\right)\right)f(\theta)_{TSGGo}d\theta$ and based on $-\ln\left(\cos\left(\frac{\theta}{2}\right)\right) = \sum_{m=1}^{\infty}\frac{2^{2m-1}(2^{2m}-1)}{m(2m)!}|B_{2t}|\left(\frac{\theta}{2}\right)^{2m}$; $|\theta| < \pi$, with the Bernoulli numbers B_{2m}. Now

$$\ddot{J}_4 = \sum_{h=1}^{\infty}\frac{2^{2h-1}(2^{2h}-1)}{h(2h)!}|B_{2h}|\int_a^b\left(\frac{\theta}{2}\right)^{2h}f(\theta)_{TSGGo}d\theta = \sum_{h=1}^{\infty}\frac{2^{2h}-1}{2h(2h)!}|B_{2h}|\,E(\theta^{2h})_{TSGGo}$$

Based on (33) with $r = 2h$, $\ddot{J}_4$ will be

$$\ddot{J}_4 = \frac{\alpha\lambda\beta}{\delta\left(\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta}-\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}\right)}$$

$$\sum_{h=1}^{\infty}\sum_{i,j,m,\ell=0}^{\infty}\frac{(-1)^{i+j+\ell}(i+1)^{j}(j+1)^{m}2^{2(h-1)}\left(2^{2h}-1\right)b_k|B_{2h}|}{j!\,m!(2h)!h}\alpha^{m}\left(\frac{\lambda}{\delta}\right)^{j}$$

$$\binom{\beta-1}{i}\binom{-\frac{1}{2}(m+3)}{\ell}e^{(i+1)\frac{\lambda}{\delta}}\frac{b^{**k+\frac{1}{2}(r+m+1)+\ell}-a^{**k+\frac{1}{2}(r+m+1)+\ell}}{k+\frac{1}{2}(2h+m+1)+\ell} \quad (42)$$

Therefore, the Shannon entropy of the $TSGGo$ distribution can be obtained as

$$SH_{TSGGo} = \ln\left(\frac{2\delta}{\alpha\lambda\beta}\right)-\frac{\lambda}{\delta}-\alpha\,\ddot{J}_1+\frac{\lambda}{\delta}\,\ddot{J}_2-(\beta-1)\,\ddot{J}_3-2\,\ddot{J}_4$$

$$+\ln\left(\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta}-\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}\right) \quad (43)$$

where $\ddot{J}_1$, $\ddot{J}_2$, $\ddot{J}_3$ and $\ddot{J}_4$ are respectively given in $(39), (40), (41)$ and (42).

3.3 Relative Entropy of TSGGo Distribution

The relative entropy of the $TSGGo$ distribution can be obtained as,

$$RE_{TSGGo} = E\left(\ln\left[\frac{f(\theta)_{TSGGo}}{f_1(\theta)_{TSGGo}}\right]\right) = \int_a^b \ln\left[\frac{f(\theta)_{TSGGo}}{f_1(\theta)_{TSGGo}}\right] f(\theta)_{TSGGo} \quad (44)$$

Taking the natural logarithm of the $f(\theta)_{TSGGo}$ with parameters $(\lambda,\beta,\delta,\alpha)$ relative of the $f_1(\theta)_{TSGGo}$ with parameters $(\lambda_1,\beta_1,\delta_1,\alpha_1)$, then

$$\ln\left[\frac{f(\theta)_{TSGGo}}{f_1(\theta)_{TSGGo}}\right] = \ln\left[\frac{\dfrac{\frac{\alpha\lambda\beta}{\delta}e^{\alpha\tan\left(\frac{\theta}{2}\right)}e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta-1}}{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta}-\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}}}{\dfrac{\frac{\alpha_1\lambda_1\beta_1}{\delta_1}e^{\alpha_1\tan\left(\frac{\theta}{2}\right)}e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{\theta}{2}\right)}-1\right)}\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{\theta}{2}\right)}-1\right)}\right]^{\beta_1-1}}{\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta_1}-\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta_1}}}\right]$$

$$= \ln\left(\frac{\alpha\lambda\beta\delta_1}{\alpha_1\lambda_1\beta_1\delta}\right) + \alpha\tan\left(\frac{\theta}{2}\right) - \frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)} - 1\right) + (\beta-1)\ln\left(1 - e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right) - \alpha_1\tan\left(\frac{\theta}{2}\right)$$

$$+ \frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{\theta}{2}\right)} - 1\right) - (\beta_1-1)\ln\left(1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{\theta}{2}\right)}-1\right)}\right)$$

$$+ \ln\left(\frac{\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta_1} - \left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta_1}}{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta} - \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}}\right)$$

$$= \ln\left(\frac{\alpha\lambda\beta\delta_1}{\alpha_1\lambda_1\beta_1\delta}\right) + (\alpha-\alpha_1)\tan\left(\frac{\theta}{2}\right) - \frac{\lambda}{\delta}e^{\alpha\tan\left(\frac{\theta}{2}\right)} + \frac{\lambda}{\delta} + \frac{\lambda_1}{\delta_1}e^{\alpha_1\tan\left(\frac{\theta}{2}\right)}$$

$$- \frac{\lambda_1}{\delta_1} + (\beta-1)\ln\left(1 - e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right) - (\beta_1-1)\ln\left(1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{\theta}{2}\right)}-1\right)}\right)$$

$$+ \ln\left(\frac{\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta_1} - \left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta_1}}{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta} - \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}}\right)$$

Now, the relative entropy of the TSGGo distribution is given by

$$RE_{TSGGo} = \ln\left(\frac{\alpha\lambda\beta\delta_1}{\alpha_1\lambda_1\beta_1\delta}\right) + \frac{\lambda}{\delta} - \frac{\lambda_1}{\delta_1} + (\alpha-\alpha_1)E\left(\tan\left(\frac{\theta}{2}\right)\right) - \frac{\lambda}{\delta}E\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}\right)$$

$$+ \frac{\lambda_1}{\delta_1}E\left(e^{\alpha_1\tan\left(\frac{\theta}{2}\right)}\right) + (\beta-1)E\left(\ln\left(1 - e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right)\right)$$

$$- (\beta_1-1)E\left(\ln\left(1 - e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{\theta}{2}\right)}-1\right)}\right)\right)$$

$$+ \ln\left(\frac{\left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta_1} - \left[1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta_1}}{\left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{b}{2}\right)}-1\right)}\right]^{\beta} - \left[1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{a}{2}\right)}-1\right)}\right]^{\beta}}\right) \quad (45)$$

where $E\left(\tan\left(\frac{\theta}{2}\right)\right)$, $E\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}\right)$, $E\left(e^{\alpha_1\tan\left(\frac{\theta}{2}\right)}\right)$, $E\left(\ln\left(1-e^{-\frac{\lambda}{\delta}\left(e^{\alpha\tan\left(\frac{\theta}{2}\right)}-1\right)}\right)\right)$ and $E\left(\ln\left(1-e^{-\frac{\lambda_1}{\delta_1}\left(e^{\alpha_1\tan\left(\frac{\theta}{2}\right)}-1\right)}\right)\right)$ is founded previously with specified parameters.

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