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The Relation Between the Degree of Monotone Approximation and the Degree of Unconstrained Approximation

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A B S T R A C T

Many researchers related to the degree of unconstrained approximation to constrained approximation, and proved the inequality: For a continuous function f on a closed interval we have

$$E_n(f) \le C\, E_n^{(h)}(f) \qquad (*)$$

where C is a positive constant. The converse of the relation $(*)$ is not achieved, so we will obtain the converse of the relation $(*)$under some conditions on f which belonging to quasi normed spaces.

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1. Introduction

Do not be surprised if I tell you that the degree of constrained approximation is worse than the degree of unconstrained approximation. This is what many researchers have proven[1], [2], [3]. In this paper, we will prove that under some conditions the two degrees can be equivalent. In [4] E. S. Bhaya, studied on the constrained and unconstrained approximation, in[5] M.S.AL-Muhja, H. Akhadkulov and N. Ahmad, introduced Estimates for constrained approximation in $L_{p,r}^{\alpha,\beta}$ space: piecewise polynomials, in[6]K.A. Kopotun, D.Leviatan and A.V.Prymak, discuss constrained spline smoothing , in[7] prove that the degree of co monotone, sequentially, coconvex approximation, of f ,through algebraic polynomials of degree $\le n,\, n \ge N$, is too $\le c(\alpha, s)n^{-q\alpha}$, where the constant N depends only on the area of the maximum , sequentially, deviation points in $(-1, 1)$ and on α.

in [8], [9] K.A. Kopotun, D. Leviatan, and I.A. Shevchuk , discuss , are the degrees of best (co)convex and unconstrained polynomial approximation the same?
Many researcher work on the upper bound of $E_n^{(h)}(f)$ is the case $h = 1$, it mean $f'(x) \geq 0$ for $a \leq x \leq b$, $a, b \in IR$. In [10] Lorentz and Zeller proved that for a continuous function on the interval $[a, b\ ]$, that satisfies

$$f^{(h)}(x) \geq 0 \text{ that } E_n(f) \leq E_n^{(h)}(f).$$

In [11] Lorenz proved that if $f^{(hi)}(x) \geq 0$ for any $a \leq x \leq b$ and $i = 1,2,3, \cdots\cdots, q$, then there exists $c > 0$ such that for positive consistent $E_n^{(h_i)}(f) \leq c\, E_n(f)$. In[12] Roulier, put conditions on (f) to insure for $n \to \infty$,

$$E_n^{(h_i)}(f) = E_n(f).$$

In [13] Roulier gave some conditions to get $\frac{E_n^{(h)}(f)}{E_n(f)} \leq c(f')$, where $c(f')$is constant depends of f'.

We obtain an estimate for the degree of best approximation once for $h = 1$ and another for any h. Then we relate $E_n^{(h)}(f)\ to\ E_n(f)$ by a constant independent on f'. These all for Lebesgue integrable functions.

Let the Lp space for $0 < p < 1$

$$L_{p[0,1]} = \left\{ f : [0,1] \to IR, with \left(\int_0^1 |f(x)|^p\, dx \right)^{\frac{1}{p}} < \infty \right\}.$$

On $L_{p[0,1]}$we define the quasi norm

$$\|f\|_p = \left(\int_0^1 |f(x)|^p\, dx \right)^{\frac{1}{p}}, 0 < p < 1.$$

For $p = \infty$, we have $x \in [0,1]$

$$\|f\|_p = \|f\|_\infty = \sup_{x \in [0,1]} |f(x)|. \ [14]$$

Before we define the degree of best approximation of $f \in L_{p[0,1]}$, let us introduce

$$\mathcal{P}_n = \{P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n\ of\ degree\ \leq n\},$$

and let

$$M_n = \{P_n(x) \in \mathcal{P}_n : P_n(x_0, x_1, x_2, \cdots\cdots, x_h) \geq 0\},$$

let

$$M = \{f \in L_{p[0,1]} : f(x_0, x_1, x_2, \cdots\cdots, x_h) \geq 0\}.$$

The space of h-monotone function.

A function $f: [0,1] \to IR$ is said to be h-monotone $h \geq 1$, on $[0,1]$ iff for each choices of $(h+1)$ distinct points $x_0, x_1 \cdots\cdots, x_h$ in [0,1] the inequality

$$f(x_0, x_1, x_2, \cdots\cdots, x_h) \geq 0,$$

$$f(x_0, x_1, x_2, \cdots\cdots, x_h) = \sum_{j=0}^{h} \frac{f(x_i)}{\omega'(x_i)}.$$

Is the h-divided difference of functions defined on the interval [0, 1] is . [14]

$$\omega(x) = \prod_{i=0}^{h} (x - x_i).$$

We define for $f \in M$ the degree of monotone approximation

$$E_n^{(h)}(f)_p = \inf_{P_n \in M_n} \|f - P_n\|_p, [15]$$

where $P_n \in M_n$ and $f \in M,$ and the degree of monotone approximation

$$E_n(f)_p = \inf_{P_n \in \mathcal{P}_n} \|f - P_n\|_p, [15]$$

Be the degree of unconstrained best approximation[16] .

For $p = \infty$, we shall denote

$$E_n(f)_p = E_n(f)$$

$$E_n^{(h)}(f)_p = E_n^{(h)}(f).$$

The ordinary modulus of continuity of $f \in L_{p[0,1]}$ is

$$\omega_1(f, \delta)_p = \sup_{0 \leq |h| \leq \delta} \|f(x+h) - f(x)\|_p.$$

2.The Main Results

In this section we present two theorem that we need in our work.

Theorem 2.1

If f is monotone in $L_{p[0,1]}$, and satisfy the condition

$$\big(f(x_2)-f(x_1)\big) \geq M(x_2-x_1),$$

where M is a positive constant, for $0 \leq x_1 < x_2 \leq 1$. Then

$$E_n(f)_p \leq \frac{c(p)}{E_n(f)_p}\left(\omega\ \left(\frac{3E_n}{M},\frac{1}{n}\right)_p + E_n(f)_p\right)$$

$$E_n^1(f)_p \leq c(p)E_n(f)_p.$$

Where $c(p)$ is constant depending only on p.

Proof.

Choose Q_n be a best approximation to $f \in L_{p[0,1]}$, i.e

$$E_n(f)_p = \|f-Q_n\|_p$$

$$Q_n(x_2)-Q_n(x_1) \geq f(x_2)-f(x_1)-|f(x_1)-Q_n(x_1)|-|f(x_2)-Q_n(x_2)|$$

$$\geq f(x_2)-f(x_1)-\|f(x_1)-Q_n(x_1)\|_p-\|f(x_2)-Q_n(x_2)\|_p$$

$$= f(x_2)-f(x_1)-2E_n(f)_p$$

$$\geq M(x_2-x_1)-2E_n(f)_p.$$

If $\big(M(x_2-x_1)-2E_n(f)_p\big) > E_n(f)_p$, then

$$M(x_2-x_1)-3E_n(f)_p > 0$$

$$(x_2-x_1) > \frac{3}{M}E_n(f)_p.$$

$$Q_n(x_2)-Q_n(x_1) > M\left(\frac{3}{M}E_n(f)_p\right)-2E_n(f)_p$$

$$= E_n(f)_p > 0.$$

Define

$$P_n(x) = \frac{M}{3E_n(f)_p}\int_{\left(1-\frac{3}{M}E_n(f)_p\right)x}^{\left(1-\frac{3E_n(f)_p}{M}+\frac{3E_n(f)_p}{M}\right)x} Q_n(t)dt,$$

we have

$$0 \leq \left(1-\frac{3E_n(f)_p}{M}+\frac{3E_n(f)_p}{M}\right)x \leq 1, \text{ and } 0 \leq \left(1-\frac{3}{M}E_n(f)_p\right)x \leq 1.$$

For $1 > x > 0$, we have $P_n'(x) > 0$.

If

$$1 - 3\frac{E_n(f)_p}{M} \leq x\ ,\ t \leq \left(1 - \frac{3}{M}E_n(f)_p + 3E_n(f)_p\right)x,$$

definition of the modulus of continuity implies

$$\left\| f(x) - f(t) \right\|_p \leq \omega\left(\frac{3}{M}E_n(f)_p,\ \left|x - t\right|\right)_p.$$

$$\left\| f(x) - P \right\|_{n_p} = \left\| f - \frac{M}{3E_n(f)_p} \int_{\left(1-\frac{3}{M}E_n(f)_p\right)x}^{\left(1-\frac{3E_n(f)p}{M}+\frac{3E_n(f)p}{M}\right)x} \frac{3}{M} Q_n(t)dt \right\|_p$$

$$= \left\| \frac{M}{3E_n(f)_p} \int_{\left(1-\frac{3}{M}E_n(f)_p\right)x}^{\left(1-\frac{3E_n(f)p}{M}+\frac{3E_n(f)p}{M}\right)x} \left(f(x) - Q_n(t)\right)dt \right\|_p$$

$$= \frac{M}{3E_n(f)_p} \left\| \int_{\left(1-\frac{3}{M}E_n(f)_p\right)x}^{\left(1-\frac{3E_n(f)p}{M}+\frac{3E_n(f)p}{M}\right)x} \left(f(x) - f(t) + f(t) - Q_n(t)\right)dt \right\|_p.$$

$$E_n(f)_p \leq \left\| f - P_n \right\|_p \leq \frac{M}{3E_n(f)_p} \left\| \int_0^1 \left(f(x) - f(t) + f(t) - Q_n(t)\right)dt \right\|_p$$

$$\leq \frac{M\ 2^{\frac{1}{p}-1}}{3E_n(f)_p}\left(\left\| \int_0^x f(x) - f(t) \right\|_p + \left\| \int_0^x f(t) - Q_n(t) \right\|_p \right)$$

$$\leq \frac{M2^{\frac{1}{p}-1}}{3E_n(f)_p}\left(\left(\int_0^1 \left| \int_0^x \left(f(x) - f(t)\right)dt \right|^p dx\right)^{\frac{1}{p}} + \left(\int_0^1 \left| \int_0^x \left|f(t) - Q_n(t)\right| dt \right|^p dx\right)^{\frac{1}{p}}\right)$$

$$\leq \frac{c(p)\,2^{\frac{1}{p}-1}}{3E_n(f)_p}\left(\left(\sum_{i=1}^{n}\frac{1}{n}\left|\int_0^x f(x_i)-f(t)dt\right|^p\right)^{\frac{1}{p}}+\int_0^x\left|f(t)-Q_n(t)\right|dt\right)$$

$$\leq \frac{c(p)\,2^{\frac{1}{p}-1}}{3E_n(f)_p}\left(\sum_{i=1}^{n}\frac{1}{n}\left|\frac{1}{n^p}\sum_{i=1}^{n}\big(f(x_i)-f(y_i)\big)\right|^p\right)^{\frac{1}{p}}$$

$$\leq \frac{c(p)\,2^{\frac{1}{p}-1}}{3E_n(f)_p}\left(\left(\sum_{i=1}^{n}\frac{1}{n.n^p}\sum_{i=1}^{n}\left|f(x_i)-f(y_i)\right|^p\right)^{\frac{1}{p}}+\left(\int_0^1\left|f(t)-Q_n(t)\right|^p dt\right)^{\frac{1}{p}}\right),$$

so

$$\left\|f-P_n\right\|_p \leq \frac{c(p)\,2^{\frac{1}{p}-1}}{3E_n(f)_p}\left(\omega\left(\frac{3E_n}{M},\frac{1}{n}\right)_p+E_n(f)_p\right).$$

This implies

$$E_n^1(f)_p \leq c(p)E_n(f)_p.$$

Theorem2-2.

Let $f \in L_{p[0,1]}$be monotone function and assume that f is positive bounded function on $L_{p[0,1]}$. Then if f is not polynomial, then

$$\lim_{n\to\infty}\frac{E_n^1(f)_p}{E_n(f)_p} \leq 4.$$

Proof :-

Assume there exists $b > 0$ such that $f(x) > b$. Let α be a positive constant greater than 1. Choose k so large that

$$E_k(f)_p < \frac{b}{(3+\alpha)}.$$

Let $P_k$be the polynomial from $\mathcal{P}_k$ of best approximation to $f \in L_{p[0,1]}$.

Let $h(x) = f(x) - P_k(x) + (1 + \alpha)E_k(f)_p$. Then we have

$$\alpha E_k(f)_p \leq f(x) - P_k(x) + (1 + \alpha)E_k(f)_p \leq (2 + \alpha)E_k(f)_p \quad (1)$$

Now let

$$\phi(x) = \int_0^x h(t)dt$$

$$(x) = f(x) - Q_{k+1}(x)\ \phi, \quad (2)$$

where

$$Q_{k+1}(x) = f(0) + \int_0^x \left(P_k(t) - (1 + \alpha)E_k(f)_p\right)dt.$$

$$\phi(x) = f(x) - f(0) + \int_0^x \left(P_k(t) - (1 + \alpha)E_k(f)_p\right)dt$$

$$\begin{aligned} Q_{k+1}(x) &= P_k(x) - (1 + \alpha)E_k(f)_p \\ &= P_k(x) - f(x) + f(x) - (1 + \alpha)E_k(f)_p \\ &\geq f(x) - (2 + \alpha)E_k(f)_p \\ &\geq b - (2 + \alpha)\tfrac{b}{(3+\alpha)} \\ &= \tfrac{b}{(3+\alpha)}. \end{aligned}$$

From (1)

$$\alpha E_k(f)_p \leq \phi(x) \leq (2 + \alpha)E_k(f)_p.$$

From theorem2.1 we get for n sufficiently large

$$E_n^1(\phi)_p \leq \left(\frac{2 + \alpha}{\alpha} + 1\right)E_n(\phi)_p.$$

That is, for n sufficiently large,

$$E_n^1(\phi)_p \le 2\left(1+\frac{1}{\alpha}\right)E_n(\phi)_p.$$

$n \ge k+1,$ we get by (2) and the monotonicity of Q_{k+1}

$$E_n(\phi)_p = E_n(f)_p$$

$$E_n(f)_p = \inf_{P_n \in \mathcal{P}_n} \left\| f - P_n \right\|_\infty$$

$$E_n^1(f)_p = \inf_{P_n \in M_n} \left\| f - P_n \right\|_p,$$

$$E_n^1(\phi)_p \ge E_n^1(f)_p.$$

$$E_n^1(\phi)_p \ge \inf_{P_n \in M_n} \left\| f - P_n \right\|_p$$

$$\frac{E_n^1(f)_p}{E_n(f)_p} \le 2\left(1+\frac{1}{\alpha}\right)$$

for n sufficiently large

$$\le 2(1+1) = 4.$$

Conclusion

usually the degree of unconstrained approximation is less than the degree of constrained approximation. In this paper we were able to obtain the opposite relation for $E_n(f) \le C\, E_n^{(h)}(f)$ under some conditions

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