Bloch sphere for four-qubit

كرة بلوخ لاربع بتات كمومية

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Abstract
In this paper, we will address the drawing and representation of four electrons in the Bloch sphere and indicate the state of the superposition between them. Each one electron here can consider as a single qubit. By using a program in Matlab to finding the superposition for four electrons which it will be sixteen states, based on the rule 2^n . For example, the state three is describe as the form$|0010\rangle$ and in a matrix as $|0010\rangle = [0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]^T$. After that, we using another program to read and drawing all states. Which appears here as a four sphere forms and each one from it represent the Bloch sphere for one electron with described the case of qubit if it was $|0\rangle$ or$|1\rangle$. Inside the sphere, the state $|0\rangle$ will be as an arrow red heading for up, and the state$|1\rangle$ will be as an arrow green heading for down. Denoted all state from these states by using the Bloch sphere, and all state can be drawing on Bloch sphere at the same time.
***Keywords**: Qubit states, the Bloch sphere, the superposition for four-qubit.*

الملخص
في ذا البحث سوف نتطرق الى رسم وتمثيل أربعة الكترونات في كرة بلوخ ونشير الى حالة التراكب بينهم. حيث يمثل كل الكترون هنا كأنه بت كمومي مفرد . نستخدم برنامج في الماتلاب لايجاد حالة التراكب. للالكترونات الأربعة والتي ستكون ستة عشر حالة معتمدة على القاعدة 2^n على سبيل المثال يمكن وصف الحالة $|0010\rangle$ كمصفوفة على النحوالتالي :
$|0010\rangle = [0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0]^T$
بعد ذلك نستخدم برنامج اخر لقراءة ورسم الحالات . وستظهر الحالة هنا على شكل اربع كرات وكل كرة منها تمثل كرة بلوخ لاكترون واحد وتصف حالة البت الكمومي فيما اذا كانت $|0\rangle$ او $|1\rangle$. في داخل الكرة ستكون الحالة $|0\rangle$ كسهم احمر اللون يشير الى الأعلى , والحالة $|1\rangle$ كسهم اخضر اللون يشير الى الأسفل . يُشار إلى كل حالة من هذه الحالات باستخدام كرة بلوخ ، ويمكن أن ترسم كل ولاية على كرة بلوخ في نفس الوقت .

1- Introduction

Quantum information is information in the condition of a quantum framework; it is the essential element of concentrate in quantum data hypothesis. Quantum information is treated using advanced computers, which controlled it, transfer from unit to another, separate and analyze them. Special algorithms suitable for these computers, known as quantum algorithms, do this. The basic unit building this information called qubit. This differs from bit (which is the basic unit of classical computers). [1]

There are quantum systems that have two levels of the state and show the quantum characteristics such as the electron spin (top or down), the polarization of the photon (vertical or horizontal), the atoms or ions that have two levels energy, and for all systems where this feature is available. A qubit expression used to represent these systems. While in classical system bit used to express systems that can be in one of the two states denoted as 0 and 1. [2]

The principle of superposition is a distinguishing characteristic of quantum mechanics. This characteristic allows qubit to be in a coherent superposition of both states, as happens in the Waves behavior in classical physics. Thus, two quantum states can added together and form another valid quantum state. Schrödinger equation is linear, so any linear set of solutions will also be a solution for it. Thus, any quantum state can represented as a sum of two or more distinct states mathematically. However, in case of measurement the system will fall and measure only one state .Worse than that, measurement here will be random, and this is one of the biggest challenges in the design of quantum algorithms. [3] The Dirac or bra-ket notation, which introduced by “English physicist Paul Dirac” in the context of quantum mechanics, used for the expression of the qubit. Quantum superposition of basic state$|0\rangle$ and $|1\rangle$ (logical qubit) considered as a unit basic used in quantum information processing. The notation $|0\rangle$ and $|1\rangle$ indicate to the quantum states, which their measurement conformity to the classical logic results 0 and 1 respectively. The qubit is different from the bit because of the superposition condition, where cannot be obtained 0 or 1 as fully. Even when multiple measurements made on qubits will not constantly give the same result. [4]

In this research, the researcher tries to build a control system that acts as a miniature model paving the way for larger models that can be built in the near future. Thus, they can provide systems that can solve problems and dilemmas. Starting from the superposition property for four-electrons, representing them in a Bloch sphere, and trying to create a small and simplified control room to deal with superposition states, and examine the possibility of manipulation of these states.

2- Realted work

In 2001, Daniel Kuan Li Oi used the geometrical form to describe quantum physical evolution on a Single-Qubit. [5] In 2004, Havel et al. they used algebra geometric to institute Bloch sphere model for a single qubit. [6] In 2009, Aron Pasieka, et al. They used an algorithm with a mathematical description to describe the single-channel qubit mathematical, intended to introduce the Bloch sphere interpretation of channels and qubit quantity and single processes. [7] In 2010, Baseman, et al. for N-qubit system, they used the “Majorana representation” to represent N-qubit cases as points on the Bloch sphere. In addition, work a comparison between geometric impersonation of N-qubit cases and the alternative representation suggested newly from current authors. [3]

3- Standard representation

In quantum mechanics and by using the computational basis we can represent the qubit by a “linear superposition” as a two-orthonormal vector$\{|0\rangle, |1\rangle\}$in “Hilbert space”. These vectors usually symbolized as: [8]

$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$, and $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

New base states can create by combining the qubit basis states by the following relationship:

$|\mathrm{A}\rangle \otimes |\mathrm{B}\rangle \otimes \cdots |n\rangle$.

As example, if we have two qubits we can represented in Hilbert space as a four-dimensional linear vector by the following product basis states:

$$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, |01\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, |10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, |11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

When we have n qubits, the superposition for these denoted by the tensor product of its counterpart qubits which represent by a linear combination of n basis vectors , each of length 2^n. [9]

4- Qubit states

A pure qubit state can define as a coherent superposition of the basis states and by a linear combination of $|0\rangle$ and $|1\rangle$, we can be describing a single qubit as:

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Where α and β both be complex numbers which called amplitudes of state. The symbol $|\alpha|^2$ refers to the probability of consequence $|0\rangle$, and the symbol $|\beta|^2$ refers to the probability of consequence $|1\rangle$. The absolute squares of the amplitudes equal to the probabilities, for that α and β must restricted by this equation: [10]

$|\alpha|^2 + |\beta|^2 = 1$

The probability $|\alpha|^2$ represent the measured for qubit state $|0\rangle$, while the $|\beta|^2$ represent the measured for qubit state $|1\rangle$. In other words, there is no way to know which of the two possible states for the formation of the superposition state. In addition, the amplitudes **α** and **β**, encodes more than the eventuality of the results of measurement. As we seen in the experiment of two-slit, the eventuality for the amplitudes **α** and **β** indicating that they represent responsible proportions for quantum interference. [3]

5- Representation the qubit in Bloch sphere

From pure qubit equation, we observe that should be four degrees of freedom (four real numbers).

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \cdots (1)$$

Two of them reduced to one-degree, the complex numbers **α**, and **β** by the normalization constraint. $|\alpha|^2 + |\beta|^2 = 1$.

Also with using an appropriate change of coordinates, one of the degrees of freedom will reduced, and the appropriate coordinates here are "Hopf coordinates": [11]

$$\alpha = e^{i\psi} cos\frac{\theta}{2} \ , \ and \ \ \beta = e^{i(\psi+\phi)} sin\frac{\theta}{2} \ .$$

Moreover, the factor $e^{i\psi}$ has no physically observable consequences on the representation of the single qubit. In case we opt **α** to be real, we will get only two degrees of freedom:

$$\alpha = cos\frac{\theta}{2} \ , \ and \ \ \beta = e^{i\phi} sin\frac{\theta}{2} \ .$$

In the same way, we can made **β** a real in the case **α** is zero. [12]

From this, we find that the best form to represent several vectors and possess two degrees of freedom is in the form of a sphere. This called "Bloch sphere", show the figure 1. Where each point on the surface of the sphere have two dimensions, and it can represent the state of quantum, (qubit) and thus we get **n** of the states on the surface of the sphere. In other words, any point on the surface of this sphere represents a single qubit. This sphere have two poles, North Pole indicate to the state $|0\rangle$ and the South Pole indicate to the state $|1\rangle$ .because of the classical bit has only two states, either 0 or 1, for that the measurement of the quantum state (qubit)can be equal with the classical bit only at the North Pole or at the South Pole. [13]

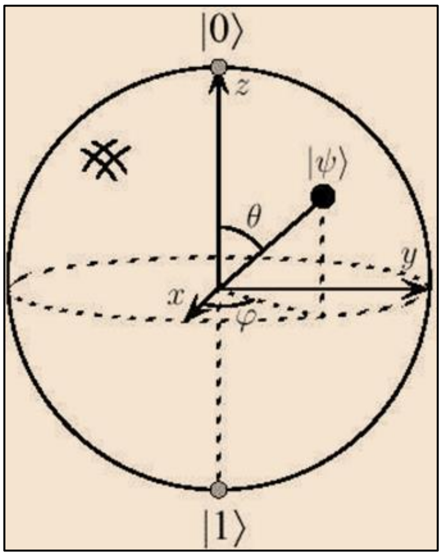

Figure 1 Bloch sphere in spherical coordinate [10]

From that, we can represent a single qubit as a point on the surface of the Bloch sphere by this equation:

$$|\psi\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\phi} sin\frac{\theta}{2}|1\rangle \quad \cdots (2)$$

Which show this state has two degrees of freedom, represented by two-angle φ and θ. [12]

6- The superposition for four-qubit

In Hilbert space for the four-qubit is the tensor product of the one-qubit Hilbert spaces $\varepsilon_1 \otimes \varepsilon_2 \otimes \varepsilon_3 \otimes \varepsilon_4$ with a direct product basis.

$$|\psi\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle) \otimes (\alpha_3|0\rangle + \beta_3|1\rangle) \otimes (\alpha_4|0\rangle + \beta_4|1\rangle) \cdots (3)$$

$where\ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{C}$, and

$$(|\alpha_1|^2 + |\beta_1|^2)(|\alpha_2|^2 + |\beta_2|^2)(|\alpha_3|^2 + |\beta_3|^2)(|\alpha_4|^2 + |\beta_4|^2) = 1 \cdots (4)$$

In matrixes, we can represent the superposition as [9]

$$\psi = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \ldots (5)$$

Each column of this matrix represents a state and is describe as follows:

$|0000\rangle = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T \cdots (6)$

Which are the state one ($|0000\rangle$) and the state six ($|0101\rangle$) is describe as follows:

$|0101\rangle = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T \cdots (7)$

7- Results and discussion

Designing a program in Matlab to show these states in the form of a single matrix as in equation (5), and another program to drawing all the states we obtained which in this case is sixteen states. The drawing as shown in Figure (2), each sphere shows the state in which the electron can be either $|0\rangle$ or $|1\rangle$. The red arrow pointing towards the top represents the state $|1\rangle$ (North Pole). The green arrow pointing towards the bottom represents the state $|1\rangle$ (South Pole). The figures (2, 3, 4, and 5) show all result that we get it from for this program.

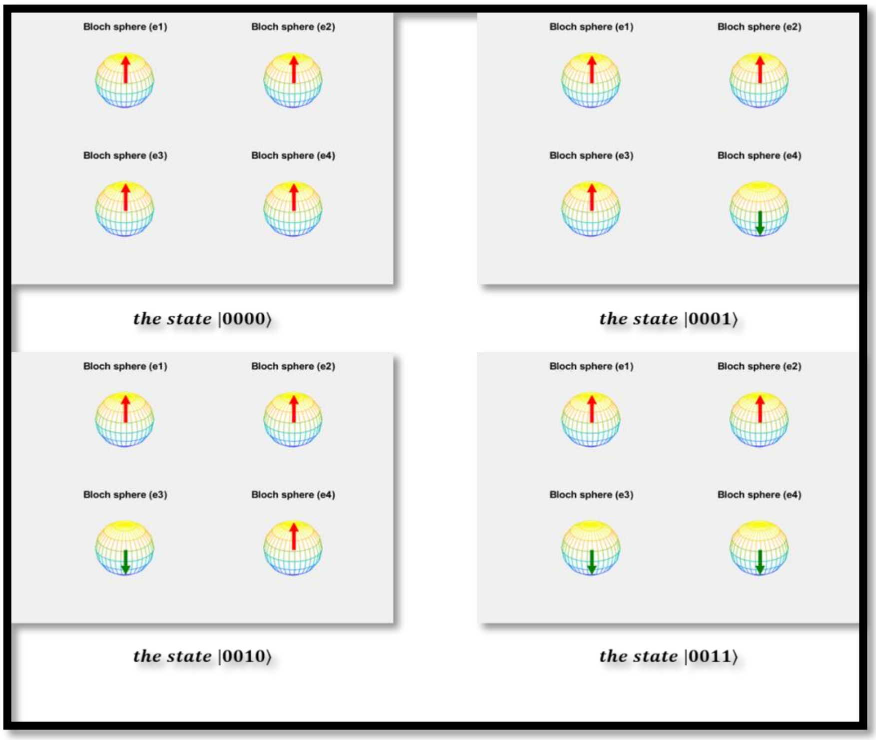

Figure 2 represent four states: $|0000\rangle, |0001\rangle, |0010\rangle,$ ***and*** $|0011\rangle$

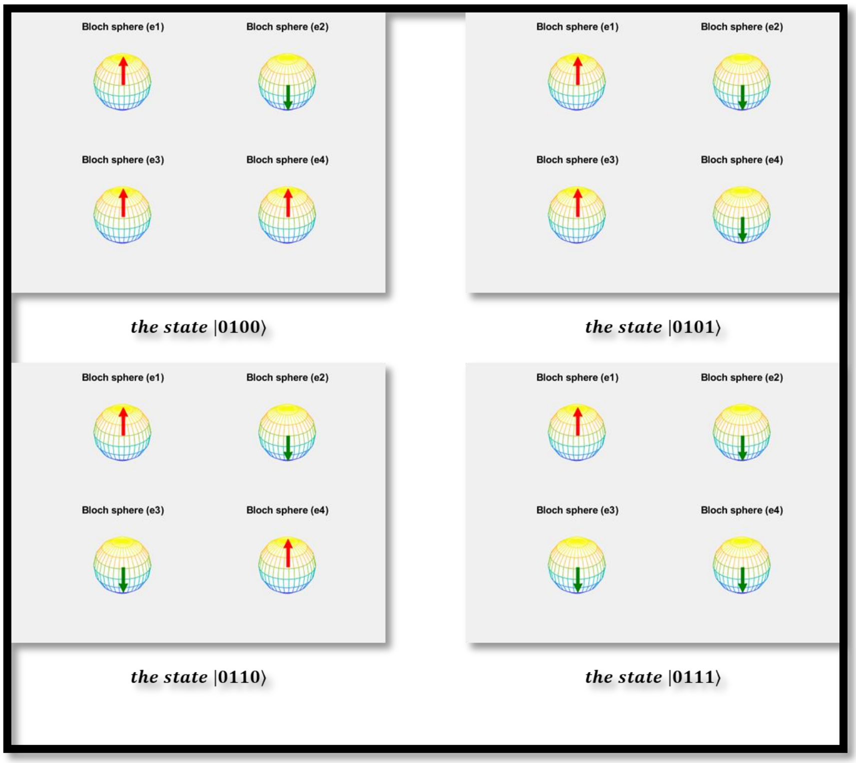

Figure 3 represent four states: $|0100\rangle, |0101\rangle, |0110\rangle,$ ***and*** $|0111\rangle$

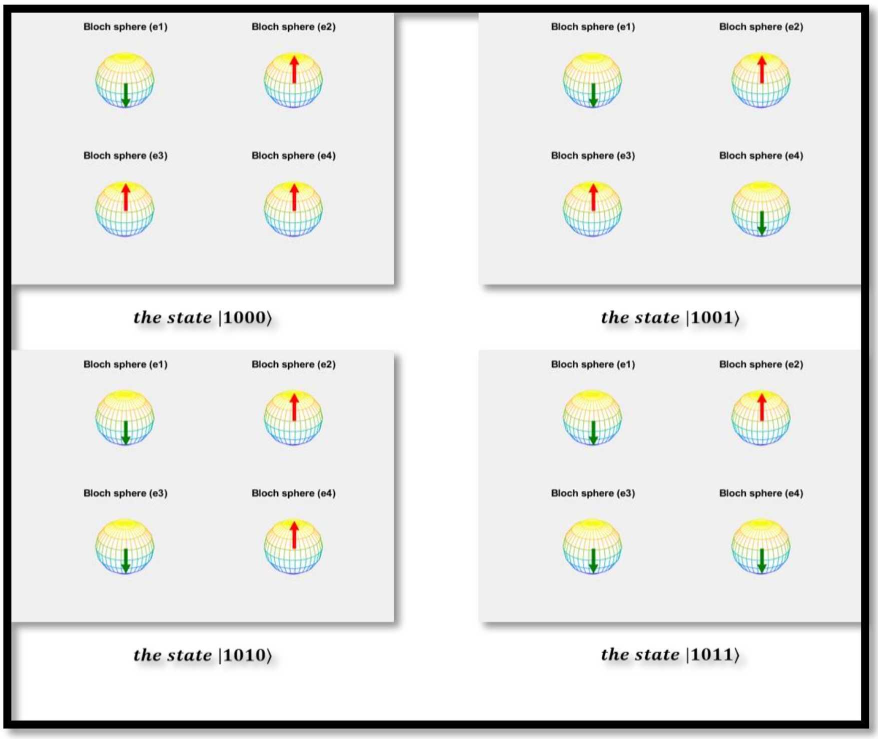

Figure 4 represent four states: $|1000\rangle, |1001\rangle, |1010\rangle,$ ***and*** $|1011\rangle$

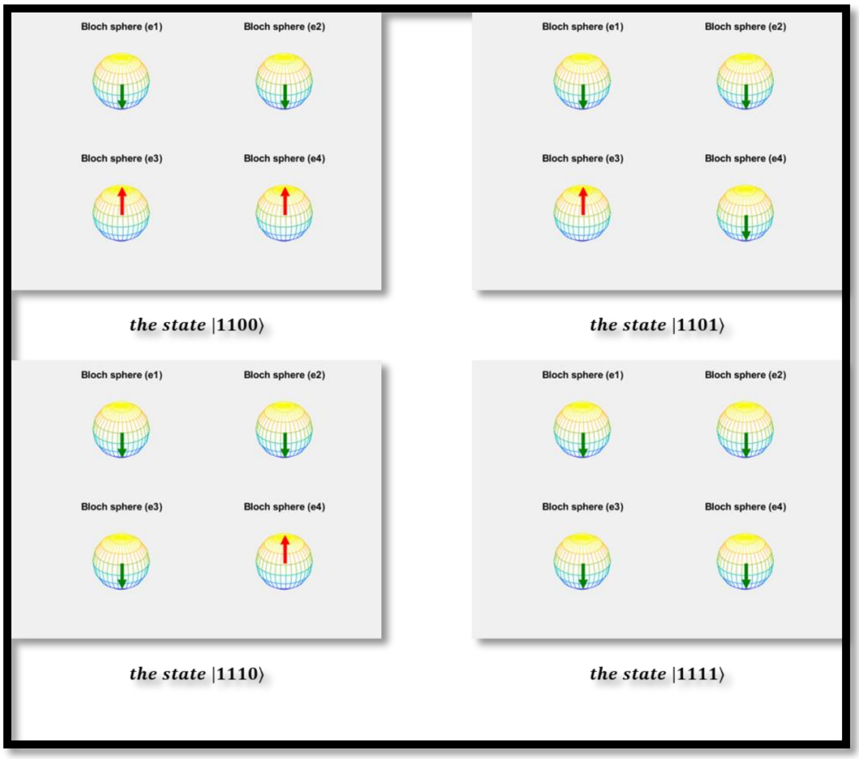

Figure 5 represent four states: $|1100\rangle, |1101\rangle, |1110\rangle$, ***and*** $|1111\rangle$

8- *Parts excerpted from programs*

The figure 4 show Parts excerpted from program that we used.

```
surf(x,y,z,'Parent',subplotk,'FaceLighting','gouraud',
...
    'EdgeLighting','flat',...
    'LineWidth',0.1,...
    'FaceColor',[1 1 1],...
    'EdgeColor','flat');
grid(subplotk,'on');
set(subplotk,'DataAspectRatio',[1 1
1],'PlotBoxAspectRatio',...
    [434 342.3 342.3]);
Figure1('arrow',[0.74 0.74],...
    [0.27 0.37],'Color',[1 0 0],...
    'LineWidth',4,...
    'HeadStyle','cback1')
title('Bloch sphere (e1)');
subplotk = subplot(2,2,k,'Parent',figure1);
axis off
hold(subplotk,'on');
surf(x,y,z,'Parent',subplotk,'FaceLighting','gouraud',
...
    'EdgeLighting','flat',...
    'LineWidth',0.1,...
    'FaceColor',[1 1 1],...
    'EdgeColor','flat');
disp('no annotation');
end
```

Figure 3 Parts excerpted from program.

9- Conclusion

This work is useful in describing the state $|0\rangle$ and $|1\rangle$ as a drawing in Bloch sphere. Then finding the superposition for four electrons and drawing all the states for the superposition in Bloch sphere. Thus, we can show the outputs as drawings when using the quantum gates or quantum circuit. In addition, that show the electron that has been allocated with the statement of change in the state of electron after the applied the quantum gates or quantum circuit on it .so, that will gives a good description for changing in any state by the gates and quantum circuits. This can be used for a quantum information processes.

References

1. Kibler, M.R., *Quantum information and quantum computing: an overview and some mathematical aspects.* arXiv preprint arXiv:1811.08499, 2018.
2. Gubernatis, James E., et al. "Emanuel Knill, Raymond Laflamme, Howard N. Barnum, Diego A. Dalvit, Jacek J. Dziarmaga." Los Alamos Science (2002): 2.
3. Baseman, Elisabeth Ann. Computing with Quantum Physics. Diss. Amherst College, 2011.
4. . Dür, Wolfgang, and Stefan Heusler. "What we can learn about quantum physics from a single qubit." arXiv preprint arXiv: 1312.1463 (2013).
5. Oi, D.K.L., The geometry of single-qubit maps. arXiv preprint quant-ph/0106035, 2001.
6. Altepeter, J.B., D.F. James, and P.G. Kwiat, 4 qubit quantum state tomography, in Quantum state estimation. 2004, Springer. p. 113-145.
7. Pasieka, A., et al., On the geometric interpretation of single qubit quantum operations on the Bloch sphere. Acta applicandae mathematicae, 2009. 108(3): p. 697.
8. Benenti, Giuliano, Giulio Casati, and Giuliano Strini. Principles of Quantum Computation and Information-Volume I: Basic Concepts. World scientific, 2004.p.100.
9. Strubell, Emma. "An introduction to quantum algorithms." COS498 Chawathe Spring 13 (2011): 19.
10. Drummond, David Edward. Adiabatic Motion of Fault Tolerant Qubits. Diss. UC Riverside, 2014.
11. Bernevig, Bogdan A., and Han-Dong Chen. "Geometry of the three-qubit state, entanglement and division algebras." Journal of Physics A: Mathematical and General 36.30 (2003): 8325.
12. Glendinning, Ian. "The Bloch sphere." QIA Meeting TechGate. 2005.
13. Williams, Colin P. Explorations in quantum computing. Springer Science & Business Media, 2010.p.17-18.