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RESEARCH ARTICLE - MATHEMATICS

Best one sided approximation of unbounded functions trigonometrical polynomials for one dimensional weighted-space

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Abstract

The main purpose of present our paper, is to obtain the best one-sided trigonometrical approximation for an unbounded function of one variable in weighted $L_{p,\alpha}\,[\pi,-\pi]$-space. Additionally, the averaged modulus of smoothness have been utilized in the weighted $L_{p,\alpha}\,[\pi,-\pi]$-space to estimate the degree of the best one-sided trigonometrical approximation. Also some certain auxiliary lemmas that will help us in obtaining our main results have been introduced. Furthermore, the direct theorem (Jackson theorem) and the inverse theorem were also proven in trigonometric polynomials, and the equivalence between them was obtained.

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1. Introduction and Preliminaries

The approximation theory is a vast field. The study of the theory of trigonometric approximation is of great mathematical interest and practical significance. The Fourier series of 2π −periodic functions on the real line can be summated linearly to obtain the most significant trigonometric polynomials utilized in approximation theory. The periodicity of the functions has contributed significantly to the advancement of the theory of trigonometric approximation. In [1] Auad and Khrajan estimated the rate for best trigonometric approximation of unbounded functions of one variable in weighted space $L_{p,\alpha}[-\pi,\pi]$ as well as studied the degree of best trigonometric approximation by modulus of smoothness in $L_{p,\alpha}[-\pi,\pi]$. In [2] R. F. Hassan etal. studied the best one-sided multiplier approximation of unbounded functions for trigonometric polynomials in weighted space $L_{p,\Psi_n}[0,2\pi]$ as well as estimate the degree of the best one-sided multiplier approximation by averaged modulus. In [3] R. Suo and Y. Ping obtain the asymptotic estimations of non-linear best m-term one-sided trigonometric approximation under the norm L_p $(1 \le p \le \infty)$ of multiplier function classes within the norm L_p used to estimate the non-linear best m-term one-sided trigonometric approximation and obtain the results of the corresponding m-term Greedy-liked one-sided trigonometric approximation. In [4] A. H. Zaboon examined the approximation of functions using the Fejer operator in the space $L_{p,w}[-\pi,\pi]$ in terms of the second-order average modulus which resulted in determining the best approximation of the function, so that the difference between the function f and its approximation $\delta_m(f)$ is equal to zero. In [5] S. K. Al-Saidy and A. H. Zuboon investigate the use of weighted trigonometric polynomials for approximating unbounded functions within locally-global weighted spaces $L_{P,\delta,w}[-\pi,\pi]$, utilizing the weighted Ditzian-Totik modulus of smoothness. In [6] Z. Cakir, C. Aykol, D. Soylemez and A. Serbetci found best approximation trigonometric polynomials in Morrey space $L_{p,\lambda}[0,2\pi]$ and the theorems both direct and inverse using trigonometric polynomials in the spaces $L_{p,\lambda}[0,2\pi]$ the closure of $C^{\infty}[0,2\pi]$ are proven and we obtain the modulus of smoothness characterization of K-functionals and provide the Bernstein type inequality for trigonometric polynomials in the spaces $L_{p,\lambda}[0,2\pi]$. In [7] S. S. Mahdi and E. S. Bhaya find the degree of the best approximation by this neural network using the k' th order of smoothness. In [8] W. A. Ajel and E. S. Bhaya obtained the relation between the degree of (monotone, unconstrained) approximation under some conditions on f which belongs to quasi normed space. On the other hand, several papers have been devoted to studying polynomial approximation with constraints. In particular in [9–12] one-sided approximation was considered and some nonlinear operators for one-sided approximation that constructions have been proposed in [13–15]. In the present work we consider rate of best one-sided trigonometrical approximation of unbounded function for one variable in weighted space in interval $[\pi,-\pi]$ in terms averaged modulus of smoothness. In addition, were presented proofs the direct theorem and inverse theorem of trigonometric polynomials and we obtained the equivalence between them.

In the last years, there has been interest in studying open problems related to one-sided approximation (see [16], [17], [18], [19] and [20]). We shall denote by $A=[-\pi,\pi]$ the set of all 2π −periodic with respect to the each variable functions of one variable $g(x)$, which are unbounded and integrable on

$$\|g\|_{L_{p,\alpha}} := \|g\|_{L_{p,\alpha}(A)} = \left\{ \int_{-\pi}^{\pi} |g(x)w_\alpha(x)|^p dx \right\}^{1/p}, \qquad 1 \le p < \infty$$

$$\|g\|_\infty := \sup\{|g(x)|: x \in A\}.$$

where $w_\alpha(x)$ is weigh function in W s.t $W=\{w|w: A \to R^+\}$.

For $f \in L_{p,\alpha}(A)$ we define the modulus of smoothness in order k of f as follows:

$$\omega_k(f,\delta) = \sup\{|\Delta_h^k f(y)| : y, y+kh \in A\}, \tag{1}$$

where

$$\Delta_h^k f(y) = \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} f(y+jh)$$

is the k −difference of the function f with step h (in direction h) .

The k −th averaged modulus of smoothness of $f \in L_{p,\alpha}$, $1 \le p < \infty$, is given by

$$\tau_k(f,\delta)_{L_{p,\alpha}(A)} = \|\omega_k(f,\cdot,\delta)\|_{L_{p,\alpha}(A)}. \tag{2}$$

Both definitions (1) and (2) in the one-dimensional case coincide with definitions of k −th averaged moduli of smoothness, given in [3], [4]. For the history of the averaged moduli of smoothness see [21], [22], [23].

In the case $p=\infty$ the k −th averaged modulus of smoothness coincides with the usual (uniform) k −th modulus of continuity. The connection between the integral moduli given by

$$\omega_k(f,\delta)_{L_{p,\alpha}(A)} = \sup\left\{ \left\|\Delta_h^k f(.)\right\|_{L_{p,\alpha}(A)} : |h| \le \delta \right\}$$

is the following:

$$\omega_k(f,\delta)_{L_{p,\alpha}(A)} \le \tau_k(f,\delta)_{L_{p,\alpha}(A)} \le \omega_k(f,\delta)_\infty.$$

We shall approximate the functions in A by trigonometrical polynomials of order n. The set of all trigonometrical polynomials of order n with respect to each of one-variables we define by

$$\pi_n = \left\{ p: p(x) = \sum_{0 \le n'+n'' \le n} a_{n',n''} \cos n'x \sin n''x \right\}.$$

The best trigonometrical approximation of order n of the function $f \in L_{p,\alpha}(A)$ is given by

$$E_n(f)_{L_{p,\alpha}(A)} = \inf\left\{ \|f-\mathcal{P}\|_{L_{p,\alpha}(A)} : \mathcal{P} \in \pi_n \right\}.$$

The degree best upper (lower) trigonometrical approximation of order n of the function $f \in L_{p,\alpha}(A)$ is respectively given by

$$E_n^+(f)_{L_{p,\alpha}(A)} = \inf\{\|\mathcal{P}-f\|_{L_{p,\alpha}(A)} : \mathcal{P} \in \pi_n, \mathcal{P}(x) \ge f(x),\ \ x \in R\}$$

$$E_n^-(f)_{L_{p,\alpha}(A)} = \inf\{\|f-\mathcal{Q}\|_{L_{p,\alpha}(A)} : \mathcal{Q} \in \pi_n, f(x) \ge \mathcal{Q}(x),\ \ x \in R\}.$$

The best one-sided trigonometrical approximation of order n of the function $f \in L_{p,\alpha}(A)$ is given by

$$\tilde{E}_n(f)_{L_{p,\alpha}(A)} = inf\left\{ \|\mathcal{P}-\mathcal{Q}\|_{L_{p,\alpha}(A)} : \mathcal{P},\mathcal{Q} \in \pi_n, \mathcal{P}(x) \ge f(x) \ge \mathcal{Q}(x), x \in R \right\}.$$

Here we enlist some of the basic properties of the defined functions, which we need below. When some of the following propositions is valid for each of $E_n^+(\cdot)_{L_{p,\alpha}(A)}, E_n^-(\cdot)_{L_{p,\alpha}(A)}$ and $\tilde{E}_n(\cdot)_{L_{p,\alpha}(A)}$.

i) $E_n(f)_{L_{p,\alpha}(A)} \le \tilde{E}_n(f)_{L_{p,\alpha}(A)},\ \tilde{E}_n(f)_{L_{p,\alpha}(A)} \le 2E_n(f)_{L_{p,\alpha}(A)}$.

ii) $\tilde{E}_n(f)_{L_{p,\alpha}(A)} = E_n^+(f)_{L_{p,\alpha}(A)} + E_n^-(f)_{L_{p,\alpha}(A)}$.

iii) If $\mathcal{P} \in \pi_n$, then $E_n(f-\mathcal{P})_{L_{p,\alpha}(A)} = E_n(f)_{L_{p,\alpha}(A)}, \tilde{E}_n(f-\mathcal{P})_{L_{p,\alpha}(A)} = \tilde{E}_n(f)_{L_{p,\alpha}(A)}$.

iv) If $\lambda > 0$, then $E(\pm\lambda f)_{L_{p,\alpha}(A)} = \lambda E_n(f)_{L_{p,\alpha}(A)}, \tilde{E}_n(\pm\lambda f)_{L_{p,\alpha}(A)} = \lambda \tilde{E}_n(f)_{L_{p,\alpha}(A)}$.

v) If $g \in L_{p,\alpha}(A)$, then

$$E_n(f+g)_{L_{p,\alpha}(A)} \le \|g\|_{L_{p,\alpha}(A)} + E_n^+(f)_{L_{p,\alpha}(A)}.$$

vi) The functions $\tilde{E}_n(\cdot)_{L_{p,\alpha}(A)}$ and $E_n(\cdot)_{L_{p,\alpha}(A)}$ are semi-additive.

vii) There exist unique $\mathcal{P}$ and $\mathcal{Q}$ in π_n for which

$$E_n^+(f)_{L_{p,\alpha}(A)} = \|\mathcal{P}-f\|_{L_{p,\alpha}(A)}, E_n^-(f)_{L_{p,\alpha}(A)} = \|f-\mathcal{Q}\|_{L_{p,\alpha}(A)}, \tilde{E}_n(f)_{L_{p,\alpha}(A)} = \|\mathcal{P}-\mathcal{Q}\|_{L_{p,\alpha}(A)}$$

and $\mathcal{P}(x) \ge f(x) \ge \mathcal{Q}(x)$ for any $x \in R$.

Proof:

i) Consider $\mathcal{P}^*, \mathcal{P}_1, \mathcal{Q}_1 \in \pi_n$ are best approximation of $f \in L_{p,\alpha}(A)$ s.t $\mathcal{Q}_1 \le f(x) \le \mathcal{P}_1(x)$

$f(x) \le \mathcal{Q}_1(x)\ \ \mathcal{P}_1(x) \le 2f(x)$

$$E_n(f)_{L_{p,\alpha}(A)} = inf\left\{\|f-\mathcal{P}\|_{L_{p,\alpha}(A)}: \mathcal{P} \in \pi_n\right\} = \|f-\mathcal{P}^*\|_{L_{p,\alpha}(A)} = \left(\int_{-\pi}^{\pi} |(f-\mathcal{P}^*)(x)w_\alpha(x)|^p dx\right)^{1/p}$$
$$\le \left(\int_{-\pi}^{\pi} |(\mathcal{P}-\mathcal{Q})(x)w_\alpha(x)|^p dx\right)^{1/p} = \|\mathcal{P}-\mathcal{Q}\|_{L_{p,\alpha}(A)} = \tilde{E}_n(f)_{L_{p,\alpha}(A)},$$

$$\tilde{E}_n(f)_{L_{p,\alpha}(A)} = inf\|\mathcal{P}-\mathcal{Q}\|_{L_{p,\alpha}(A)} = \|\mathcal{P}_1-\mathcal{Q}_1\|_{L_{p,\alpha}(A)} = \left(\int_{-\pi}^{\pi} |(\mathcal{P}_1-\mathcal{Q}_1)(x)w_\alpha(x)|^p dx\right)^{1/p}$$
$$= \left(\int_{-\pi}^{\pi} |[\mathcal{P}_1(x)-\mathcal{Q}_1(x)]w_\alpha(x)|^p dx\right)^{1/p} \le \left(\int_{-\pi}^{\pi} |[2f(x)-f(x)]w_\alpha(x)|^p dx\right)^{1/p}$$
$$= \left(\int_{-\pi}^{\pi} |[2f(x)-\mathcal{P}^*(x)+\mathcal{P}^*(x)-f(x)]w_\alpha(x)|^p dx\right)^{1/p}$$
$$= \left(\int_{-\pi}^{\pi} |[2f(x)-\mathcal{P}^*(x)]w_\alpha(x)|^p dx\right)^{1/p} + \left(\int_{-\pi}^{\pi} |[f(x)-\mathcal{P}^*(x)]w_\alpha(x)|^p dx\right)^{1/p}$$
$$= \|2f-\mathcal{P}^*\|_{L_{p,\alpha}(A)} + \|f-\mathcal{P}^*\|_{L_{p,\alpha}(A)} \le 2E_n(f)_{L_{p,\alpha}(A)}.$$

ii)

$$\tilde{E}_n(f)_{L_{p,\alpha}(A)} = inf\|\mathcal{P}-\mathcal{Q}\|_{L_{p,\alpha}(A)} = inf\left(\int_{-\pi}^{\pi} |(\mathcal{P}-\mathcal{Q})(x)w_\alpha(x)|^p dx\right)^{1/p} = inf\left(\int_{-\pi}^{\pi} |[\mathcal{P}(x)-\mathcal{Q}(x)]w_\alpha(x)|^p dx\right)^{1/p}$$
$$= inf\left(\int_{-\pi}^{\pi} |[\mathcal{P}(x)-f(x)+f(x)-\mathcal{Q}(x)]w_\alpha(x)|^p dx\right)^{1/p}$$
$$\le inf\left(\int_{-\pi}^{\pi} |[\mathcal{P}(x)-f(x)]w_\alpha(x)|^p dx\right)^{1/p} + inf\left(\int_{-\pi}^{\pi} |[f(x)-\mathcal{Q}(x)]w_\alpha(x)|^p dx\right)^{1/p}$$
$$= inf\left(\int_{-\pi}^{\pi} |(\mathcal{P}-f)(x)w_\alpha(x)|^p dx\right)^{1/p} + inf\left(\int_{-\pi}^{\pi} |(f-\mathcal{Q})(x)w_\alpha(x)|^p dx\right)^{1/p}$$
$$= \|\mathcal{P}-f\|_{L_{p,\alpha}(A)} + \|f-\mathcal{Q}\|_{L_{p,\alpha}(A)} = E_n^+(f)_{L_{p,\alpha}(A)} + E_n^-(f)_{L_{p,\alpha}(A)}.$$

iii) $E_n(f-\mathcal{P})_{L_{p,\alpha}(A)} = inf\|(f-\mathcal{P})-\mathcal{P}\|_{L_{p,\alpha}(A)} = inf\|f-2\mathcal{P}\|_{L_{p,\alpha}(A)} = E_n(f)_{L_{p,\alpha}(A)},$

$$\tilde{E}_n(f-\mathcal{P})_{L_{p,\alpha}(A)} = inf\|(f-\mathcal{P})-\mathcal{Q}\|_{L_{p,\alpha}(A)} = inf\|(f-(\mathcal{P}+\mathcal{Q})\|_{L_{p,\alpha}(A)} = \tilde{E}_n(f)_{L_{p,\alpha}(A)}.$$

iv) $E_n(\pm\lambda f)_{L_{p,\alpha}(A)} = inf\|(\pm\lambda f) - (\pm\lambda\mathcal{P})\|_{L_{p,\alpha}(A)} = inf\left\{\left(\int_{-\pi}^{\pi}\left|\left((\pm\lambda)(f-\mathcal{P})\right)(x)w_\alpha(x)\right|^p dx\right)^{1/p}\right\} = \lambda\, inf\left\{\left(\int_{-\pi}^{\pi}|(f-\mathcal{P})(x)w_\alpha(x)|^p dx\right)^{1/p}\right\} = \lambda E_n(f)_{L_{p,\alpha}(A)}$,

$$\tilde{E}_n(\pm\lambda f)_{L_{p,\alpha}(A)} = inf\|(\pm\lambda\mathcal{P}) - (\pm\lambda\mathcal{Q})\|_{L_{p,\alpha}(A)} = inf\left\{\left(\int_{-\pi}^{\pi}\left|\left((\pm\lambda)(\mathcal{P}-\mathcal{Q})\right)(x)w_\alpha(x)\right|^p dx\right)^{1/p}\right\}$$
$$= \lambda\, inf\left\{\left(\int_{-\pi}^{\pi}|(\mathcal{P}-\mathcal{Q})(x)w_\alpha(x)|^p dx\right)^{1/p}\right\} = \lambda\tilde{E}_n(f)_{L_{p,\alpha}(A)},$$

v) $E_n(f+g)_{L_{p,\alpha}(A)} = inf\|(f+g)-\mathcal{P}\|_{L_{p,\alpha}(A)} = inf\left\{\left(\int_{-\pi}^{\pi}\left|\left((f+g)-\mathcal{P}\right)(x)w_\alpha(x)\right|^p dx\right)^{1/p}\right\} = inf\left\{\left(\int_{-\pi}^{\pi}|[(f+g)(x)-\mathcal{P}(x)]w_\alpha(x)|^p dx\right)^{1/p}\right\} = inf\left(\int_{-\pi}^{\pi}|[f(x)+g(x)-\mathcal{P}(x)]w_\alpha(x)|^p dx\right)^{1/p} \leq inf\left(\int_{-\pi}^{\pi}|g(x)w_\alpha(x)|^p dx\right)^{1/p} + inf\left(\int_{-\pi}^{\pi}|[\mathcal{P}(x)-f(x)]w_\alpha(x)|^p dx\right)^{1/p} = inf\left(\int_{-\pi}^{\pi}|g(x)w_\alpha(x)|^p dx\right)^{1/p} + inf\left(\int_{-\pi}^{\pi}|[(\mathcal{P}-f)(x)]w_\alpha(x)|^p dx\right)^{1/p} = \|g\|_{L_{p,\alpha}(A)} + E_n^+(f)_{L_{p,\alpha}(A)}$.

vi) We note from (iii) and (v) that $\tilde{E}_n(\cdot)_{L_{p,\alpha}(A)}$ and $E_n(\cdot)_{L_{p,\alpha}(A)}$ are semi-additive.

vii) If $\mathcal{Q}(x) \leq f(x) \leq \mathcal{P}(x)$ we need to prove that the a uniqueness $\mathcal{Q}(x)$ and $\mathcal{P}(x)$,

so, let $\mathcal{Q}'(x) \leq f(x) \leq \mathcal{P}'(x) \ni \mathcal{Q}'(x) \leq \mathcal{Q}_n(x)$ & $\mathcal{P}'(x) \leq \mathcal{P}(x)$

$$\Longrightarrow \mathcal{Q}'(x) \leq \mathcal{Q}(x) \leq f(x) \leq \mathcal{P}'(x) \leq \mathcal{P}(x)$$

$\Rightarrow \mathcal{P}(x)$ has a unique sub polynomial it is itself

$$\Longrightarrow \mathcal{P}'(x) = \mathcal{P}(x) \;\&\; \mathcal{Q}'(x) = \mathcal{Q}(x).$$

Let f and g be 2π −periodic integrable functions. Then convolution of f and g is

$$g * f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x, x-u) f(x,u)du.$$

We shall use the following property of convolutions: if $g = g(x), g \in L_1(A), f \in L_{p,\alpha}(A)$, then

$$\|g * f\|_{L_{p,\alpha}(A)} \leq \frac{1}{2\pi}\ \|g\|_{L_1(A)}\|f\|_{L_{p,\alpha}(A)}.$$

The first Bernoulli functions denoted by $B_\alpha(x)$, $\alpha \geq 0$ where

$$B_0(s) \equiv 1,$$
$$B_1(s) = 2\sum_{v=1}^{\infty}\frac{\sin vs}{s} = \begin{cases}\pi - s, & 0 < s < 2\pi \\ 0, & s = 0.\end{cases}$$

2. Auxiliary lemmas

Lemma 2.1. If $B_\alpha f \in L_1(A)$ then

$$f(x) = a_0 + B_\alpha * Df(x),$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)w_\alpha(x)dx.$$

Lemma 2.2. Let $Df \in L_{p,\alpha}(A)$ then

$$\|B_\alpha * Df\|_{L_{p,\alpha}(A)} \leq c\|f\|_{L_{p,\alpha}(A)},$$

where c is positive constant and $*$ is the convolution of B_α and Df.

The proofs of these two lemmas are routine and can be made by induction.

Lemma 2.3. Let n be a natural number. Then $\tilde{E}_n(B_1)_1 = 4\pi^2/n$, i.e. there exist two trigonometrical polynomials T_n and t_n of order $n-1$ such that:

(i) $T_n(s) \geq B_1(s) \geq t_n(s),\ \ s \in R$,

(ii) $\|T_n - t_n\|_{L_{p,\alpha}(A)} = 4\pi^2/n$.

Proof. In order to prove the lemma we shall construct trigonometrical polynomials with certain interpolator properties. Afterwards we show that these are the extremal polynomials.

Now T_n is defined by the following conditions if $B_1(s) = \pi - s,\ 0 < s < 2\pi$,

$$T_n(2\pi k/n) = B_1(2\pi k/n), \qquad k = 1,2, \dots, n,$$
$$T'_n(2\pi k/n) = B'_1(2\pi k/n) = 1, \qquad k = 1,2, \dots, n-1.$$

The existence of a polynomial with these properties is evident. The polynomial t_n is defined by

$$t_n(2\pi k/n) = B_1(2\pi k/n), \qquad k = 1,2, \dots, n-1,$$
$$t'_n(2\pi k/n) = B'_1(2\pi k/n) = 1, \qquad k = 1,2, \dots, n-1,$$

or more simply by the observation that

$$t_n(s) = -T_n(2\pi - s)$$

must hold.

For future reference we remark that it follows from the definitions of T_n and t_n that

$$T_n(s) - t_n(s) = 2\pi F_n(s) = 2\pi n^{-2}\left(\sin\frac{1}{2}s\right)^{-2}\left(\sin\frac{1}{2}ns\right)^2.$$

This is evident since $(T_n - t_n)/2\pi$ has double zeros at the points $2\pi k/n\,, k = 1,2, \dots, n-1$, and equals 1, for $s = 0$, and these properties characterize the Fejér kernel F_n among the trigonometrical polynomials of order $n-1$.

Lemma 2.4. we have

$$B_\alpha - (t_n)_\alpha = B_\beta(B_\alpha - (t_n)_\alpha),$$

such that $\alpha, \beta \geq 0$ and $\alpha < \beta$.

The proof of this combinatory Lemma can be done by induction.

The following Lemma is a well-known tool for intermediate approximation in the classical approximation theory:

Lemma 2.5. Let $f \in L_{p,\alpha}(A)$ and k and n be natural numbers and $\delta > 0$. There exists $f_{k,n} \in L_{p,\alpha}(A)$ with the properties:

a)$\left|f_{k,n}(x) - f(x)\right| \leq \omega_k(f, x, 2/n), x \in R$,

b)$\left\|f_{k,n} - f\right\|_{L_{p,\alpha}(A)} \leq c(k,\delta)\omega_k(f, 1/n)_{L_{p,\alpha}(A)}$,

c)we have

$$\left\|Df_{k,n}\right\|_{L_{p,\alpha}(A)} \leq c(k,\delta)n\omega_k(f, 1/n)_{L_{p,\alpha}(A)}.$$

We shall use an analogue of this Lemma with the restriction that the function $f_{k,n}$ is over the function f. As a consequence at the place of $\omega_r(f, 1/n)_{L_{p,\alpha}(A)}$ will appear $\tau_r(f, 1/n)_{L_{p,\alpha}(A)}$.

Lemma 2.6. Let $f \in L_{p,\alpha}(A)$ and $1 \leq p < \infty$. For any natural numbers k and n there is a function $F_{k,n}$ with the properties:

a) $0 \leq F_{k,n}(x) - f(x) \leq 2^{k\delta - k + 1}\omega_k(f, x, (2k\delta + 8\pi)/kn)$ and

b) $\left\|F_{k,n} - f\right\|_{L_{p,\alpha}(A)} \leq c(k,\delta)\tau_k(f, 1/n)_{L_{p,\alpha}(A)}$

c) we have

$$\left\|DF_{k,n}\right\|_{L_{p,\alpha}(A)} \leq c(k,\delta)n\tau_k(f, 1/n)_{L_{p,\alpha}(A)}.$$

Proof. Let $A_n = \{a: a = \left(\frac{2\pi}{n}a\right), a - \text{integer}\}$. Let ψ_n be an infinitely many times differentiable function with the properties:

1. $0 \le \psi_n(x) \le 1$ for every $x \in R$,
2. $\psi_n(x) = 0$ for $|x| \ge 2\pi/n$,
3. $$\sum_{a\in A_n} \psi_n(x-a) = 1.$$
4. we have $$|D\psi_n(x)| \le cn.$$

Such functions exist, for example we can take

$$\psi_n(x) = \Phi_n(x) \Big/ \sum_{a\in A_n} \Phi_n(x-a),$$

where

$$\Phi_n(x) = \begin{cases} \exp(-1/(1-4\pi^2x^2/n^2)), & |x| < 2\pi/n\,, \\ 0, & |x| \ge 2\pi/n\,. \end{cases}$$

Let $v = k\delta$ and let us consider the function

$$F_{k,n}(x) = f_{v,n}(x) + \sum_{a\in A_n} \omega_v(f, a, (2v+4\pi)/vn)\psi_n(x-a),$$

where $f_{v,n}$ is the corresponding function from Lemma 2.5.

The function $F_{k,n}$ is well-defined for every x, because for every x in the sum on the right-hand side only a finite number of terms are different from zero (see property 2 of the function ψ_n). Let us show that the function $F_{k,n}$ satisfies the conditions of the Lemma.

Let $x \in R$. Denoting $A_n(x) = \left\{a: a \in A_n, |x-a| \le \frac{2\pi}{n}\right\}$ we have

$$F_{k,n}(x) - f(x) = f_{v,n}(x) - f(x) + \sum_{a\in A_n(x)} \omega_v(f, a, (2v+4\pi)/vn)\psi_n(x-a) \ge$$

(since $\psi_n(x-a) \ne 0$ only for $a \in A_n(x)$)

$$\ge -\omega_v(f, x, 2/\pi) + \min_{a\in A_n(x)} \omega_v(f, a, (2v+4\pi)/vn)$$

$$\ge -\omega_v(f, x, 2/n) + \omega_v(f, x, 2/n) = 0$$

since for every $a \in A_n(x)$.

On the other hand,

$$F_{k,n}(x) - f(x) \le \omega_v(f, x, 2/n) + \max_{a\in A_n(x)} \omega_v(f, a, (2v+4\pi)/vn)$$

$$\le \omega_v(f, x, 2/n) + \omega_v(f, x, (2v+8\pi)/vn) \le 2^{k\delta-k+1}\omega_k(f, x, (2\delta k+8\pi)/kn).$$

Then b) follows from a) taking the $L_{p,\alpha}(A)$ $-$norm of both sides. At the end, using property 4) of ψ_n we have

$$\left|DF_{k,n}(x)\right| \le \left|Df_{v,n}(x)\right| + \sum_{a\in A_n(x)} \omega_v(f, a, (2v+4\pi)/vn\,|D\psi_n(x-a)|$$

$$\le \left|Df_{v,n}(x)\right| + c\omega_v(f, x, (2v+8\pi)/vn) \sum_{a\in A_n(x)} 1.$$

Taking $L_{p,\alpha}(A)$ −norm from the both sides, using the property c) of $f_{v,n}$ and the fact that we have only a finite number of terms on the right hand side, we obtain c), with constant $c = c(k, \delta)$.

Lemma 2.7. Let $f \in L_{p,\alpha}(A)$, $1 \le p < \infty$. Then

$$E_n^+(f)_{L_{p,\alpha}(A)} \le c\tau_1(f, 1/n)_{L_{p,\alpha}(A)}.$$

Proof. Set $x_i = i\pi n^{-1}, i = 0, \dots, 2n$, $y_i = (x_{i-1} + x_i)/2$, $i = 1, \dots, 2n$, $y_{2n+1} = y_1$ and define the 2π −periodic functions S_n and J_n as follows:

$$S_n(x) = \begin{cases} \sup\limits_{t \in [x_{i-1}, x_i]} f(t) \quad for \quad x = y_i, & i = 1, \dots, 2n, \\ \max\{S_n(y_i), S_n(y_{n+1})\} \ \ for \ \ x = x_i, & i = 1, \dots, 2n, \\ S_n(-\pi) = S_n(\pi), & \\ linear\ and\ continuous\ for\ \ x \in [x_{i-1}, y_i] & \\ and\ x \in [y_i, x_i], & i = 1, \dots, 2n\ . \end{cases}$$

$$J_n(x) = \begin{cases} \inf\limits_{t \in [x_{i-1}, x_i]} f(t) \quad for \quad x = y_i, & i = 1, \dots, 2n, \\ \min\{J_n(y_i), J_n(y_{n+1})\} \ \ for \ \ x = x_i, & i = 1, \dots, 2n, \\ J_n(-\pi) = J_n(\pi), & \\ linear\ and\ continuous\ for\ \ x \in [x_{i-1}, y_i] & \\ and\ x \in [y_i, x_i], & i = 1, \dots, 2n. \end{cases}$$

Clearly, we have

$$J_n(x) \le f(x) \le S_n(x), \ \ x \in [-\pi, \pi]. \tag{3}$$

The derivatives $S'_n(x)\ and\ J'_n(x)\ of\ S_n\ and\ J_n$ exist at each point of the interval $[-\pi, \pi]$ except the points x_i, $i = 0, \dots, 2n$, y_i, $i = 1, \dots, 2n$. Moreover, using the definitions of the functions S_n and J_n, we immediately have

$$|S'_n(x)| \le 2n\pi^{-1}\omega_1(f, x; 4\pi n^{-1}), x \ne x_i, y_i, \tag{4}$$

$$|J'_n(x)| \le 2n\pi^{-1}\omega_1(f, x; 4\pi n^{-1}), x \ne x_i, y_i,$$

(e.g. , if $x \in (y_i, x_i)$, then as S_n is linear we have

$|S'_n(x)| \le 2n\pi^{-1}|S_n(y_{i+1}) - S_n(y_i)| \le 2n\pi^{-1}\omega_1(f, x; 4\pi n^{-1})$),

and moreover,

$$0 \le S_n(x) - J_n(x) \le \omega_1(f, x; 2\pi n^{-1}). \tag{5}$$

It follows from (3) that

$$\|S'_n(x)\|_{L_{p,\alpha}(A)} \le 2n\pi^{-1}\tau_1(f; 4\pi n^{-1})_{L_{p,\alpha}(A)}, \tag{6}$$

$$\|J'_n(x)\|_{L_{p,\alpha}(A)} \le 2n\pi^{-1}\tau_1(f; 4\pi n^{-1})_{L_{p,\alpha}(A)}.$$

Moreover, (4) gives

$$\|S_n - J_n\|_{L_{p,\alpha}(A)} \le \tau_1(f; 2\pi n^{-1})_{L_{p,\alpha}(A)}. \tag{7}$$

Using $E_n^+(f)_{L_{p,\alpha}(A)} \le cn^{-1}\|f\|_{L_{p,\alpha}(A)}$we obtain from (5)

$$E_n^+(S_n)_{L_{p,\alpha}(A)} \le c(1)\tau_1(f; 4\pi n^{-1})_{L_{p,\alpha}(A)}; \quad E_n^+(J_n)_{L_{p,\alpha}(A)} \le c(1)\tau_1(f; 4\pi n^{-1})_{L_{p,\alpha}(A)}. \tag{8}$$

The following inequality is obvious:

$$E_n^+(f)_{L_{p,\alpha}(A)} \le E_n^+(S_n)_{L_{p,\alpha}(A)} + \|S_n - J_n\|_{L_{p,\alpha}(A)} + E_n^+(J_n)_{L_{p,\alpha}(A)}. \tag{9}$$

From (7)-(9) we obtain

$$E_n^+(f)_{L_{p,\alpha}(A)} \le 2c(1)\tau_1(f; 4\pi n^{-1})_{L_{p,\alpha}(A)} + \tau_1(f; 2\pi n^{-1})_{L_{p,\alpha}(A)} \le c\tau_1(f; n^{-1})_{L_{p,\alpha}(A)}.$$

Lemma 2.8. [24]**.** Let r be a natural number, $r \geq m$, and $D^{\alpha} f \in L_p$ for $|\alpha| \leq r, 1 \leq p < \infty$. Then there exists a polynomial $S \in \pi_n$ such that

$$\left\| D^{\beta}(f - S) \right\|_{L_p} \leq c(m, r) n^{|\beta| - r} \sum_{|\alpha| = r} \| D^{\alpha} f \|_{L_p}$$

For every multi-index β such that $|\beta| \leq r$.

Remark. Similarly for every two functions $f^+, f^-, f^+ \geq f \geq f^-$ we obtain

Lemma 2.9. Let $f \in L_{p,\alpha}(A), f^+ \in L_{p,\alpha}(A), f^- \in L_{p,\alpha}(A)$ and $f^+ \geq f \geq f^-$. Then

$$\tau_k(f, \delta)_{L_{p,\alpha}(A)} \leq \tau_k(f^+, \delta)_{L_{p,\alpha}(A)} + \tau_k(f^-, \delta)_{L_{p,\alpha}(A)} + 2^{k-1} \tau_1(f^+ - f^-, k\delta)_{L_{p,\alpha}(A)} + 2^{k-1} \| f^+ - f^- \|_{L_{p,\alpha}(A)}.$$

we have:

$$\tau_1(\mathcal{P}_n - \mathcal{Q}_n, k\delta)_{L_{p,\alpha}(A)} \leq 2(k\delta) \| D(\mathcal{P}_n - \mathcal{Q}_n) \|_{L_{p,\alpha}(A)}. \tag{10}$$

Since $\| D(\mathcal{P}_n - \mathcal{Q}_n) \|_{L_{p,\alpha}(A)} \leq n \| \mathcal{P}_n - \mathcal{Q}_n \|_{L_{p,\alpha}(A)}$ ($\mathcal{P}_n, \mathcal{Q}_n \in \pi_n$) we obtain from (5):

$$\tau_1(\mathcal{P}_n - \mathcal{Q}_n, k\delta)_{L_{p,\alpha}(A)} \leq 2(k\delta n) \| \mathcal{P}_n - \mathcal{Q}_n \|_{L_{p,\alpha}(A)} = 2\tilde{E}_n(f)_{L_{p,\alpha}(A)}(k\delta n). \tag{11}$$

From (4) we obtain:

$$\tau_k(f, \delta)_{L_{p,\alpha}(A)} \leq \tau_k(\mathcal{P}_n, \delta)_{L_{p,\alpha}(A)} + \tau_k(Q_n, \delta)_{L_{p,\alpha}(A)} + 2^{k-1} \left(1 + 2(k\delta n)\tilde{E}_n(f)\right)_{L_{p,\alpha}(A)}. \tag{12}$$

Let us estimate $\tau_k(\mathcal{P}_n, \delta)_{L_{p,\alpha}(A)}$ (the estimation of $\tau_k(\mathcal{Q}_n, \delta)_{L_{p,\alpha}(A)}$ follows the same way). Setting $n = 2^{s_0}$ and using property of τ_k, we have:

$$\tau_k(\mathcal{P}_n, \delta)_{L_{p,\alpha}(A)} \leq \sum_{i=0}^{s_0} \tau_k\left(\mathcal{P}_{2^i} - \mathcal{P}_{2^{i-1}}, \delta\right)_{L_{p,\alpha}(A)} + \tau_k(\mathcal{P}_1 - \mathcal{P}_0, \delta)_{L_{p,\alpha}(A)}. \tag{13}$$

Since $g = \mathcal{P}_{2^i} - \mathcal{P}_{2^{i-1}} \in \pi_n$, we obtain:

$$\tau_k(g, \delta)_{L_{p,\alpha}(A)} \leq c(k, \delta)\delta \| Dg \|_{L_{p,\alpha}(A)} \leq c(k, \delta)\delta 2^i \| g \|_{L_{p,\alpha}(A)}. \tag{14}$$

Let $\delta = n^{-1} = 2^{-s_0}$. Then $\delta \leq 2^{-i}$ for $i \leq s_0$ and $\delta 2^i \leq 1$. Therefore from (7) we obtain

$$\tau_k\left(\mathcal{P}_{2^i} - \mathcal{P}_{2^{i-1}}, \delta\right)_{L_{p,\alpha}(A)} \leq c_1(k, \delta)\delta^k 2^{ik} \left\| \mathcal{P}_{2^i} - \mathcal{P}_{2^{i-1}} \right\|_{L_{p,\alpha}(A)} \leq 2c_1(k, \delta)\delta^k 2^{ik} \tilde{E}_{2^{i-1}}(f)_{L_{p,\alpha}(A)}, \tag{15}$$

where the constant $c_1(k, \delta)$ depends only on k and δ.

From (6) and (8) we obtain ($\tilde{E}_{-1} \equiv \tilde{E}_0, \delta = n^{-1}$)

$$\begin{aligned} \tau_k(\mathcal{P}_n, n^{-1})_{L_{p,\alpha}(A)} &\leq \sum_{i=0}^{s_0} 2c_1(k, \delta) n^{-k} 2^{ik} \, \tilde{E}_{2^{i-1}}(f)_{L_{p,\alpha}(A)} \\ &\leq c_2(k, \delta) n^{-k} \sum_{s=0}^{n} (s+1)^{k-1} \tilde{E}_s(f)_{L_{p,\alpha}(A)}, \end{aligned} \tag{16}$$

where the constant $c_2(k, \delta)$ depends only k and m.

Similarly

$$\tau_k(\mathcal{Q}_n, n^{-1})_{L_{p,\alpha}(A)} \leq c_3(k, \delta) n^{-k} \sum_{s=0}^{n} (s+1)^{k-1} \tilde{E}_s(f)_{L_{p,\alpha}(A)}. \tag{17}$$

Inequalities (4), (9) and (10) (for $\delta = n^{-1}$) give Theorem 3.2 in the case when $n = 2^{s_0}$. Transition to arbitrary n is standard.

3. Main results

We shall prove the following Jackson 's type theorem for the best one variable one-sided approximations:

Theorem 3.1. Let $f \in L_{p,\alpha}(A)$ and $1 \leq p < \infty$. For every natural number k there exists a constant $c(k, \delta)$ depending only on k and the one dimension δ, such that

$$\tilde{E}_n(f)_{L_{p,\alpha}(A)} \le c(k,\delta)\tau_k(f,1/n)_{L_{p,\alpha}(A)}$$

Proof

First we shall prove the case $k \ge 1$. Using the function $F_{k,n}$ from Lemma 2.6, we get

$$E_n^+\left(f - F_{k,n} + F_{k,n}\right)_{L_{p,\alpha}(A)} \le \left\|f - F_{k,n}\right\|_{L_{p,\alpha}(A)} + E_n^+\left(F_{k,n}\right)_{L_{p,\alpha}(A)} \le c(k,\delta)\tau_k(f,1/n)_{L_{p,\alpha}(A)} + E_n^+\left(F_{k,n}\right)_{L_{p,\alpha}(A)}.$$

To estimate $E_n^+\left(F_{k,n}\right)_{L_{p,\alpha}(A)}$ we use the polynomial from Lemma 2.8, Lemma 2.7, the estimate of the first averaged modulus by means of mixed derivatives, Lemma 2.8 and Lemma 2.6. We get

$$E_n^+\left(F_{k,n}\right)_{L_{p,\alpha}(A)} = E_n^+\left(F_{k,n} - S\right)_{L_{p,\alpha}(A)} \le c(\delta)\tau_1\left(F_{k,n} - S, 1/n\right)_{L_{p,\alpha}(A)} \le c(\delta)n^{-1}\left\|D\left(F_{k,n} - S\right)\right\|_{L_{p,\alpha}(A)}$$
$$\le c(k,\delta)n^{-k}\left\|DF_{k,n}\right\|_{L_{p,\alpha}(A)} \le c(k,\delta)\tau_k(f,1/n)_{L_{p,\alpha}(A)}.$$

We have obtained that for $k \ge \delta$ we have

$$E_n^+(f)_{L_{p,\alpha}(A)} \le c(k,\delta)\tau_k(f,1/n)_{L_{p,\alpha}(A)}.$$

In the case when $k < \delta$ we use the fact that $\tau_\delta(f,1/n)_{L_{p,\alpha}(A)} \le c(k,\delta)\tau_k(f,1/n)_{L_{p,\alpha}(A)}$ and therefore we have the needed inequality again.

We end the proof with the following:

$$\tilde{E}_n(f)_{L_{p,\alpha}(A)} = E_n^+(f)_{L_{p,\alpha}(A)} + E_n^-(f)_{L_{p,\alpha}(A)} = E_n^+(f)_{L_{p,\alpha}(A)} + E_n^+(-f)_n$$
$$\le c(k,\delta)\tau_k(f,1/n)_{L_{p,\alpha}(A)} + c(k,\delta)\tau_k(-f,1/n)_{L_{p,\alpha}(A)} = 2c(k,\delta)\tau_k(f,1/n)_{L_{p,\alpha}(A)}.$$

Theorem 3.2. Let $f \in L_{p,\alpha}(A)$. For every natural number k there exists a constant $c(k,\delta)$ depending only on k and δ such that

$$\tau_k(f,1/n)_{L_{p,\alpha}(A)} \le c(k,\delta)n^{-k}\sum_{v=0}^{n}(v+1)^{k-1}\tilde{E}_v(f)_{L_{p,\alpha}(A)}.$$

Proof. Let for every natural number n the trigonometrical polynomials $\mathcal{P}_n \in \pi_n, \mathcal{Q}_n \in \pi_n$ be such that

$$\tilde{E}_n(f)_{L_{p,\alpha}(A)} = \|\mathcal{P}_n - \mathcal{Q}_n\|_{L_{p,\alpha}(A)}$$
$$\mathcal{Q}_n(x) \le f(x) \le \mathcal{P}_n(x), \qquad x \in R.$$

Let $x \in A$ be fixed and $y, y + kh \in A$.

If $0 \le \Delta_h^k f(y)$, then

$$0 \le \Delta_h^k f(y) = \sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}f(y+jh) \le \sum_{\substack{j=0 \\ j\equiv k(mod\ 2)}}^{k}\binom{k}{j}\mathcal{P}_n(y+jh) - \sum_{\substack{j=0 \\ j\equiv k-1(mod\ 2)}}^{k}\binom{k}{j}\mathcal{Q}_n(y+jh)$$
$$= \Delta_h^k\mathcal{P}_n(y) - \sum_{\substack{j=0 \\ j\equiv k-1(mod\ 2)}}^{k}\binom{k}{j}\left(\mathcal{Q}_n(y+jh) - \mathcal{P}_n(y+jh)\right)$$
$$= \Delta_h^k\mathcal{P}_n(y) + \sum_{\substack{j=0 \\ j\equiv k-1(mod\ 2)}}^{k}\binom{k}{j}\left[\mathcal{P}_n(y+jh) - \mathcal{Q}_n(y+jh) - \left(\mathcal{P}_n(x) - \mathcal{Q}_n(x)\right)\right]$$
$$+ \sum_{\substack{j=0 \\ j\equiv k-1(mod\ 2)}}^{k}\binom{k}{j}\left(\mathcal{P}_n(x) - \mathcal{Q}_n(x)\right)$$
$$\le \Delta_h^k\mathcal{P}_n(y) + 2^{k-1}\omega_1(\mathcal{P}_n - \mathcal{Q}_n, x, k\delta) + 2^{k-1}\left(\mathcal{P}_n(x) - \mathcal{Q}_n(x)\right),$$

i.e.

$$0 \le \Delta_h^k f(y) \le \Delta_h^k \mathcal{P}_n(y) + 2^{k-1}\omega_1(\mathcal{P}_n - \mathcal{Q}_n, x, k\delta) + 2^{k-1}\big(\mathcal{P}_n(x) - \mathcal{Q}_n(x)\big). \qquad (18)$$

Analogically in the case $\Delta_h^k f(y) \le 0$ we obtain:

$$0 \le -\Delta_h^k f(y) \le \left|\Delta_h^k \mathcal{Q}_n(y)\right| + 2^{k-1}\omega_1(\mathcal{P}_n - \mathcal{Q}_n, x, k\delta) + 2^{k-1}\big(\mathcal{P}_n(x) - \mathcal{Q}_n(x)\big). \qquad (19)$$

From (13) and (14) it follows :

$$\omega_k(f, x, \delta) \le \omega_k(\mathcal{P}_n, x, \delta) + \omega_k(\mathcal{Q}_n, x, \delta) + 2^{k-1}\omega_1(\mathcal{P}_n - \mathcal{Q}_n, x, k\delta) + 2^{k-1}\big(\mathcal{P}_n(x) - \mathcal{Q}_n(x)\big),$$

e. g.

$$\tau_k(f, \delta)_{L_{p,\alpha}(A)} \le \tau_k(\mathcal{P}_n, \delta)_{L_{p,\alpha}(A)} + \tau_k(\mathcal{Q}_n, \delta)_{L_{p,\alpha}(A)} + 2^{k-1}\tau_1(\mathcal{P}_n - \mathcal{Q}_n, k\delta)_{L_{p,\alpha}(A)} + 2^{k-1}\tilde{E}_n(f)_{L_{p,\alpha}(A)}. \quad (20)$$

Theorems 3.1 and 3.2 give us

Corollary 3.3. Let $f \in L_{p,\alpha}(A)$. For $0 < \alpha < k, 1 \le p < \infty$, the following two conditions are equivalent:

i) $\tau_k(f, \delta)_{L_{p,\alpha}(A)} = O(\delta^\alpha)$

ii) $\tilde{E}_n(f)_{L_{p,\alpha}(A)} = O(n^{-\alpha})$.

This Corollary gives characterization of the best one-sided trigonometrical approximations in $L_{p,\alpha}(A), 1 \le p < \infty$, by means of the averaged moduli of smoothness in the one-dimensional case. This characterization is similar to the classical characterization of the best trigonometrical approximations in $L_{p,\alpha}(A)$ by means of the classical integral moduli of smoothness $\omega_k(f, \delta)_{L_{p,\alpha}(A)}$.

4. Conclusions

In this study, we have found the degree of best one-sided trigonometrical approximation of unbounded functions for one variable in weighted $L_{p,\alpha}(A)$ −space in terms averaged modulus of smoothness. The direct theorem and inverse theorem of trigonometric polynomials have been proven, we obtain the equivalence.

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