Mustansiriyah Journal of Pure and Applied Sciences, Print ISSN: 2957-9910. Online ISSN: 2957-9929, Vol. 2, No. 3, 2024, Pages 91-101
DOI:

MJPAS

MUSTANSIRIYAH JOURNAL OF PURE AND APPLIED SCIENCES

Journal homepage:
https://mjpas.uomustansiriyah.edu.iq/index.php/mjpas

RESEARCH ARTICLE - MATHEMATICS

Comparing Bayesian estimation of two-parameters Lomax Reliability using Lindley method Estimation

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Article Info.

Article history:

Received
11 April 2024

Accepted
19 May 2024

Publishing
30 June 2024

Abstract

Bayesian analysis of two-parameters Lomax distribution's reliability has been examined in this search using three prior functions [Gamma, assumption &Inverted Levy] , the estimation was produced under squared errors loss function used idea of Lindley. The simulation study was carried out to assess the performance of the three alternative estimators of reliability function by 6 assumed experiments for different values of distributions parameters with small, moderate, and large sample sizes using mean squared error criteria. The best performance, according to the reported results showed that the best performance was for Inverted Levy prior with squared error loss function.

The official journal published by the College of Education at Mustansiriya University

***Keywords**:* Bayes, Lomax distribution, Lindley approximation, Reliability, Simulation

1. Introduction

The Lomax distribution(1954) [1] , also known as the pareto distribution of second class with two parameters, has gotten a lot of interest from theorists and statisticians because of its application in reliability and lifespan testing research. It's been used in the biological sciences and even for modeling the size distribution of computer files on servers [2]. The study concept necessitates the use of statistical distributions as failure models. Lindley approximation estimation method were utilized to estimate the reliability function for two unknown parameters of Lomax distribution with pdf given by [3]:

$$f(x;\gamma,\beta)=\frac{\gamma}{\beta}(1+\frac{x}{\beta})^{-(\gamma+1)} \qquad x>0 \qquad \gamma,\beta>0 \tag{1}$$

Then the cumulative distribution function for Lomax distribution is given by:

$$F(x;\gamma,\beta)=1-(1+\frac{x}{\beta})^{-\gamma} \tag{2}$$

Where: γ,β are the shape and scale parameters respectively .

and the Reliability function is given by:

$$R(x) = \left(1 + \frac{x}{\beta}\right)^{-\gamma} \tag{3}$$

2. Lindley approximate

Lindley [4] proposed his procedure to approximate the ration of the two integrals such as in equation (3), which is used to obtain the approximate Bayes estimator of reliability is define: $\hat{R}(x) \cong \Phi(\gamma_{ML} + \beta_{ML}) + \frac{1}{2}(A + L_{30}B_{12} + L_{03}B_{21} + L_{21}C_{12} + L_{12}C_{21}) + P_1A_{12} + P_2A_{21}$ (4)

$\Phi(\gamma_{ML} + \beta_{ML})$ the function of maximum likelihood for the parameters γ, β.

$$A = \sum_{i=0}^{n} \sum_{j=0}^{n} w_{ij}\, \tau_{ij}$$

$$A_{ij} = w_i\, \tau_{ii} + w_j \tau_{ji} \tag{5}$$

$$B_{ij} = (\, w_i \tau_{ii} + w_j \tau_{ij}\,)\tau_{ii} \tag{6}$$

$$C_{ij} = 3w_i \tau_{ii} \tau_{ij} + w_j(\, \tau_{ii} \tau_{jj} + 2\tau^2_{ij}\,) \tag{7}$$

$$w_{ij} = \frac{\partial^2 \phi(\gamma,\beta)}{\partial\gamma\partial\beta} \tag{8}$$

$\tau_{ii} = \frac{-I_{jj}}{(I_{ii}I_{jj} - I_{ij}I_{ji})}$, $\tau_{jj} = \frac{-I_{ii}}{(I_{ii}I_{jj} - I_{ij}I_{ji})}$, $\tau_{ij} = \frac{-I_{ij}}{(I_{ii}I_{jj} - I_{ij}I_{ji})}$, and $\tau_{ij} = -I^{-1}$, where:

$$I = \begin{bmatrix} \frac{\partial^2 \ln L}{\partial \gamma^2} & \frac{\partial^2 \ln L}{\partial\gamma\partial\beta} \\ \frac{\partial^2 \ln L}{\partial\gamma\partial\beta} & \frac{\partial^2 \ln L}{\partial \beta^2} \end{bmatrix} \tag{9}$$

Now :

$$L_{ij} = \frac{\partial^{i+j} lnL(x|\gamma,\beta)}{\partial\gamma^i \beta^j} \qquad \text{i,j = 0, 1, 2, 3 and i+j = 3} \tag{10}$$

Then: $L_{30} = \frac{\partial^3 lnL(x|\gamma,\beta)}{\partial\gamma^2}$, $L_{03} = \frac{\partial^3 lnL(x|\gamma,\beta)}{\partial\beta^3}$, $L_{21} = \frac{\partial^3 \ln L(x|\gamma,\beta)}{\partial\gamma^2\partial\beta}$, $L_{12} = \frac{\partial^3 \ln L(x|\gamma,\beta)}{\partial\gamma\partial\beta^2}$

later : $$p_i = \frac{\partial \ln h(\gamma,\beta)}{\partial\gamma\partial\beta} \tag{11}$$

3. Bayes estimator

The joint posterior density function can be written as: if the unknown parameter or reliability is treated as a random variable with a joint function as [5] :

$$Q(\gamma, \beta|x) = \frac{Lf(x;\gamma,\beta)g_{1(\gamma)}.g_2(\beta)}{\iint_0^\infty Lf(x;\gamma,\beta)g(\gamma).g\,(\beta)d\gamma d\beta} \tag{12}$$

As a result, the Bayes estimator for the joint function of $(\gamma,\beta,)$ for a squared error loss function is [6] :

$$E_{\gamma,\beta}h(\gamma,\beta)=\frac{\iint_0^\infty h(\gamma,\beta)Lf(x;\gamma,\beta)g_1(\gamma)g_2(\beta)d\gamma d\beta}{\iint_0^\infty Lf(x;\gamma,\beta)g_1(\gamma)g_2(\beta)d\gamma d\beta} \tag{13}$$

then based on the likelihood function of the observed data by equation (1):

$$Lf(x;\gamma,\beta)=\frac{\gamma^n}{\beta^n}e^{-(\gamma+1)\sum_{i=1}^n \ln(1+\frac{x_i}{\beta})} \tag{14}$$

$$\ln L = n\ln\gamma - n\ln\beta-(\gamma+1)\sum_{i=1}^n \ln(1+\frac{x_i}{\beta}) \tag{15}$$

$$\frac{\partial lnL}{\partial\gamma}=\frac{n}{\gamma}-\sum_{i=1}^n \ln(1+\frac{x_i}{\beta}) \tag{16}$$

$$\hat{\gamma}_{ML}=\frac{n}{\sum_{i=1}^n \ln(1+\frac{x_i}{\beta})} \tag{17}$$

Now : $$\frac{\partial lnL}{\partial\beta}=-\frac{n}{\beta}-\gamma\sum_{i=1}^n \frac{\frac{-x_i}{\beta^2}}{1+\frac{x_i}{\beta}}-\sum_{i=1}^n \frac{\frac{-x_i}{\beta^2}}{1+\frac{x_i}{\beta}} \tag{18}$$

$$\rightarrow \hat{\beta}_{ML}=[(\gamma+1)\sum_{i=1}^n \frac{x_i}{\beta^2(1+\frac{x_i}{\beta})}]/n \tag{19}$$

to obtain the MLE for this parameter , solve the equation above numerically using the Newten – Raphson method, then compensated in equation (17) for $\hat{\gamma}_{ML}$ as [7] : take initial value for β as β_k, then compensated β_k in equation (17) and later taking the derivative for equation (19) , then :

$\beta_{k+1}=\beta_k-\frac{\hat{\beta}_{ML}}{(\hat{\beta}_{ML})^{/}}$, when the absolute difference $|\beta_{k+1}-\beta_k| < \epsilon$ is very small, we stop and use the corrected $\hat{\beta}_{MLE}$ in equation (17) to get the MLE of γ.

now by Lindley idea ,and equations (16), (18):

$$\frac{\partial^2 \ln L}{\partial\gamma^2}=\frac{-n}{\gamma^2} \tag{20}$$

$$\frac{\partial^2 \ln L}{\partial\beta^2}=\frac{n}{\beta^2}+(\gamma+1)\sum_{i=1}^n \frac{-x_i(2\beta+x_i)}{(\beta^2+\beta x_i)^2} \tag{21}$$

$$\frac{\partial lnL}{\partial\gamma\partial\beta}=\sum_{i=1}^n \frac{x_i}{\beta^2(1+\frac{x_i}{\beta})} \tag{22}$$

Then for equations (9)and (20,21,22) :

$$I=\begin{bmatrix} \frac{-n}{\gamma} & \sum_{i=1}^n \frac{x_i}{\beta^2(1+\frac{x_i}{\beta})} \\ \sum_{i=1}^n \frac{x_i}{\beta^2(1+\frac{x_i}{\beta})} & \frac{n}{\beta^2}-(\gamma+1)\sum_{i=1}^n \frac{x_i(2\beta+x_i)}{(\beta^2+\beta x_i)^2} \end{bmatrix}$$

Now for reliability estimation, and by equation (3&8) we get :

$$w_1 = \frac{\partial R(x)}{\partial \gamma} = (1+\frac{x}{\beta})^{-\gamma}[\ln\left(1+\frac{x}{\beta}\right)](-1)$$

$$\rightarrow w_1 = -\left(1+\frac{x}{\beta}\right)^{-\gamma}\left[\ln\left(1+\frac{x}{\beta}\right)\right]$$

$$w_{11} = \frac{\partial^2 R(x)}{\partial \gamma^2} \quad \rightarrow w_{11} = \left[\ln\left(1+\frac{x}{\beta}\right)\right](-1)\left[\left(1+\frac{x}{\beta}\right)^{-\gamma}\right]\ln\left(1+\frac{x}{\beta}\right)(-1) = \left[\ln\left(1+\frac{x}{\beta}\right)\right]^2\left(1+\frac{x}{\beta}\right)^{-\gamma}$$

$$w_2 = \frac{\partial R(x)}{\partial \beta} \rightarrow w_2 = \theta\left(1+\frac{x}{\beta}\right)^{-(\gamma+1)}\left(\frac{x}{\beta^2}\right)$$

$$w_{22} = \frac{\partial^2 R(x)}{\partial \beta^2}$$

$$\rightarrow w_{22} = -\gamma\left(1+\frac{x}{\beta}\right)^{-(\gamma+1)}\left(\frac{2x}{\beta^3}\right) + \left(\frac{x}{\beta^2}\right)\left[\gamma(\gamma+1)\left(1+\frac{x}{\beta}\right)^{-(\gamma+2)}\left(\frac{x}{\beta^2}\right)\right]$$

$$w_{12} = \frac{\partial^2 R(x)}{\partial \gamma \partial \beta}$$

$$\rightarrow w_{12} = \left(1+\frac{x}{\beta}\right)^{-\gamma}\left[\frac{x}{\beta^2\left(1+\frac{x}{\beta}\right)}\right] + \left[\ln\left(1+\frac{x}{\beta}\right)\right]\left[\gamma\left(1+\frac{x}{\beta}\right)^{-(\gamma+1)}\right]\left[\frac{x}{\beta^2}\right]$$

$$w_{21} = \frac{\partial^2 R(x)}{\partial \beta \partial \gamma}$$

$$\rightarrow w_{21} = \left[\frac{x}{\beta^2}\right]\left[-\gamma\left(1+\frac{x}{\beta}\right)^{-(\gamma+1)}\ln\left(1+\frac{x}{\beta}\right) + \left(1+\frac{x}{\beta}\right)^{-(\gamma+1)}\right]$$

Now by equation(10) ,(15) we get :

$$L_{30} = \frac{\partial^3 lnL(x|\gamma,\beta)}{\partial \gamma^3} \rightarrow L_{30} = \frac{2n}{\gamma^3}$$

$$L_{03} = \frac{\partial^3 lnL(x|\gamma,\beta)}{\partial \beta^3}$$

$$\rightarrow L_{03} = \frac{-2n}{\beta^3} - (\gamma+1)\sum_{i=1}^{n}\frac{(\beta^2+\beta x_i)(2x_i)-(2\beta x_i+x_i^2)(4\beta+4x_i)}{(\beta^2+\beta x_i)^3}$$

$L_{21} = \frac{\partial^3 lnL(x|\gamma,\beta)}{\partial \gamma^2 \partial \beta} \rightarrow L_{21} = 0$, and $\quad L_{12} = \frac{\partial^3 lnL(x|\gamma,\beta)}{\partial \gamma \partial \beta^2} \quad \rightarrow L_{21} = \sum_{i=1}^{n}\frac{-x_i(2\beta+x_i)}{(\beta^2+\beta x_i)^2}$

Later and by equations (5),(6),(7), we find:

$A_{12} = w_1\tau_{11} + w_2\tau_{21}$ and $A_{21} = w_2\tau_{22} + w_1\tau_{12}$

$B_{12} = (w_1\tau_{11} + w_2\tau_{12})\tau_{11}$ and $B_{21} = (w_2\tau_{22} + w_1\tau_{21})\tau_{22}$

$C_{12} = 3w_1\tau_{11}\tau_{12} + w_2(\tau_{11}\tau_{22} + 2\tau_{12}^2)$ and $C_{21} = 3w_2\tau_{22}\tau_{21} + w_1(\tau_{22}\tau_{11} + 2\tau_{21}^2)$

3.1 Posterior function under Gamma prior:

The Gamma prior function for (γ,β) defined as :

$$g_1(\gamma)=\gamma^{a-1}e^{-b\gamma} \quad , \quad g_2(\beta)=\beta^{c_1-1}e^{-c_2\beta} \tag{23}$$

Where $(\gamma,\beta,a,b,c_1,c_2>0)$, then the joint function for $(\gamma\&\beta)$ given as:

$$h_1(\gamma,\beta)=\gamma^{a-1}\beta^{c_1-1}e^{-b\gamma}e^{-c_2\beta} \tag{24}$$

$$\ln h_1(\gamma,\beta)=(a-1)\ln\gamma+(c_1-1)\ln\beta-b\gamma-c_2\beta$$

By equation (11); $p_{G_1}=\frac{\alpha-1}{\gamma}-b \qquad p_{G_2}=\frac{c_1-1}{\beta}-c_2$

by equation (14 & 24):

$$L(x/\gamma,\beta)h_1(\gamma,\beta)=\gamma^{a-1}\beta^{c_1-1}e^{-b\gamma}e^{-c_2\beta}\frac{\gamma^n}{\beta^n}e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}$$

$$L(x/\gamma,\beta)h_1(\gamma,\beta)=\frac{\gamma^{a+n-1}}{\beta^{n-c_1+1}}e^{-(b\gamma+c_2\beta)}e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)} \tag{25}$$

Then the joint posterior function say (Q_G) under Gamma prior is;

$$Q_G(\gamma,\beta/x)=\frac{\frac{\gamma^{a+n-1}}{\beta^{n-c_1+1}}e^{-(b\gamma+c_2\beta)}e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}}{\int_0^\infty\int_0^\infty\frac{\gamma^{a+n-1}}{\beta^{n-c_1+1}}e^{-(b\gamma+c_2\beta)}e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}d\gamma d\beta} \tag{26}$$

3.2 Posterior function under assumption prior:

The assumption prior function for (γ,β) which non informative defined as:

$$g(\gamma)=k\gamma^2 \quad , \quad g(\beta)=k\beta^2 \qquad k,\ \beta,\gamma>0 \tag{27}$$

then the joint function for $(\gamma\&\beta)$ given as:

$$h_2(\gamma,\beta)=k^2\gamma^2\beta^2 \tag{28}$$

$$lnh_2(\gamma,\beta)=2lnk+2ln\gamma+2ln\beta$$

By equation (11); $P_1=\frac{2}{\gamma} \quad , \quad P_2=\frac{2}{\beta}$

Now by equation (14 , 28):

$$L(x/\gamma,\beta)h_2(\gamma,\beta)=k^2\gamma^2\beta^2\frac{\gamma^n}{\beta^n}e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}=k^2\frac{\gamma^{n+2}}{\beta^{n-2}}e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}$$

Then the joint posterior unction say (Q_S) under squared error loss function is:

$$Q_S(\gamma,\beta/x)=\frac{k^2\,\frac{\gamma^{n+2}}{\beta^{n-2}}\,e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}}{k^2\int_0^\infty\int_0^\infty\frac{\gamma^{n+2}}{\beta^{n-2}}e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}d\gamma d\beta} \tag{29}$$

3.3 Posterior function under Inverted Levy prior:

The Inverted Levy prior for (γ,β) defined as [8] :

$$g(\gamma)=\sqrt{\frac{\delta}{2\pi}}\gamma^{-\frac{1}{2}}e^{-\frac{\delta\gamma}{2}}\quad ,g(\beta)=\sqrt{\frac{\delta}{2\pi}}\beta^{-\frac{1}{2}}e^{-\frac{\delta\beta}{2}}\qquad \delta,\gamma,\beta>0 \tag{30}$$

then the joint function for $(\gamma\&\beta)$ given as:

$$h_3(\gamma,\beta)=(\sqrt{\frac{\delta}{2\pi}})^2\gamma^{-\frac{1}{2}}e^{-\frac{\delta\gamma}{2}}\beta^{-\frac{1}{2}}e^{-\frac{\delta\beta}{2}}$$

$$h_3(\gamma,\beta)=\frac{\delta}{2\pi}\gamma^{-\frac{1}{2}}\beta^{-\frac{1}{2}}e^{-\frac{1}{2}\delta(\gamma+\beta)} \tag{31}$$

$lnh_3(\gamma,\beta)=\ln\frac{\delta}{2\pi}-\frac{1}{2}ln\gamma-\frac{1}{2}ln\beta-\frac{1}{2}\delta(\gamma+\beta)$ By equations (14,31) :

$$L(x/\gamma,\beta)h_3(\gamma,\beta)=\frac{\delta}{2\pi}\gamma^{-\frac{1}{2}}\beta^{-\frac{1}{2}}e^{-\frac{1}{2}\delta(\gamma+\beta)}\frac{\gamma^n}{\beta^n}e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}$$

$$=\frac{\delta}{2\pi}\frac{\gamma^{n-\frac{1}{2}}}{\beta^{n+\frac{1}{2}}}\;e^{-\frac{1}{2}\delta(\gamma+\beta)}e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}$$

Then the joint posterior function say (Q_L) under squared error loss function is:

$$Q_L(\gamma,\beta/x)=\frac{\frac{\delta}{2\pi}\frac{\gamma^{n-\frac{1}{2}}}{\beta^{n+\frac{1}{2}}}e^{-\frac{1}{2}\delta(\gamma+\beta)}e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}}{\frac{\delta}{2\pi}\int_0^\infty\int_0^\infty\frac{\gamma^{n-\frac{1}{2}}}{\beta^{n+\frac{1}{2}}}e^{-\frac{1}{2}\delta(\gamma+\beta)}e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}d\gamma d\beta} \tag{32}$$

A loss function $[L(\hat{R},R)]$ represent losses incurred when estimate the reliability (R) by $(\hat{R})$ [9] [10] .Here we used squared error loss function which is from symmetric type given by $L(\hat{R},R)=(\hat{R}-R)^2$, and the approximate Bayes reliability estimator is :

$\hat{R}_S=E(R|\underline{x})$ which is define as:

a. using Gamma prior

by equations (3, 26) is :

$$\hat{R}_G(x)=\frac{\int_0^\infty\int_0^\infty\left(1+\frac{x_i}{\beta}\right)^{-\gamma}\frac{\gamma^{a+n-1}}{\beta^{n-c_1+1}}e^{-(b\gamma+c_2\beta)}e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}d\gamma d\beta}{\int_0^\infty\int_0^\infty\frac{\gamma^{a+n-1}}{\beta^{n-c_1+1}}e^{-(b\gamma+c_2\beta)}e^{-(\gamma+1)\sum_{i=1}^{n}\ln\left(1+\frac{x_i}{\beta}\right)}d\gamma d\beta} \tag{33}$$

Where, $\hat{R}_G(x)$ is approximate Bayes reliability estimator under Gamma prior and squared error loss function.

b. using assumption prior

by equations(3,29) is :

$$\hat{R}_S(x) = \frac{\int_0^\infty \int_0^\infty \left(1+\frac{x_i}{\beta}\right)^{-\gamma} \frac{\gamma^{n+2}}{\beta^{n-2}} e^{-(\gamma+1)\sum_{i=1}^{n} \ln\left(1+\frac{x_i}{\beta}\right)} d\gamma d\beta}{\int_0^\infty \int_0^\infty \frac{\gamma^{n+2}}{\beta^{n-2}} e^{-(\gamma+1)\sum_{i=1}^{n} \ln\left(1+\frac{x_i}{\beta}\right)} d\gamma d\beta} \tag{34}$$

Where $\hat{R}_S(x)$ is approximate Bayes reliability estimator under assumption prior and squared loss function.

c. using Inverted Levy prior

by equations (3,32):

$$\hat{R}_L(x) = \frac{\int_0^\infty \int_0^\infty (1+\frac{x_i}{\beta})^{-\gamma} \frac{\delta}{2\pi} \frac{\gamma^{n-\frac{1}{2}}}{\beta^{n+\frac{1}{2}}} e^{-\frac{1}{2}\delta(\gamma+\beta)} e^{-(\gamma+1)\sum_{i=1}^{n} \ln\left(1+\frac{x_i}{\beta}\right)^{-\gamma}} d\gamma d\beta}{\int_0^\infty \int_0^\infty \frac{\delta}{2\pi} \frac{\gamma^{n-\frac{1}{2}}}{\beta^{n+\frac{1}{2}}} e^{-\frac{1}{2}\delta(\gamma+\beta)} e^{-(\gamma+1)\sum_{i=1}^{n} \ln\left(1+\frac{x_i}{\beta}\right)^{-\gamma}} d\gamma d\beta} \tag{35}$$

Where $\hat{R}_L(x)$ is approximate Bayes reliability estimator under Inverted Levy prior and squared loss function.

4.Simulation

In this section, we utilized a simulation research [11]to compare the reliability estimators that were derived previously. The study produced data that was distributed uniformly and converted it into a two-parameter Lomax distribution ,by generated values for r.v.'s of size (n=15,25,50,100) for two parameters Lomax distribution with default values of parameters as(γ =0.5, 0.7, 0.9) (β=0.4, 0.6, 1), in times ($t_i = 1.0,1.2,1.4,1.6,1.8,2.0$), where the values of the prior function parameters are:$a = 2, b = 1.5, c_1 = 2.3, c_2 = 1.5$, $\delta = 2.5$, and take the number of repeat sample (L =1000), and generated data from eq.(2) as:

Let $[u = F(x)]$, where (u) is a random variable constant on interval (0,1).

$$\Rightarrow 1 - u = \left[1 + \frac{x}{\beta}\right]^{-\gamma}, \ (1-u)^{-\frac{1}{\gamma}} = \left[1 + \frac{x}{\beta}\right], \text{ and } \ (1-u)^{-\frac{1}{\gamma}} - 1 = \frac{x}{\beta}$$

Then: $x = \beta\left[(1-u)^{-\frac{1}{\gamma}} - 1\right]$ (36)

The Monte – Carlo simulation study used to compare between the Bayes estimators by MSE criteria (MATLAB 2020) where: $MSE(\hat{R}) = \frac{1}{L}\sum_{i=1}^{L}(\hat{R}_i - R)^2$.

Tables (1-6) provide the results of simulations as:

Table I. MSE values of Bayesian est. Reliability(exp.1&2)

(Exp. 1). γ= 0.5 β=0.4					
n	t_i	**RG**	**RS**	**RL**	**best**
15	1.0	0.0121	0.0723	0.0301	RG
	1.2	0.0101	0.0712	0.0298	RG
	1.4	0.0112	0.0662	0.0245	RG
	1.6	0.0125	0.0590	0.0221	RG
	1.8	0.0130	0.0523	0.0218	RG
	2.0	0.0233	0.0200	0.0210	RS
25	1.0	0.0072	0.0098	0.0070	RL
	1.2	0.0073	0.0101	0.0070	RL
	1.4	0.0072	0.0070	0.0076	RS
	1.6	0.0075	0.0071	0.0077	RS
	1.8	0.0081	0.0088	0.0076	RL
	2.0	0.0080	0.0086	0.0074	RL
50	1.0	0.0031	0.0033	0.0030	RL
	1.2	0.0032	0.0035	0.0030	RL
	1.4	0.0032	0.0035	0.0031	RL
	1.6	0.0031	0.0036	0.0031	RG,RL
	1.8	0.0031	0.0036	0.0030	RL,
	2.0	0.0030	0.0035	0.0030	RG,RL
100	1.0	0.0019	0.0022	0.0017	RL
	1.2	0.0021	0.0021	0.0018	RL
	1.4	0.0021	0.0022	0.0021	RG,RL
	1.6	0.0021	0.0022	0.0021	RG,RL
	1.8	0.0021	0.0021	0.0022	RG,RS
	2.0	0.0021	0.0021	0.0021	ALL
(Exp.2) γ= 0.5 β=0.6					
n	t_i	**RG**	**RS**	**RL**	**best**
15	1.0	0.0080	0.0543	0.0238	RG
	1.2	0.0087	0.0509	0.0229	RG
	1.4	0.0090	0.0488	0.0222	RG
	1.6	0.0100	0.0482	0.0220	RG
	1.8	0.0110	0.0443	0.0113	RG
	2.0	0.0117	0.0429	0.0115	RL
25	1.0	0.0072	0.0085	0.0062	RL
	1.2	0.0077	0.0088	0.0070	RL
	1.4	0.0079	0.0090	0.0073	RL
	1.6	0.0081	0.0093	0.0074	RL
	1.8	0.0081	0.0098	0.0079	RL
	2.0	0.0082	0.0099	0.0080	RL
50	1.0	0.0033	0.0036	0.0031	RL
	1.2	0.0036	0.0041	0.0033	RL
	1.4	0.0039	0.0042	0.0037	RL
	1.6	0.0042	0.0046	0.0042	RG,RL
	1.8	0.0042	0.0048	0.0041	RL,
	2.0	0.0044	0.0049	0.0043	RL
100	1.0	0.0016	0.0016	0.0016	ALL
	1.2	0.0018	0.0018	0.0018	ALL
	1.4	0.0020	0.0022	0.0018	RL
	1.6	0.0021	0.0023	0.0021	RG,RL
	1.8	0.0023	0.0024	0.0022	RL
	2.0	0.0023	0.0026	0.0022	RL

Table 2 MSE values of Bayesian est. Reliability (exp.3&4).

(Exp.3) γ=0.5 β=1					
n	t_i	RG	RS	RL	best
15	1.0	0.0048	0.0280	0.0151	RG
	1.2	0.0057	0.0282	0.0156	RG
	1.4	0.0066	0.0285	0.0163	RG
	1.6	0.0077	0.0292	0.0168	RG
	1.8	0.0088	0.0294	0.0177	RG
	2.0	0.0095	0.0299	0.0182	RG
25	1.0	0.0060	0.0050	0.0049	RL
	1.2	0.0064	0.0059	0.0055	RL
	1.4	0.0071	0.0066	0.0063	RL
	1.6	0.0076	0.0072	0.0071	RL
	1.8	0.0080	0.0076	0.0080	RG,RL
	2.0	0.0082	0.0085	0.0081	RL
50	1.0	0.0032	0.0030	0.0030	RS,RL
	1.2	0.0035	0.0033	0.0032	RL
	1.4	0.0036	0.0034	0.0032	RL
	1.6	0.0038	0.0035	0.0033	RL
	1.8	0.0040	0.0041	0.0039	RL,
	2.0	0.0040	0.0042	0.0040	RG,RL
100	1.0	0.0011	0.0011	0.0011	ALL
	1.2	0.0014	0.0014	0.0013	RL
	1.4	0.0016	0.0016	0.0016	ALL
	1.6	0.0017	0.0017	0.0017	ALL
	1.8	0.0018	0.0019	0.0018	RG,RL
	2.0	0.0020	0.0021	0.0019	RL
(Exp.4) γ= 0.7 β=0.4					
n	t_i	RG	RS	RL	best
15	1.0	0.0162	0.1132	0.0539	RG
	1.2	0.0144	0.0968	0.0517	RG
	1.4	0.0150	0.0856	0.0452	RG
	1.6	0.0161	0.0818	0.0429	RG
	1.8	0.0180	0.0780	0.0392	RG
	2.0	0.0100	0.0762	0.0320	RG,RL
25	1.0	0.0082	0.0109	0.0082	RG,RL
	1.2	0.0080	0.0109	0.0077	RL
	1.4	0.0078	0.0106	0.0075	RL
	1.6	0.0072	0.0103	0.0076	RG
	1.8	0.0069	0.0101	0.0070	RG
	2.0	0.0064	0.0100	0.0060	RL
50	1.0	0.0026	0.0042	0.0025	RL
	1.2	0.0025	0.0040	0.0024	RL
	1.4	0.0024	0.0038	0.0023	RL
	1.6	0.0023	0.0037	0.0022	RL
	1.8	0.0021	0.0036	0.0021	RG,RL
	2.0	0.0020	0.0034	0.0020	RG,RL
100	1.0	0.0019	0.0020	0.0017	RL
	1.2	0.0017	0.0019	0.0017	RG,RL
	1.4	0.0015	0.0019	0.0015	RG,RL
	1.6	0.0013	0.0019	0.0013	RG,RL
	1.8	0.0013	0.0016	0.0013	RG,RL
	2.0	0.0011	0.0014	0.0010	RL

Table 3 MSE values of Bayesian est. Reliability (exp.5&6).

n	t_i	RG	RS	RL	best
		(Exp.5) γ= 0.7 β=0.6			
15	1.0	0.0321	0.1343	0.0518	RG
	1.2	0.0344	0.1194	0.0405	RG
	1.4	0.0377	0.1011	0.0467	RG
	1.6	0.0404	0.0982	0.0470	RG
	1.8	0.0428	0.0973	0.0504	RG
	2.0	0.0453	0.0941	0.0639	RG
25	1.0	0.0083	0.0098	0.0081	RL
	1.2	0.0084	0.0101	0.0083	RL
	1.4	0.0084	0.0103	0.0084	RG,RL
	1.6	0.0083	0.0103	0.0084	RG
	1.8	0.0084	0.0103	0.0083	RL
	2.0	0.0083	0.0102	0.0080	RL
50	1.0	0.0026	0.0032	0.0027	RG,RL
	1.2	0.0027	0.0033	0.0027	RG,RL
	1.4	0.0027	0.0034	0.0028	RG
	1.6	0.0030	0.0034	0.0036	RG
	1.8	0.0033	0.0034	0.0032	RL
	2.0	0.0034	0.0036	0.0035	RG
100	1.0	0.0019	0.0020	0.0019	RG,RL
	1.2	0.0020	0.0021	0.0019	ALL
	1.4	0.0020	0.0019	0.0019	ALL
	1.6	0.0020	0.0021	0.0020	RG,RL
	1.8	0.0020	0.0022	0.0020	RG,RL
	2.0	0.0019	0.0022	0.0018	RL
		(Exp.6) γ= 0.7 β= 1			
n	t_i	RG	RS	RL	best
15	1.0	0.0478	0.1161	0.0904	RG
	1.2	0.0482	0.1020	0.0851	RG
	1.4	0.0485	0.0981	0.0834	RG
	1.6	0.0472	0.0976	0.0798	RG
	1.8	0.0590	0.0969	0.0777	RG
	2.0	0.0613	0.0745	0.0850	RS
25	1.0	0.0070	0.0078	0.0067	RL
	1.2	0.0072	0.0087	0.0072	RG,RL
	1.4	0.0078	0.0094	0.0079	RG
	1.6	0.0082	0.0099	0.0083	RG
	1.8	0.0083	0.0111	0.0080	RL
	2.0	0.0083	0.0105	0.0081	RL
50	1.0	0.0033	0.0032	0.0029	RL
	1.2	0.0034	0.0037	0.0034	RG,RL
	1.4	0.0035	0.0037	0.0035	RG,RL
	1.6	0.0038	0.0040	0.0038	RG,RL
	1.8	0.0039	0.0042	0.0039	RG,RL
	2.0	0.0039	0.0042	0.0040	RG
100	1.0	0.0012	0.0012	0.0012	ALL
	1.2	0.0016	0.0016	0.0016	ALL
	1.4	0.0018	0.0018	0.0018	ALL
	1.6	0.0019	0.0020	0.0019	RG,RL
	1.8	0.0020	0.0020	0.0020	ALL
	2.0	0.0020	0.0020	0.0020	ALL

5. Conclusion

From the tables 1-3 above; in the case of small sample size (n=15), it found that **RG** the beast performance among the three estimators in all experiments, while both **RL** and **RG** indexed as best in most cases in the moderate sample size (n=25,50), in big sample sizes (n= 100) all priors as best in some cases ,but **RS** did not give the best performance mostly. In general, **RL** was the best estimator for reliability function among the three Bayesian estimators.

Acknowledgement

The author would like to thank Mustansiriyah University (www.uo Mustansiriyah.edu.iq) Baghdad- Iraq for its support in the present work.

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