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On Some New Types of Partitions Associated with e^s -Abacus

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ABSTRACT

In this work a new method called Sequences method used to establish new diagram which called e^s-abacus with four different movements, namely $(1,1,\ldots,1)$. Then, a new rule constrict to find e^s-abacus diagram constrict in case $\ell_2, \ell_3, \ldots, \ell_e$ from e^s-abacus diagram in case ℓ_1. Further, several examples are given to illustrate the new method.

Keywords: *β-numbers, e-abacus, Partition, Sequences method, abacus configuration.*

1. Introduction

A new convenient way to represented any partition of positive integer number using β-numbers which defined from Littlewod, 1951 was founded. The idea of e-abacus has been used to solve many problem which related with Iwahori-Hecke algebras and q-Schur algebras [1,2,3,4,5]. Mathas, James and Sinead (2005), introduced the idea of e-abacus configuration with k beads (ℓ_1) (with $k+1$ beads), (ℓ_2), (with $k+2$ beads),…, (ℓ_e) (with $k+e$ beads) [5]. Numerous diagrams have been generated based on different methods [7-11]. The idea of intersection of e-abacus configuration was intrudes in [6]. In this paper we introduced new method called Sequences method to find e-abacus configuration (in case $\ell_2, \ell_3, \cdots, \ell_e$) from e-abacus (case ℓ_1), this new abacus denoted e^s-abacus diagram, and research for relation between the intersection of the new and normal diagram.

1. Background and Notation

Let $\gamma = (\gamma_1, \gamma_2, \gamma_3, \ldots \gamma_n)$ be a partition of k, where γ_i is a sequence of integers such that $|\gamma| = \sum_{i=1}^{n} \gamma_i = k, \forall\, i \geq 1$. Let be integer greater than or equal the number of non–zero parts γ. Now consider a graphical representation of partition called e-abacus configuration with e columns, labeled from $0 \ldots e-1$ as (0,1,2,…,e-1) from left to right, and label the bead position as well as empty bead position on column j as $j, j+e, j+2e, \ldots$ begin from top. There is a bijection between partitions γ with at most ℓ position and sequence of integer numbers $\beta_1 > \beta_2 > \cdots > \beta_j$ called beta number where $\beta_j = \gamma_i + \ell - j$ as shown in Figure 1.

run.0	run.1		run.e-1
0	1	… …	$e-1$
e	$e+1$	… …	$2e-1$
$2e$	$2e+1$	… …	$3e-1$
.	.	.	.

***e*-abacus configuration**

Let $\mu=(8^2,6,3,2,1^4)$, d = 31,

1. if e = 2 then there are two e-abacus configuration,
   - if $\ell_1=9$ = 9 then β = {1,2,3,4,6,8,12,15,16}
   - if $\ell_2=10$ = 10 then β = {0,2,3,4,5,7,9,13,16,17}.
2. if e = 3 then there are thee e-abacus configuration,
   - if $\ell_1=9$ then β = {1,2,3,4,6,8,12,15,16}.
   - if $\ell_2=10$ then β = {0,2,3,4,5,7,9,13,16,17}.
   - If $\ell_3=11$ then β = {0,1,3,4,5,6,8,10,1,17,18}.

Theorem [7]: Let $m.d._{\ell_s}$ be an abacus configuration in case $\ell_1=\gamma_1-1+$ k, $\ell_2=\gamma_2-2+$ k, $\ldots,\ell_e=\gamma_k$

With k beads

1. for τ_κ = 1, then $\#\cap_{s=1}^{e} m._{\ell_s}$=1 = ϕ

2.For $\tau_\kappa \geq e$ for some κ, then: $\#\cap_{s=1}^{e} m._{\ell_s}$ = [$\sum_{t=1}^{R}\tau_t-$ R $(e-1)$] where r the number of parts of γ.

Consider $\gamma=(8^2,6,3,2,1^4)$ then $\#\cap_{s=1}^{e} m._{\ell_s}$ = [$\sum_{t=1}^{R}\tau_t-$ R $(e-1)$]= [4 − 2]==2

Remark 1[13]: Let $\alpha_{mn}^{A\ell s}$ be a position in e-abacus configuration located in column n and rows m where s number of beads, n=1,2,…,e and m= 1, 2, …., $\left(\text{fix}\left(\frac{\beta_i}{2}\right)+1\right)+(e-1)$ = r.

Sequences movement $\boldsymbol{\beta}$–numbers:

In this section, e^s-abacus configuration was interfused by application the Sequences movement on e-abacus configuration.

Role 2: Let $\alpha_{mn}^{\ell_s}$ be a position in e-abacus, if x =1 then

$$\rightarrow \begin{cases} \alpha^{e^s \ell_s}_{(m-1)n} & \text{if} \quad (e-n+1) < m \leq (r-e+n), \begin{cases} \frac{2+e}{2} \leq n \leq e & \text{if} \quad n\ is\ even \\ \frac{3+e}{2} \leq n \leq e & \text{if} \quad n\ is\ odd \end{cases} \\ \alpha^{e^s \ell_s}_{(1+m)n} & \text{if} \quad n \leq m < (r-n+1), \begin{cases} 1 \leq n \leq \frac{e}{2} & \text{if} \quad n\ is\ even \\ 1 \leq n < \frac{e-1}{2} & \text{if} \quad n\ is\ odd \end{cases} \\ \alpha^{e^s \ell_s}_{m(n-1)} & \text{if} \quad m = u, u < n \leq e-u+1, \begin{cases} 1 \leq u \leq \frac{e}{2} & \text{if} \quad n\ is\ even \\ 1 \leq u < \frac{e-1}{2} & \text{if} \quad n\ is\ odd \end{cases} \\ \alpha^{e^s \ell_s}_{m(1+n)} & \text{if} \quad m = (r-u+1), u \leq n < e-u+1, \begin{cases} 1 \leq u \leq \frac{e}{2} & \text{if} \quad n\ is\ even \\ 1 \leq u < \frac{e-1}{2} & \text{if} \quad n\ is\ odd \end{cases} \\ \alpha^{e^s \ell_s}_{(1+m)n} & \text{if} \quad n = \frac{e-1}{2},\ \frac{1+e}{2} \leq m < \frac{2r-e-1}{2} \\ \alpha^{e^s \ell_s}_{(\frac{3+e}{2})(\frac{1+e}{2})} & \text{if} \quad n = \frac{1+e}{2}, m = \frac{2r-e-1}{2} \end{cases}$$

Where s is the number of beads, n=1,2,…,e and m= 1, 2, …., r, for $r = \left(\text{fix}\left(\frac{\beta_1}{2}\right) + 1\right) + e - 1.$

5.1 Sequences movement $\boldsymbol{\beta}$–numbers in case (1, 1, 1,…)

To find e^s -abacus configuration in case $\ell_2, \ell_3, \dots, \ell_e$ from e-abacus configuration we need to division e-abacus in case ℓ_1 into several parts:

A- If e is even

Remark 3: e-abacus configuration in case b_1 will be divided in to three parts:

1- $H_k =$

$$\begin{cases} \left\{ \alpha^{e^s \ell_s}_{21} \right\} & \text{if} \quad k = 1 \\ \left\{ \alpha^{e^s \ell_s}_{1(k-1)}, \alpha^{e^s \ell_s}_{2(k-1)}, \dots, \alpha^{e^s \ell_s}_{(k-1)(k-1)} \right\} & \text{if} \quad k = 2,3,\dots,\frac{e}{2} \\ \left\{ \alpha^{e^s b_s}_{1\left(\frac{e+2u}{2}\right)}, \alpha^{e^s b_s}_{2\left(\frac{e+2u}{2}\right)}, \dots, \alpha^{e^s \ell_s}_{\left(\frac{e-2u}{2}\right)\left(\frac{e+2u}{2}\right)} \right\} & \text{if} \quad \mathrm{k} = \frac{e+2u+2}{2}, \mathrm{u} = 0, 1, \dots, \frac{e-4}{2}. \\ \left\{ \alpha^{e^s \ell_s}_{1(e-1)}, \alpha^{e^s \ell_s}_{1\mathrm{e}} \right\} & \text{if} \quad k = e \end{cases}$$

$$H'_k = \begin{cases} \left\{\alpha_{31}^{e^s\ell_s}, \alpha_{41}^{e^s\ell_s}, \dots, \alpha_{(r-1)1}^{e^s\ell_s}\right\} & \text{if} \quad k=1 \\ \left\{\alpha_{(k+1)k}^{e^s\ell_s}, \alpha_{(k+2)k}^{e^s\ell_s}, \dots, \alpha_{(r-k+1)k}^{e^s\ell_s}\right\} & \text{if} \quad k = 2,3,\dots,\frac{e}{2} \\ \left\{\alpha_{\left(\frac{e-2u}{2}\right)\left(\frac{e+2u+2}{2}\right)}^{e^s\ell_s}, \alpha_{\left(\frac{e-2u+2}{2}\right)\left(\frac{e+2u+2}{2}\right)}^{e^s\ell_s}, \dots, \alpha_{\left(\frac{2r-e+2u}{2}\right)\left(\frac{e+2u+2}{2}\right)}^{e^s\ell_s}\right\} & \text{if} \quad \mathrm{k} = \frac{e+2u+2}{2}, \mathrm{u} = 0,1,\dots,\frac{e-4}{2}. \\ \left\{\alpha_{2\mathrm{e}}^{e^s\ell_s}, \alpha_{3\mathrm{e}}^{e^s\ell_s}, \dots,, \alpha_{(\mathrm{r}-4)e}^{e^s\ell_s}\right\} & \text{if} \quad k=e \end{cases}$$

Rule 4: The *e*-abacus configuration in case ℓ_1 plays a main role to find the e^s-abacus configuration in case $\ell_1, \ell_2, \ell_3, \dots, \ell_e$ as follows:

1.H_1 in e^s-abacus configuration in case $\ell_1 \to H_2$ in case ℓ_2 then $H_2 \cup \left\{\alpha_{32}^{e^s\ell_s}\right\} \to H_3$ in case ℓ_3 then $H_3 \cup \left\{\alpha_{43}^{e^s\ell_s}\right\} \to \cdots \to H_{\left(\frac{e+2}{2}\right)}$ in case $\ell_{\left(\frac{e+2}{2}\right)}$ then $H_{\left(\frac{e+2}{2}\right)} \setminus \left\{\alpha_{(\frac{e}{2})(\frac{e}{2})}^{e^s\ell_s}\right\} \to H_{\left(\frac{e+4}{2}\right)}$ in case $\ell_{\left(\frac{e+4}{2}\right)}$ then $H_{\left(\frac{e+4}{2}\right)} \setminus \left\{\alpha_{(\frac{e-2}{2})(\frac{e+}{2})}^{e^s\ell_s}\right\} \to \cdots \to H_e$ in case ℓ_{e}.

Rule 4: The *e*-abacus configuration in case ℓ_1 plays a main role to find the e^s-abacus configuration in case $\ell_1, \ell_2, \ell_3, \dots, \ell_e$ as follows:

1.H_1 in e^s-abacus configuration in case $\ell_1 \to H_2$ in case ℓ_2 then $H_2 \cup \left\{\alpha_{32}^{e^s\ell_s}\right\} \to H_3$ in case ℓ_3 then $H_3 \cup \left\{\alpha_{43}^{e^s\ell_s}\right\} \to \cdots \to H_{\left(\frac{e+2}{2}\right)}$ in case $\ell_{\left(\frac{e+2}{2}\right)}$ then $H_{\left(\frac{e+2}{2}\right)} \setminus \left\{\alpha_{(\frac{e}{2})(\frac{e}{2})}^{e^s\ell_s}\right\} \to H_{\left(\frac{e+4}{2}\right)}$ in case $\ell_{\left(\frac{e+4}{2}\right)}$ then $H_{\left(\frac{e+4}{2}\right)} \setminus \left\{\alpha_{(\frac{e-2}{2})(\frac{e+}{2})}^{e^s\ell_s}\right\} \to \cdots \to H_e$ in case ℓ_{e}.

2.H_k in e^s-abacus configuration in case ℓ_1 then $H_k \cup \left\{\alpha_{(k+1)k}^{e^s\ell_1}\right\} \to H_{k+1}$ in case ℓ_2 then $H_{k+1} \cup \left\{\alpha_{(k+2)(k+1)}^{e^s\ell_s}\right\} \to H_{k+2}$ in case ℓ_3 then $H_{k+2} \cup \left\{\alpha_{(k+3)(k+2)}^{e^s\ell_s}\right\} \to \cdots \to H_{\left(\frac{e}{2}+1\right)}$ in case $\ell_{\left(\frac{e-2k}{2}+2\right)}$ then $H_{\left(\frac{e+2}{2}\right)} \setminus \left\{\alpha_{(\frac{e}{2})(\frac{e}{2})}^{e^s\ell_s}\right\} \to H_{\left(\frac{e}{2}+2\right)}$ in case $\ell_{\left(\frac{e-2k}{2}+3\right)}$ then $H_{\left(\frac{e}{2}+2\right)} \setminus \left\{\alpha_{(\frac{e}{2}-1)(\frac{e}{2}+1)}^{e^s\ell_s}\right\} \to \to H_{\left(\frac{e}{2}+3\right)}$ in case $\ell_{\left(\frac{e-2k}{2}+4\right)}$ then $H_{\left(\frac{e}{2}+3\right)} \setminus \left\{\alpha_{(\frac{e}{2}-2)(\frac{e}{2}+1)}^{e^s\ell_s}\right\} \to \cdots \to H_{\left(\frac{e}{2}+n\right)}$

in case $\ell_{\left(\frac{e-2k}{2}+(n+1)\right)}$ then $H_{\left(\frac{e}{2}+n\right)} \setminus \left\{ \alpha^{e^s\ell_1}_{\left(\frac{e}{2}-n+1\right)\left(\frac{e}{2}+n\_1\right)} \right\}$ in case $\ell_{\mathrm{e-k+1}}$ $\rightarrow$ $H_1 = \{o\}$ in case $\ell_{\mathrm{e-k+2}}$ $\rightarrow H_2$ in case $\ell_{\mathrm{e-k+3}}$ $\rightarrow H_3$ in case $\ell_{\mathrm{e-k+4}}$ then $H_3 \cup \left\{ \alpha^{e^s\ell_1}_{32} \right\} \rightarrow \cdots \rightarrow H_{(k-1)}$ in case ℓ_{e} where $\kappa = 2, 3, . . , e - 1, n = 1,2, . . , \frac{e-2}{2}$.

3.H_e in e^s-abacus configuration in case ℓ_1 then $H_e \cup H'_e \rightarrow H'_1$ in case $\ell_2 \rightarrow H_2$ in case ℓ_3 …

4.H'_1 in e^s-abacus configuration in case b_1 $\rightarrow$ H'_2 in case ℓ_2 then $H'_2 \setminus \left\{ \alpha^{e^s\ell_1}_{32}, \alpha^{e^s\ell_1}_{(r-1)2} \right\} \rightarrow H'_3$ in case ℓ_3 then $H'_3 \setminus \left\{ \alpha^{e^s\ell_1}_{43}, \alpha^{e^s\ell_1}_{(r-2)3} \right\} \rightarrow$ $\cdots$ $H'_{\left(\frac{e}{2}\right)}$ in case $\ell_{\left(\frac{e}{2}\right)}$ then $\left\{ H'_{\left(\frac{e}{2}\right)} \setminus \left\{ \alpha^{e^s\ell_1}_{\left(\frac{e+2}{2}\right)\left(\frac{e}{2}\right)} \right\} \right\} \cup \left\{ \alpha^{e^s\ell_1}_{\left(\frac{2r-e+2}{2}\right)\left(\frac{e+2}{2}\right)} \right\} \rightarrow$ $H'_{\left(\frac{e+2}{2}\right)}$ in case $\ell_{\left(\frac{e+2}{2}\right)}$ then $\{H'_{\left(\frac{e+2}{2}\right)} \cup \left\{ \alpha^{e^s\ell_1}_{\left(\frac{e}{2}\right)\left(\frac{e}{2}\right)} \right\} \cup \left\{ \alpha^{e^s\ell_1}_{\left(\frac{2r-e+4}{2}\right)\left(\frac{e+2}{2}\right)} \right\}\} \rightarrow$ H'_e in case ℓ_{e} .

5.H'_k in e^s-abacus configuration in case b_1 then $H'_k \setminus \left\{ \alpha^{e^s\ell_1}_{\left(\frac{e}{2}\right)\left(\frac{e+2}{2}\right)}, \alpha^{e^s\ell_1}_{\left(\frac{2r-e}{2}\right)\left(\frac{e+2}{2}\right)} \right\}$ $\rightarrow H'_{k+1}$ in case ℓ_2 then $H'_{(k+2)} \setminus$ $\left\{ \alpha^{e^s\ell_1}_{\left(\frac{e-2}{2}\right)\left(\frac{e+4}{2}\right)}, \alpha^{e^s\ell_1}_{\left(\frac{2r-e+2}{2}\right)\left(\frac{e+4}{2}\right)} \right\} \rightarrow \cdots \rightarrow H'_{\left(\frac{e}{2}\right)}$ in case $\ell_{\left(\frac{e-2k+2}{2}\right)}$ then $\left\{ H'_{\left(\frac{e}{2}\right)} \setminus \right.$ $\left. \left\{ \alpha^{e^s\ell_1}_{\left(\frac{e+2}{2}\right)\left(\frac{e+2}{2}\right)} \right\} \right\} \cup \left\{ \alpha^{e^s\ell_1}_{\left(\frac{2r-e+2}{2}\right)\left(\frac{e+2}{2}\right)} \right\} \rightarrow H'_{\left(\frac{e+2}{2}\right)}$ in case $\ell_{\left(\frac{e-2k+4}{2}\right)}$ then $H'_{\left(\frac{e+2}{2}\right)} \cup \left\{ \alpha^{A^{tc}b_1}_{\left(\frac{e}{2}\right)\left(\frac{e}{2}\right)} \right\} \cup \left\{ \alpha^{e^s\ell_1}_{\left(\frac{2r-e+4}{2}\right)\left(\frac{e+4}{2}\right)} \right\} \rightarrow H'_{\left(\frac{e+4}{2}\right)}$ in case $\ell_{\left(\frac{e-2k+6}{2}\right)}$ $\rightarrow$ $\cdots \rightarrow H'_e$ in case $\ell_{\mathrm{e-k+1}}$then $H'_{\mathrm{e}} \cup H_e \rightarrow H'_1$ in case $\ell_{\mathrm{e-k+2}} \rightarrow \cdots \rightarrow$ $H'_{(\mathrm{k-1})}$in case ℓ_e where $k = 2, 3, . . , e - 1$.

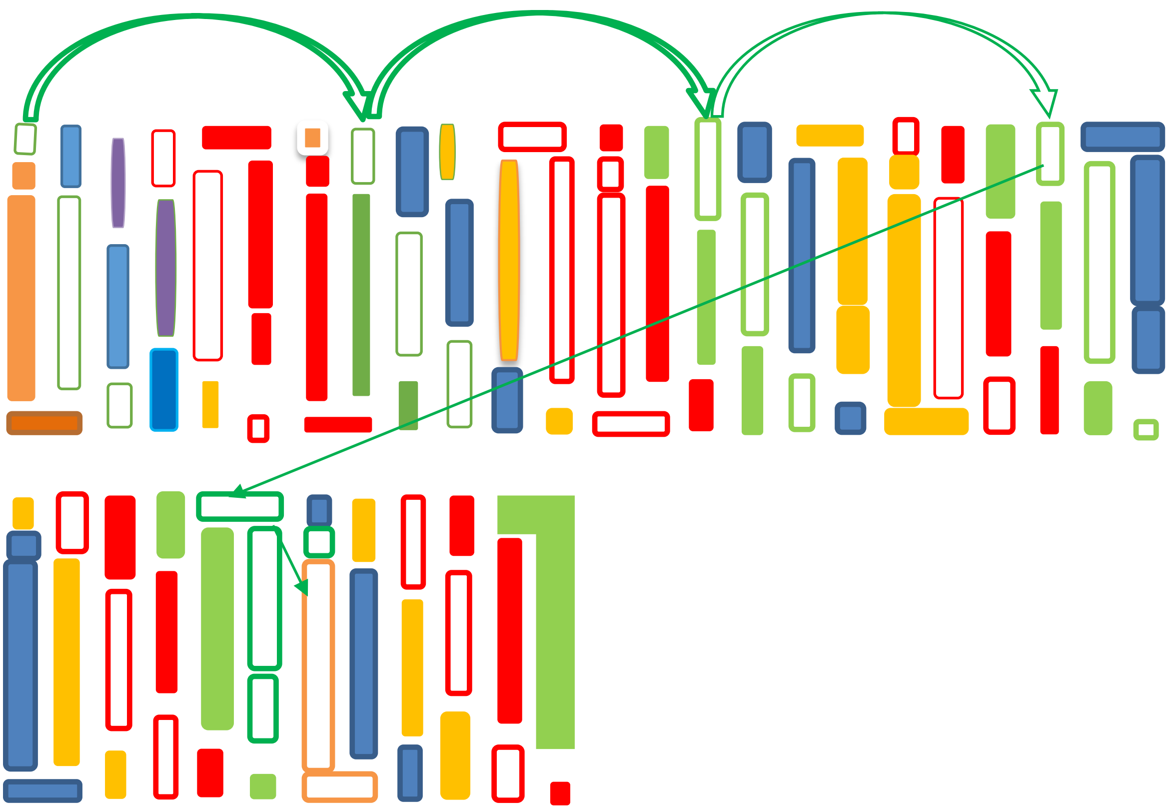
6.H'_e in e^s-abacus configuration in case ℓ_1 then $H_e \cup H'_e \rightarrow H'_1$ in e-abacus configuration in case ℓ_2 ….

7.H''_1 in e^s-abacus configuration in case ℓ_1 $\rightarrow$ H''_2 in case ℓ_2 then $H''_2 \cup \left\{ a^{e^s\ell_1}_{2(r-1)} \right\} \rightarrow H''_3$ in case ℓ_3 then $H''_3 \cup \left\{ a^{e^s\ell_1}_{3(r-2)} \right\} \rightarrow \cdots \rightarrow H''_{\left(\frac{e}{2}\right)}$ in case $\ell_{\left(\frac{e}{2}\right)}$ then $\left\{H''_{\left(\frac{e}{2}\right)} \setminus \left\{\alpha^{e^s \ell_1}_{\left(\frac{2r-e+2}{2}\right)\left(\frac{e+2}{2}\right)}\right\}\right\} \to H''_{\left(\frac{e+2}{2}\right)}$ in case $\ell_{\left(\frac{e+2}{2}\right)}$ then $H'_{\left(\frac{e+2}{2}\right)} \setminus \left\{\alpha^{e^s \ell_1}_{\left(\frac{2r-e+4}{2}\right)\left(\frac{e+4}{2}\right)}\right\} \to \cdots \to H''_e$ in case ℓ_e.

8.H''_k in e^s-abacus configuration in case ℓ_1 then $H''_k \cup \left\{\alpha^{A^{tc}b_1}_{k(r-k+1)}\right\}$ $\to H''_{k+1}$ in case ℓ_2 then $H''_{k+1} \cup \left\{\alpha^{A^{tc}b_1}_{(k+1)(r-k)}\right\} \to \cdots \to H''_{\left(\frac{e}{2}\right)}$ in case $\ell_{\left(\frac{e-2k+2}{2}\right)}$ then $\left\{H''_{\left(\frac{e}{2}\right)} \setminus \left\{\alpha^{A^{tc}b_1}_{\left(\frac{2r-e+2}{2}\right)\left(\frac{e+2}{2}\right)}\right\}\right\} \to H''_{\left(\frac{e+2}{2}\right)}$ in case $\ell_{\left(\frac{e-2k+4}{2}\right)}$ then $H''_{\left(\frac{e+2}{2}\right)} \setminus \left\{\alpha^{A^{tc}b_1}_{\left(\frac{2r-e+4}{2}\right)\left(\frac{e+4}{2}\right)}\right\} \to \cdots \to H''_e$ in case $\ell_{e-k+1} \to H''_1$ in case $\ell_{e-k+2} \to \cdots \to H''_{k-1}$ in case ℓ_e where $k = 2, 3,..,e-1$.

9.H''_e in e^s-abacus configuration in case ℓ_1 then $\to H''_1$ in e^s-abacus configuration in case ℓ_2 …

for the above example, where $\gamma = (8^2, 6, 3, 2, 1^4)$ and $e = 6$.



b_1						b_2						b_3						b_4					
o	o	-	o	o	-	-	o	o	-	o	o	o	-	o	o	-	-	o	o	-	o	o	-
-	-	-	o	-	-	o	o	-	-	o	-	o	-	o	-	o	o	o	o	-	o	-	o
-	o	o	o	o	o	o	-	o	-	o	o	o	o	-	o	-	o	-	o	o	-	o	-
o	-	o	-	-	-	-	o	-	-	-	-	o	-	o	-	-	-	-	o	-	-	-	-
-	o	-	-	-	-	-	-	o	-	-	-	-	-	-	-	-	-	o	-	-	-	-	-
-	-	-	-	-	-	o	-	-	-	-	-	o	o	-	-	-	-	o	o	o	-	-	-
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-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

b_5						b_6					
o	o	o	-	o	o	o	o	o	o	-	-
o	o	o	-	o	-	o	-	o	o	-	o
o	-	-	-	-	o	o	o	o	-	-	-
-	-	o	o	-	-	o	-	-	o	o	-
o	o	-	-	-	-	-	o	o	-	-	-
-	o	o	-	-	-	o	-	o	-	-	-
-	-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-	-	-

Figure 5. e^s-abacus configuration in case ℓ_1.

Theorem 5: Let $\gamma = (\gamma_1^{t_1}, \gamma_2^{t_2}, \dots, \gamma_m^{t_m})$ be a partition of k and Let $\# \cap_{s=1}^{e} m._{\ell_s}$ be a numerical value in e^s-abacus configuration to the beads positions which located in common bead positions in abacus configuration. Then $\# \cap_{s=1}^{e} m._{\ell_s}$ in e^s-abacus configuration = $\# \cap_{s=1}^{e} m._{\ell_s}$ in e-abacus= $\# \cap_{s-1}^{e} m_{\ell_s} = [\sum_{k=1}^{\sigma} \Gamma_\kappa - \sigma(e-1)]$.

Based on Rule 2 all beads position will be stay in common positions in e^s- abacus configuration, hence $\# \cap m._{\ell_s}$ in e-abacus = $\# \cap m_{\ell_s}$ in e^s- abacus configuration.

These results in theorem 5 is clear in diagram 11 comparing it with diagram 3 for e=3, for our example when $\gamma = (8^2, 6,3,2, 1^4)$ for e=3.

$\ell_4 = 9$	$\ell_4=10$	$\ell_4= 11$	$\ell_4= 11$	$\cap_{s=1}^{e} m._{\ell_s}$
• • • −	− • • •	• − • −	• • − •	− − − −
− • − −	• − − −	• • • −	• • − •	− − − −
• − − •	• • − −	• • • −	• • − •	• − − −
• − − −	− • − −	• − • −	− • • •	− − − −
• − − −	− • − −	− − − −	− − − −	− − − −
• − − −	• • − −	− • − −	− − − −	− − − −
− − − −	− − − −	− − − −	− − − −	− − − −
− − − −	− − − −	− − − −	− − − −	− − − −

Figure 6. intersection e^s-abacus configuration in case b_1

Definition 6 [12]. Let $\lambda = (\lambda_1,\lambda_2, \dots)$, $\gamma = (\gamma_1,\gamma_2, \dots)$ be a two partitions , and let ${\beta_1}^{\lambda} < \cdots < {\beta_k}^{\lambda}$, ${\beta_1}^{\gamma} < \cdots < {\beta_k}^{\gamma}$, be Beta numbers to partition λ and γ sequently, then $\lambda \trianglerighteq \mu$ if and only if $\sum_{s=1}^{k} \beta_s^{\lambda} \geq \sum_{s=1}^{k} \beta_s^{\gamma}$.

Remark 7: Let ${\beta_i}^{e}$ be a set of Beta numbers for e-abacus and Let ${\beta_i}^{e^s}$ be a set of Beta numbers for e^s-abacus, since ${\beta_i}^{e} = n + me - e - 1$, where $i=1,2,..,\ell$.

$$\text{Then } {\beta_i}^{e^s} = \begin{cases} {\beta_i}^{e} - e & \text{if} \quad (1+e-n) < m \leq (r+n-e), \frac{2+e}{2} \leq n \leq e\,. \\ {\beta_i}^{e} + e & \text{if} \quad n \leq m < (1+r-n), 1 \leq n \leq \frac{e}{2}. \\ {\beta_i}^{e} - 1 & \text{if} \quad m = u, u < n \leq 1+e-u, 1 \leq u \leq \frac{e}{2}. \\ {\beta_i}^{e} + 1 & \text{if} \quad m = (1+r-u), u \leq n \leq e-u, 1 \leq u \leq \frac{e}{2}. \end{cases}$$

Not: Let $N^{\downarrow}$ be numerical values of ${\beta_i}^{e}$ where $(1+e-n) < m \leq (n+r-e), \frac{2+e}{2} \leq n \leq e$ and $N^{\uparrow}$ be numerical values of ${\beta_i}^{e}$ where $n \leq m < (r-n+1), 1 \leq n \leq \frac{e}{2}$. Let $N^{\leftarrow}$ be numerical values of ${\beta_i}^{e}$ where $m = u, u < n \leq e-u+1, 1 \leq u \leq \frac{e}{2}$ and $N^{\rightarrow}$ be numerical values of ${\beta_i}^{e}$ where $m = (r-u+1), u \leq n \leq e-u, 1 \leq u \leq \frac{e}{2}$.

Theorem 9: Let $= (\lambda_1,\lambda_2, \dots)$, $\gamma = (\gamma_1,\gamma_2, \dots)$ be two partitions and let ${\beta_1}^{\lambda} < \cdots < {\beta_k}^{\lambda}$, ${\beta_1}^{\gamma} < \cdots < {\beta_k}^{\gamma}$, be positions of beads on the e-abacus and e^s-abacus respectively, then $\lambda \triangleright \mu$ if and only if $N^{\downarrow} - (\text{fix}\left(\frac{N^{\leftarrow}}{e}\right)) \geq N^{\uparrow}$.

Proof: Since $N^{\downarrow} - (\text{fix}\left(\frac{N^{\leftarrow}}{e}\right)) \geq N^{\uparrow}$, then $eN^{\downarrow} - N^{\leftarrow} \geq eN^{\uparrow}$, then $eN^{\downarrow} - N^{\leftarrow} - eN^{\uparrow} \geq 0$. Then $\sum_{i=1}^{b} {\beta_i}^{e^s} = \sum_{i=1}^{b} {\beta_i}^{e} + eN^{\downarrow} - N^{\leftarrow} - eN^{\uparrow}$, then $\sum_{i=1}^{b} {\beta_i}^{e^s} \geq \sum_{i=1}^{b} {\beta_i}^{e}$. Based on Definition 9 then $\lambda \trianglerighteq \mu$.

Following e^{s}-abacus in case ℓ_4 will be divided in to three part:

$1-\Gamma_k =$

$$\begin{cases} \{\,\alpha_{21}^{e^s\ell_s}\} & \text{if} \quad \kappa = 1. \\ \{\,\alpha_{1(k-1)}^{e^s\ell_s}, \alpha_{2(k-1)}^{e^s b_s}, \dots, \alpha_{(k-1)(k-1)}^{e^s\ell_s}\,\} & \text{if} \quad \kappa = 2,3,\dots.,\frac{e-1}{2}. \\ \{\,\alpha_{1(\frac{e+2u+1}{2})}^{e^s\ell_s}, \alpha_{2(\frac{e+2u+1}{2})}^{e^s\ell_s}, \dots, \alpha_{(\frac{e-2u-1}{2})(\frac{e+2u+1}{2})}^{e^s\ell_s}\} & \text{if} \quad \kappa = \frac{e+2u+3}{2}, \text{u} = 0,1,\dots,\frac{e-5}{2}. \\ \{\,\alpha_{1(\frac{e-1}{2})}^{e^s\ell_s}, \alpha_{2(\frac{e-1}{2})}^{e^s\ell_s}, \dots., \alpha_{(\frac{e-1}{2})(\frac{e-1}{2})}^{e^s\ell_s}\,\} & \text{if} \quad \kappa = \frac{e+1}{2}. \\ \{\,\alpha_{1(e-1)}^{e^s\ell_s}, \alpha_{1\text{e}}^{e^s b_s}\,\} & \text{if} \quad \kappa = e. \end{cases}$$

$2-\Gamma_k'$

$$= \begin{cases} \{\,a_{31}^{e^s\ell_s}, a_{41}^{e^s\ell_s}, \dots, a_{(r-1)1}^{e^s\ell_s}\} & \text{if} \quad \kappa = 1. \\ \{a_{(k+1)k}^{e^s b_s}, a_{(k+2)k}^{e^s b_s}, \dots, a_{(r-k+1)k}^{e^s b_s}\} & \text{if} \quad \kappa = 2,3,\dots.,\frac{e-1}{2} \\ \{a_{\left(\frac{e-2u-1}{2}\right)\left(\frac{e+2u+3}{2}\right)}^{e^s\ell_s}, a_{\left(\frac{e-2u+1}{2}\right)\left(\frac{e+2u+3}{2}\right)}^{e^s\ell_s}, \dots, a_{\left(\frac{2r-e+2u+1}{2}\right)\left(\frac{e+2u+3}{2}\right)}^{e^s\ell_s}\} & \text{if} \quad \kappa = \frac{e+2u+3}{2}, \text{u} = 0,1, \\ \{a_{\left(\frac{e+1}{2}\right)\left(\frac{e+1}{2}\right)}^{e^s\ell_s}, a_{\left(\frac{e+3}{2}\right)\left(\frac{e+1}{2}\right)}^{e^s\ell_s}, \dots, a_{\left(\frac{2r-e-1}{2}\right)\left(\frac{e+1}{2}\right)}^{e^s\ell_s}, a_{(\frac{2r-e+1}{2})\left(\frac{e+1}{2}\right)}^{e^s\ell_s}\} & \text{if} \quad \kappa = \frac{e+1}{2}. \\ \{a_{2\text{e}}^{e^s\ell_s}, a_{3\text{e}}^{e^s\ell_s}, \dots,, a_{(\text{r}-4)e}^{e^s\ell_s}\} & \text{if} \quad \kappa = e. \end{cases}$$

$3-\Gamma_k''$

$$= \begin{cases} \{\,\alpha_{r1}^{e^s\ell_s}, a_{r2}^{e^s\ell_s}\} & \text{if} \quad \kappa = 1. \\ \{\alpha_{(r-\kappa+1)(\kappa+1)}^{e^s\ell_s}, \alpha_{(r-\kappa+2)(\kappa+1)}^{e^s b_s}, \dots, \alpha_{r(\kappa+1)}^{e^s\ell_s}\} & \text{if} \quad \kappa = 2,3,\dots.,\frac{e-1}{2}. \\ \{\alpha_{\left(\frac{2r-e+2u+5}{2}\right)\left(\frac{e+2u+5}{2}\right)}^{e^s\ell_s}, \alpha_{\left(\frac{2r-e+2u+7}{2}\right)\left(\frac{e+2u+5}{2}\right)}^{e^s\ell_s}, \dots, \alpha_{r\left(\frac{e+2u+5}{2}\right)}^{e^s\ell_s}\} & \text{if} \quad \kappa = \frac{e+2u+3}{2}, \text{u} = 0,1,\dots,\frac{e-5}{2}. \\ \{\alpha_{(\frac{2r-e+3}{2})(\frac{e+3}{2})}^{e^s\ell_s}, \quad \alpha_{(\frac{2r-e+5}{2})(\frac{e+3}{2})}^{e^s\ell_s}, \dots, \alpha_{r(\frac{e+3}{2})}^{e^s\ell_s}\} & \text{if} \quad \kappa = \frac{e+1}{2}. \\ \{\alpha_{(r-3)e}^{e^s\ell_s}, \alpha_{(r-2)e}^{e^s\ell_s}\} & \text{if} \quad \kappa = e. \end{cases}$$

Rule 10. The *e*-abacus configuration in case ℓ_1 Plays a essential role to find e^s-abacus configuration in case $\ell_2, \ell_3, \dots, \ell_e$asfollows:

$\Gamma_1$in e^s- abacus configuration case $\ell_1 \rightarrow \Gamma_2$ in case ℓ_2 then $\Gamma_2 \cup \left\{ \alpha_{32}^{e^s \ell_1} \right\} \rightarrow \Gamma_3$ in case ℓ_3 then $\Gamma_3 \cup \left\{ \alpha_{43}^{e^s \ell_1} \right\} \rightarrow \cdots \rightarrow \Gamma_{\left(\frac{e-1}{2}\right)}$ in case $\ell_{\left(\frac{e-1}{2}\right)}$ then $\Gamma_{\left(\frac{e-1}{2}\right)} \cup \left\{ \alpha_{\left(\frac{e+1}{2}\right)\left(\frac{e-1}{2}\right)}^{e^s \ell_1} \right\} \rightarrow \Gamma_{\left(\frac{e+1}{2}\right)}$ in case $\ell_{\left(\frac{e+1}{2}\right)} \rightarrow \Gamma_{\left(\frac{e+3}{2}\right)}$ in case $\ell_{\left(\frac{e+3}{2}\right)}$ then $\Gamma_{\left(\frac{e+3}{2}\right)} \setminus \left\{ \alpha_{\left(\frac{e-1}{2}\right)\left(\frac{e+1}{2}\right)}^{e^s \ell_1} \right\} \rightarrow \Gamma_{\left(\frac{e+5}{2}\right)}$ in case $\ell_{\left(\frac{e+5}{2}\right)}$ then $\Gamma_{\left(\frac{e+5}{2}\right)} \setminus \left\{ \alpha_{\left(\frac{e-3}{2}\right)\left(\frac{e+3}{2}\right)}^{e^s \ell_1} \right\} \rightarrow \cdots \rightarrow \Gamma_e$ I n case ℓ_e.

$\Gamma_k$in e^s-abacus configuration case ℓ_1 then $\Gamma_k \cup \left\{ \alpha_{(k+1)k}^{e^s \ell_1} \right\} \rightarrow \Gamma_{k+1}$ in case ℓ_2 then $\Gamma_{k+1} \cup \left\{ \alpha_{(k+2)(k+1)}^{e^s \ell_1} \right\} \rightarrow \Gamma_{k+2}$ in case ℓ_3 then $\Gamma_{k+2} \cup \left\{ \alpha_{(k+3)(k+2)}^{e^s \ell_1} \right\} \rightarrow \cdots \rightarrow \Gamma_{\left(\frac{e-1}{2}\right)}$in case $\ell_{\left(\frac{e-2k+1}{2}\right)}$ then $\Gamma_{\left(\frac{e-1}{2}\right)} \setminus \left\{ \alpha_{\left(\frac{e+1}{2}\right)\left(\frac{e-1}{2}\right)}^{e^s \ell_1} \right\} \rightarrow \Gamma_{\left(\frac{e+1}{2}\right)}$ in case $\ell_{\left(\frac{e-2k+3}{2}\right)} \rightarrow \Gamma_{\left(\frac{e+3}{2}\right)}$ in case $\ell_{\left(\frac{e-2k+5}{2}\right)}$ then $\Gamma_{\left(\frac{e+3}{2}\right)} \setminus \left\{ \alpha_{\left(\frac{e-1}{2}\right)\left(\frac{e+1}{2}\right)}^{e^s \ell_1} \right\} \rightarrow \cdots \rightarrow \Gamma_e$ in case ℓ_{e-k+1} then $\Gamma_e \cup \Gamma'_e \rightarrow \Gamma'_1$ in case ℓ_{e-k+2}.

If $e = 3$

Γ'_1 in e^s- abacus configuration case b_1 before shift this case the order of all element set Γ'_1 will be change and add $\alpha_{r1}^{e^s}$ in up and be ={ $\alpha_{r1}^{e^s}$, $\alpha_{21}^{e^s}, \alpha_{31}^{e^s}, \dots, \alpha_{(r-1)1}^{e^s}$} $\rightarrow$ Γ'_2 in e^s- abacus configuration case ℓ_2.

Γ'_1 in e^s- abacus configuration case $\ell_1 \rightarrow \Gamma'_3$ in e^s- abacus configuration case ℓ_3.

Γ'_2 in e^s- abacus configuration case ℓ_1 before shift this case the order of all element set Γ'_2 will be change and add $\alpha_{r1}^{e^s}$ in up and be ={ $\alpha_{11}^{e^s}$, $\alpha_{(r-1)2}^{e^s}$, $\alpha_{22}^{e^s}, \alpha_{32}^{e^s}, \dots, \alpha_{(r-2)2}^{e^s}, \alpha_{r3}^{e^s}$ } $\rightarrow$ H'_3 in e^s- abacus configuration case $\ell_2 \rightarrow$ Γ'_1 in e^s- abacus configuration case ℓ_3 and add o above.

Γ'_3 in e^s- abacus configuration case $b_1 \rightarrow$ Γ'_1 in e^s- abacus configuration case b_2 and add o above before shift this case the order of all element set Γ'_3 will be change and be ={ o, $\alpha_{13}^{e^s}$, $\alpha_{23}^{e^s}$, $\alpha_{33}^{e^s}, \dots, \alpha_{(r-3)3}^{e^s}, \alpha_{12}^{e^s}, \alpha_{(r-2)3}^{e^s}$ } $\rightarrow$ Γ'_2 in e^s- abacus configuration case $\ell_2 \rightarrow \Gamma'_2$ in e^s- abacus configuration case ℓ_2 .

If $e > 3$

Γ'_1 in case $\ell_1 \rightarrow \Gamma'_2$ in case b_2 then $\Gamma'_2 \setminus \left\{ \alpha_{32}^{e^s \ell_1}, \alpha_{(r-1)2}^{e^s \ell_1} \right\} \rightarrow \Gamma'_3$ in case ℓ_3 then $\Gamma'_3 \setminus \left\{ \alpha_{43}^{e^s \ell_1}, \alpha_{(r-2)3}^{e^s \ell_1} \right\} \rightarrow \cdots \rightarrow \Gamma'_{\left(\frac{e-1}{2}\right)}$ in case $\ell_{\left(\frac{e-1}{2}\right)}$, before shift to next case the order of all element set $\left\{ \Gamma'_{\left(\frac{e-1}{2}\right)} \setminus \left\{ \alpha_{\left(\frac{e+1}{2}\right)\left(\frac{e-1}{2}\right)}^{e^s \ell_1} \right\} \right\}$ will be change and be

$\left\{\alpha^{e^s\ell_1}_{\left(\frac{2r-e+3}{2}\right)\left(\frac{e-1}{2}\right)}, \alpha^{e^s\ell_1}_{\left(\frac{e+3}{2}\right)\left(\frac{e-1}{2}\right)}, \alpha^{e^s\ell_1}_{\left(\frac{e+5}{2}\right)\left(\frac{e-1}{2}\right)}, \dots, \alpha^{e^s\ell_1}_{\left(\frac{2r-e+1}{2}\right)\left(\frac{e-1}{2}\right)}\right\} \to \Gamma'_{\left(\frac{e+1}{2}\right)}$ in case $\ell_{\left(\frac{e+1}{2}\right)}$ before shift to next case the order of all element set $\Gamma'_{\left(\frac{e+1}{2}\right)}$ will be change and be { $\alpha^{e^s\ell_1}_{\left(\frac{2r-e+1}{2}\right)\left(\frac{e+1}{2}\right)}, \alpha^{e^s\ell_1}_{\left(\frac{e+1}{2}\right)\left(\frac{e+1}{2}\right)}, \alpha^{e^s\ell_1}_{\left(\frac{e+3}{2}\right)\left(\frac{e+1}{2}\right)}, \dots, \alpha^{e^s\ell_1}_{\left(\frac{2r-e-1}{2}\right)\left(\frac{e+1}{2}\right)}$} and add $\alpha^{e^s\ell_1}_{\left(\frac{2r-e+3}{2}\right)\left(\frac{e+3}{2}\right)}$ down $\to \Gamma'_{\left(\frac{e+3}{2}\right)}$ in case $\ell_{\left(\frac{e+3}{2}\right)}$ then $\left\{\alpha^{e^s\ell_1}_{\left(\frac{e-1}{2}\right)\left(\frac{e+1}{2}\right)}\right\} \cup \Gamma'_{\left(\frac{e+3}{2}\right)} \to \Gamma'_{\left(\frac{e+5}{2}\right)}$ in case $\ell_{\left(\frac{e+5}{2}\right)}$ then $\left\{\alpha^{e^s\ell_1}_{\left(\frac{e-3}{2}\right)\left(\frac{e+3}{2}\right)}\right\} \cup \Gamma'_{\left(\frac{e+5}{2}\right)} \to \cdots \to \Gamma'_{\mathrm{e}}$ in case b_e

Γ'_k in case b_1 then $\Gamma'_k \backslash \left\{\alpha^{e^s\ell_1}_{\left(\frac{e}{2}\right)\left(\frac{e+2}{2}\right)}, \alpha^{e^s\ell_1}_{\left(\frac{2r-e}{2}\right)\left(\frac{e+2}{2}\right)}\right\} \to \Gamma'_{(k+1)}$ in case ℓ_2 then $\Gamma'_{(k+2)}\backslash$ $\{\alpha^{e^s\ell_1}_{\left(\frac{e-2}{2}\right)\left(\frac{e+4}{2}\right)}, \alpha^{e^s\ell_1}_{\left(\frac{2r-e+2}{2}\right)\left(\frac{e+4}{2}\right)}\} \to \cdots \to \Gamma'_{\left(\frac{e-1}{2}\right)}$ in case $\ell_{\left(\frac{e-2k+1}{2}\right)}$, before shift to next case the order of all element set

$\{\Gamma'_{\left(\frac{e-1}{2}\right)} \backslash \left\{\alpha^{e^s\ell_1}_{\left(\frac{e+1}{2}\right)\left(\frac{e-1}{2}\right)}\right\}\}$ will be change and be $\left\{\alpha^{e^s\ell_1}_{\left(\frac{2r-e+3}{2}\right)\left(\frac{e-1}{2}\right)}, \alpha^{e^s\ell_1}_{\left(\frac{e+3}{2}\right)\left(\frac{e-1}{2}\right)}, \alpha^{e^s\ell_1}_{\left(\frac{e+5}{2}\right)\left(\frac{e-1}{2}\right)}, \dots, \alpha^{e^s\ell_1}_{\left(\frac{2r-e+1}{2}\right)\left(\frac{e-1}{2}\right)}\right\} \to \Gamma'_{\left(\frac{e+1}{2}\right)}$ in case $\ell_{\left(\frac{e-2k+3}{2}\right)}$ before shift to next case the order of all element set $\Gamma'_{\left(\frac{e+1}{2}\right)}$ will be change and be { $\alpha^{e^s\ell_1}_{\left(\frac{2r-e+1}{2}\right)\left(\frac{e+1}{2}\right)}, \alpha^{e^s\ell_1}_{\left(\frac{e+1}{2}\right)\left(\frac{e+1}{2}\right)}, \alpha^{e^s\ell_1}_{\left(\frac{e+3}{2}\right)\left(\frac{e+1}{2}\right)}, \dots, \alpha^{e^s\ell_1}_{\left(\frac{2r-e-1}{2}\right)\left(\frac{e+1}{2}\right)}$} and add $\alpha^{e^s\ell_1}_{\left(\frac{2r-e+3}{2}\right)\left(\frac{e+3}{2}\right)}$ down $\to \Gamma'_{\left(\frac{e+3}{2}\right)}$ in case $\ell_{\left(\frac{e\pm 2k+5}{2}\right)}$ then $\left\{\alpha^{e^s\ell_1}_{\left(\frac{e-1}{2}\right)\left(\frac{e+1}{2}\right)}\right\} \cup \Gamma'_{\left(\frac{e+3}{2}\right)} \to \Gamma'_{\left(\frac{e+5}{2}\right)}$ in case $\ell_{\left(\frac{e-2k+7}{2}\right)}$ then $\left\{\alpha^{e^s\ell_1}_{\left(\frac{e-3}{2}\right)\left(\frac{e+3}{2}\right)}\right\} \cup \Gamma'_{\left(\frac{e+5}{2}\right)} \to \cdots \to \Gamma'_{\mathrm{e}}$ in case ℓ_{e-k+1} then $\Gamma_e \cup$ $\{\Gamma'_e \backslash\, a^{e^s b_1}_{(r-4)e}\} \to \Gamma'_1$ in case ℓ_{e-k+2}.

Γ''_1 in case $\ell_1 \to \Gamma''_2$ in case ℓ_2 then $\Gamma''_2 \cup \left\{\alpha^{e^s\ell_1}_{(r-1)2}\right\} \to \Gamma''_3$ in case ℓ_3 then $\Gamma''_3 \cup$ $\left\{\alpha^{e^s\ell_s}_{(r-2)3}\right\} \to \cdots \to \Gamma''_{\left(\frac{e-1}{2}\right)}$ in case $\ell_{\left(\frac{e-1}{2}\right)}$ then $\to \Gamma''_{\left(\frac{e+1}{2}\right)}$ in case $\ell_{\left(\frac{e+1}{2}\right)}$ then $\Gamma'_{\left(\frac{e+1}{2}\right)} \backslash$ $\left\{\alpha^{e^s\ell_1}_{\left(\frac{2r-e+3}{2}\right)\left(\frac{e+3}{2}\right)}\right\} \to \cdots \to \Gamma''_e$ in case ℓ_e.

<u>Theorem</u> 11: Let $\lambda = (\lambda_1^{d_1}, \lambda_2^{d_2}, \ldots, \lambda_m^{d_m})$ be a partition of t. Then $\# \cap m.d._{\ell_s}$ in e-abacus configuration = $\# \cap m.d._{b_s}$ in e^s-abacus configuration.

These results in theorem (11) is clear in Figure 9 where μ = (8^2, 6,3,2, 1^4) and for e=3.

e = 3			b = 9			b = 10			b=11			$\cap_{s=1}^{3} m.d.$		
0	1	2	•	•	−	−	•	•	•	−	•	−	−	−
3	4	5	−	−	−	•	-	−	•	−	•	−	−	−
6	7	8	•	•	•	•	•	−	•	•	−	•	•	−
9	10	11	•	−	−	-	•	−	•	−	•	−	−	−
12	13	14	−	−	−	•	-	•	−	•	•	−	−	−
15	16	17	•	−	−	−	•	−	−	−	−	−	−	−
18	19	20	•	•	−	−	•	−	•	−	−	−	−	−

Figure 9. Intersection e^s- abacus configurations

Remark 12 : Let $\beta_i{}^e$ be a set of beta-numbers for e-abacus configurations and let $\beta_i{}^{e^s}$ be a set of beta-numbers for e^s-abacus configurations .

Remark 13: Let $\beta_i{}^e$ set of beta-numbers for e-abacus configurations and Let $\beta_i{}^{e^s}$ set of beta-numbers

Γ''_k in case $\ell_1 \to \Gamma''_{k+1}$ in case ℓ_2 then $\Gamma''_{k+1} \cup \left\{\alpha^{e^s\ell_1}_{(r-1)2}\right\} \to \cdots \to \Gamma''_{\left(\frac{e-1}{2}\right)}$ in case $\ell_{\left(\frac{e-2k+1}{2}\right)}$ then $\to \Gamma''_{\left(\frac{e+1}{2}\right)}$ in case $\ell_{\left(\frac{e-2k+3}{2}\right)}$ then $\Gamma'_{\left(\frac{e+1}{2}\right)} \setminus \left\{\alpha^{e^s\ell_1}_{\left(\frac{2r-e+3}{2}\right)\left(\frac{e+3}{2}\right)}\right\} \to \ldots \to \Gamma''_e$ in case $\ell_{e-k+1} \to \Gamma''_1$ in case $\ell_{e-k+2} \to \cdots \to \Gamma''_{k-1}$ in case ℓ_e.

for e^s-abacus, since $\beta_i{}^e = me + n - e - 1$, where $i = 1,2,\ldots,\ell$.

Then

$$\beta_i{}^{e^s} = \begin{cases} \beta_i{}^e - e & \text{if} \quad (1+e-n) < m \le (1+r-e+n), \frac{e+3}{2} \le n \le e\,. \\ \beta_i{}^e + e & \text{if} \quad n \le m < (1+r-n), 1 \le n \le \frac{e-1}{2}. \\ \beta_i{}^e - 1 & \text{if} \quad m = u, u \le n \le e-u, 1 \le u \le \frac{e-1}{2}. \\ \beta_i{}^e + 1 & \text{if} \quad m = (1+r-u), u \le n \le 1+e-u, 1 \le u \le \frac{e-1}{2}. \\ \beta_i{}^e - e(r-e) & \text{if} \quad n = \frac{e+1}{2}, m = \frac{2r-e+1}{2}. \\ \beta_i{}^e + e & \text{if} \quad n = \frac{e+1}{2}, m = \frac{e+1}{2}, \ldots, \frac{2r-e-1}{2}. \end{cases}$$

Remark 14. Let $N^{\downarrow\downarrow}$ is the number of $\beta_i{}^e$ where $k = \frac{e+1}{2}, n = \frac{e+1}{2}, \ldots, \frac{2r-e-1}{2}$, $N^{\uparrow\uparrow} = (r-e)$.

Theorem 15. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\gamma = (\gamma_1, \gamma_2, \ldots)$ be a partitions and let $\beta_1{}^\lambda < \cdots < \beta_k{}^\lambda$, $\beta_1{}^\gamma < \cdots < \beta_k{}^\gamma$ be the position of beta numbers for e-abacus and e^s-abacus respectively, then $\gamma \rhd \lambda$ if and only if

$$N^{\downarrow} + fix(\frac{N^{\rightarrow}}{e}) \geq N^{\uparrow} + fix(\frac{N^{\leftarrow}}{e}).$$

Proof:

Since $N^{\downarrow} + fix(\frac{N^{\rightarrow}}{e}) \geq N^{\uparrow} + fix(\frac{N^{\leftarrow}}{e})$, then $eN^{\downarrow} + N^{\rightarrow} \geq eN^{\uparrow} + N^{\leftarrow}$. Thus $eN^{\downarrow} - N^{\leftarrow} - eN^{\uparrow} + N^{\leftarrow} \geq 0$. By remark (13) $\sum_{i=1}^{b} \beta_i{}^{e^s} = \sum_{i=1}^{b} \beta_i{}^e + eN^{\downarrow} - N^{\leftarrow} - eN^{\uparrow} + eN^{\downarrow\downarrow} - e(r-e)$. Then $\sum_{i=1}^{b} \beta_i{}^{e^s} \geq \sum_{i=1}^{b} \beta_i{}^e$. By Definition 6 then $\lambda \trianglerighteq \mu$.

CONCLUSION

In this work, a method is proposed to create an e-abacus that represents the sequence movement of betanumber in the e-abacus configurations. A special case when the length $(1,1,\ldots,1)$ was revealed to illustrate the rule for designing a new e-abacus, a rule to findthe new diagram of $\ell_2, \ell_3, \ldots, \ell_e$, was presented where e integers number, $e \geq 2$.

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