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The Dynamics of lozi map

ديناميكية دالة لوزا

Wafaa H. Al-Hilli
University of AL-Qadisiyah , Education College of Pure Science
E-mail: wafaahadi23@yahoo.com

Abstract:

In this research , we study the dynamic properties that lead to transformations into chaos .in there , we find the point of resistance and the points of proximity to it with prove demonstrated areas of contraction and expansion of the map as well the point of attracting and repelling and we touched to the chaotic properties as defined of Gulike and w-chaotic ,which rely on sensitivity of the initial conditions by use matlab program to draw the figures as well as Lyapunov expansion . finally found lyapunov dimension .

Key words:-
Fixed point , sensitive dependence on initial conditions ,lyapunov exponents .

الخلاصة:-

في هذا البحث سوف ندرس الخصائص الديناميكية التي تؤدي الى الفوضى . وفيها ندرس نقاط الثابتة ونقاط القريبة منها مع اثبات مناطق التقلص والتوسع للدالة. وقد تطرقنا للخصائص الفوضوية المتمثلة بالحساسية المقيدة بالظروف الاولية وتوسيع ليبانوف والتعدي حسب تعاريف كل من Gulik & Wiggin . حيث استخدم برنامج الماتلاب لأثبات كل من الحساسية والتعدي . واخيراً درسنا بعد ليبانوف..

الكلمات المفتاحية: - النقطة الصامدة , الحساسية المقيدة بالشروط الاولية , توسيع ليبانوف

1.**Introduction:-**

Chaotic theory is one of the most important theories in mathematics. It is a very exciting theory that has recently spread and has received widespread interest in its interpretation of many complex systems such as climate disturbance, sea wave movement, heart rate fluctuation and other , which chaos theory has interpreted as a regular movement but difficult to see it. In this research the La is two-dimensional and two parameter

$$L_{a,b}\binom{x}{y}=\binom{1+y-a|x|}{bx} \quad .$$

In this research the La is two-dimensional and two parameter , we prove by the definition in
(3. 2) and (4.4) the lozi map is chaotic if b≠0 but if b=0 we find Lozi is not chaotic

Preliminaries:-

Let $G:R^2 \rightarrow R^2$ such that $G\binom{x}{y} = \binom{h(x,y)}{g(x,y)}$ be a map .Any pair $\binom{r}{s}$ for where $h\binom{r}{s}$ =r , g $\binom{r}{s} = s$ is say a fixed point of the 2- dimensional dynamical system .Let V be a subset of R^2 $and\ V_0 = \binom{r}{s}$ be any element in v Consider $La:V\rightarrow R^2$ a map. Furthermore , assume that the first partials of the coordinate maps La_1 and La_2 of La exist at v_0 is the linear map $DLa(v_0)$ defined on R^2 by $DLa(v_0) = \begin{bmatrix} \frac{\partial la1(v_0)}{\partial x} & \frac{\partial la1(v_0)}{\partial y} \\ \frac{\partial la2(v_0)}{\partial x} & \frac{\partial la2(v_0)}{\partial y} \end{bmatrix}$ For every v_0 in V the determine of $D_{La=}(v_0)$ is called the Jacoban of La at v_0 and is called J=det $DLa(v_0)$. Let La: $R^2 \rightarrow R^2$ be a map and $v_0 \in R^2$, if $|JLa(v_0)| < 1$,then La is say area contracting at v_0, if $|JLa(v_0)| > 1$ then La is say area expanding at v_o .The La is say a diffeomorpism if achieved:(one - to - one , onto ,C^∞, and its inverse)[2] .

Some Properties of Lozi Map:-

Proposition(1.1):-

1. If b=0 and $-1\leq a \leq 1$ then the lozi map has one fixed point $p=\begin{pmatrix} \frac{1}{1+a} \\ 0 \end{pmatrix}$.
2. If b=0 and a>1 the lozi map has two fixed point $p=\begin{pmatrix} \frac{1}{1+a} \\ 0 \end{pmatrix}, q = \begin{pmatrix} \frac{1}{1-a} \\ 0 \end{pmatrix}$.
3. If $b\neq 0$, b<a+1 and b>-a+1 then the lozi map has two fixed point $p = \begin{pmatrix} \frac{1}{1+a-b} \\ \frac{b}{1+a-b} \end{pmatrix}, q = \begin{pmatrix} \frac{1}{1-a-b} \\ \frac{b}{1-a-b} \end{pmatrix}$.
4. If $b\neq 0$, b<1+a and b<-a+1 then the lozi map has one fixed point $p=\begin{pmatrix} \frac{1}{1+a-b} \\ \frac{b}{1+a-b} \end{pmatrix}$.
5. If $b\neq 0$, b>1+a and b>-a+1 then the lozi map has one fixed point $p=\begin{pmatrix} \frac{1}{1-a-b} \\ \frac{b}{1-a-b} \end{pmatrix}$.

Proof:-

1. If b=0 and $-1 \leq a \leq 1$, since $\begin{pmatrix} 1+y-a|x| \\ bx \end{pmatrix} = \binom{x}{y}$

so y=0 this implies $x = \frac{1}{1+a}$ therefore $p=\begin{pmatrix} \frac{1}{1+a} \\ 0 \end{pmatrix}$ is fixed point of $L_{a,0}$

2. Since $\begin{pmatrix} 1+y-a|x| \\ bx \end{pmatrix} = \binom{x}{y}$ so y=0 then if b=0 , x<0 and a>1 then $x = \frac{1}{1+a}$ therefor $p=\begin{pmatrix} \frac{1}{1+a} \\ 0 \end{pmatrix}$

So if b=0 , x>0 and a>1 then $x = \frac{1}{1-a}$ therefor $q = \begin{pmatrix} \frac{1}{1-a} \\ 0 \end{pmatrix}$. hence p, q are two fixed point of $L_{a,0}$

3. Since $\begin{pmatrix} 1+y-a|x| \\ bx \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$ then $bx=y$ this implies $1+bx-ax=x\,, x-bx+ax=1$ therefore $x=\frac{1}{1-b+a}$

If b≠0 and b<a+1 then $y=\frac{b}{1-b+a}$ therefor $p=\begin{pmatrix} \frac{1}{1+a-b} \\ \frac{b}{1+a-b} \end{pmatrix}$,

And if $b\neq 0$ and $b>-a+1$ then $x=\frac{1}{1-b-a}$ $hence\ y=\frac{b}{1-b-a}$ therefor $q=\begin{pmatrix} \frac{1}{1-a-b} \\ \frac{b}{1-a-b} \end{pmatrix}$.

Then p,q are two fixed point.

4. Since $\begin{pmatrix} 1+y-a|x| \\ bx \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$

If $b\neq 0,\ b<a+1\ and\ b<1-a+1$ then $x=\frac{1}{1-b+a}$ $this\ implise\ y=\frac{b}{1-b+a}$ $hence$ p=$\begin{pmatrix} \frac{1}{1+a-b} \\ \frac{b}{1+a-b} \end{pmatrix}$ $is\ one$ fixed point of La .

5. Since $\begin{pmatrix} 1+y-a|x| \\ bx \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$

If $b\neq 0, b>a+1$ and $b<-a+1$ then x= $\frac{1}{1-b-a}$ $this\ implise\ y=\frac{b}{1-b-a}$ $hence$ q = $\begin{pmatrix} \frac{1}{1-a-b} \\ \frac{b}{1-a-b} \end{pmatrix}$

Proportion(1.2):-

1. $If\ b=0\,, a>1\ and\ -1<a<1\ then\ Jacobian\ of\ lozi\ map\ is\ 0$
2. $If\ b\neq 0\ \ then\ Jacobian\ of\ la\ \ is-b$

Proof :-

If b≠0 then Jacobin of L_a is $-b$

DLa(V_0)= $\begin{bmatrix} \frac{\partial f1}{\partial x} & \frac{\partial f1}{\partial y} \\ \frac{\partial g1}{\partial x} & \frac{\partial g1}{\partial y} \end{bmatrix}=\begin{bmatrix} \frac{-a\,\partial|x|}{\partial x} & 1 \\ b & 0 \end{bmatrix}$

Then $J=det\ D\ La,b(V0)=-b$

Remark(1.3):-

If b=0 , then La is only area contracting

Proposition (1.4):-

1. If $b \neq 0$, $|-b| > 1$ then La is area expanding .
2. If $b \neq 0, |-b| < 1$ then La is area contracting.

Proof:-

1. If $b \neq 0, |-b| > 1$ then by proposition (1.2) $L_{a,b}\binom{x}{y}$ is area expanding map and if $b \neq 0, |-b| < 1$ this implies that $L_{a,b}\binom{x}{y}$ is an area contracting.

Proposition (1.5):-

1. $If\ b = 0\ and\ a > 1, -1 < a < 1\ then\ La, b\ is\ not\ one\ to\ one\ and\ not\ onto$
2. $if\ b \neq o\ , b < a + 1\ and\ b > -a + 1\ is\ one\ to\ one\ and\ onto.$
3. $If\ b \neq 0\ and\ b < a + 1\ and\ b < 1 - a + 1\ is\ one\ to\ one\ and\ onto$
4. $If\ b \neq 0\ and\ b < a + 1\ and\ b < -a + 1\ is\ one\ to\ one\ and\ onto.$

Proof:-

1. $Let\ T(x, y) = (1 + y - a|x|, bx)$

$T(1,0) = (1 - a, b), T(0,1) = (2,0)\ then\ L_{a,b}\begin{pmatrix} 1-a & b \\ 2 & 0 \end{pmatrix}$

If $b = 0$ and $-1 \leq 1 - a \leq 1$ then $0 \leq -a \leq 2$

$so\ L_{a,b}\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} using\ Row\ Echelon\ form, we\ get\ L_{a,b}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\ we\ have\ no\ pivot\ in\ every\ collem$

$then\ L_{a,b}\ is\ not\ one\ to\ one,$ And we have no pivot in every row then La is not onto.

2. $If\ b \neq 0\ , b < a + 1\ and\ b > -a + 1\ we\ choose\ a = 1\ and\ b = 1\ so$

$L_{a,b}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} using\ Row\ Echelon\ form, we\ get\ L_{a,b}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ we\ have\ pivot\ in\ every\ collem$

$then\ L_a\ is\ one\ to\ one,$ And we have pivot in every row then La is onto .

The same way (3,4)

Proposition (1.6):- La is C^{∞}

Proof:-

$$L_a\binom{x}{y} = \binom{1 + y - a|x|}{bx}$$

Then all first partial derivatives exist and continues

Note that $\frac{\partial^n f(x,y)}{\partial x^n} = 0\ \forall n \epsilon N\ ,\ and\ \ \frac{\partial^n f(x,y)}{\partial y^n} = 0\ ,\ \frac{\partial^n g(x,y)}{\partial x^n} = 0\ \forall n \geq 2\ ,\ and\ \ \frac{\partial^n g(x,y)}{\partial y^n} = 0$ we get that all its L_a exist $k - th$ partial derivatives exist and continues . for K from definition of diffeomorfism.

Remark(1.7):-

1. $If\ b = 0\ then\ L_{a,0} is\ not\ diffeomorphism$.
2. $If\ b \neq 0$ then $L_{a,b}$ is diffeomorphism

Proposition (1.8):-

If b=0 then The eigenvalues of lozi is $\lambda_1 = -a\,, \lambda_2 = 0$

Proof:-

To find the eigenvalues of lozi map then

DLa(V_0)= $\begin{bmatrix} \frac{\partial f1}{\partial x} & \frac{\partial f2}{\partial y} \\ \frac{\partial g1}{\partial x} & \frac{\partial g2}{\partial y} \end{bmatrix} = \begin{bmatrix} -a-\lambda & 1 \\ b & 0-\lambda \end{bmatrix} = 0 \; therefore \; (-a-\lambda)(-\lambda) - b = 0$

Since $\lambda_{1,2} = \frac{-a \pm \sqrt{4b+a^2}}{2}$, if $b = 0$ then $\lambda_1 = -a\,, \lambda_2 = 0$ are the eigenvalues of lozi map.

Proposition (1.9):-

If b≠0 then The eigenvalues of lozi are $\lambda_{1,2} = \frac{-a \pm \sqrt{4b+a^2}}{2}$

Proof:-

To find the eigenvalues of lozi map if b≠0 then

DLa(V_0)= $\begin{bmatrix} \frac{\partial f1}{\partial x} & \frac{\partial f2}{\partial y} \\ \frac{\partial g1}{\partial x} & \frac{\partial g2}{\partial y} \end{bmatrix} = \begin{bmatrix} a-\lambda & 1 \\ b & 0-\lambda \end{bmatrix} = 0 \; therefore \; (a-\lambda)(-\lambda) - b = 0$

Since $\lambda_{1,2} = \frac{a \pm \sqrt{4b+a^2}}{2}$ are the eigenvalues of lozi map.

2. Sensitive Dependence On Initial Conditions :-

The most important threads that Gulik addresses in his definition of chaos in the definition of sensitive on initial conditions which ensures.

Definition (2.1):[1][5]:-

A map $F: X \to X$ is said to be Sensitive Dependence On Initial Conditions(sdic) if there exists $\varepsilon > 0$ such that for any $x_0 \in X$ and any open set $U \subset X$ containing x_0 there exists $y_0 \in U$ and $n \in Z^+$ such that $d(f^n(x_0), f^n(y_0)) > \varepsilon$.

That is $\exists \varepsilon > 0, \forall x, \forall \delta > 0, \exists y \in B_\delta(x), \exists n : d(f^n(x_0), f^n(y_0)) > \varepsilon$.

We prove (sdic) by use matlab program

Figure (1):- If b≠0 then (sdic) is no satisfy.

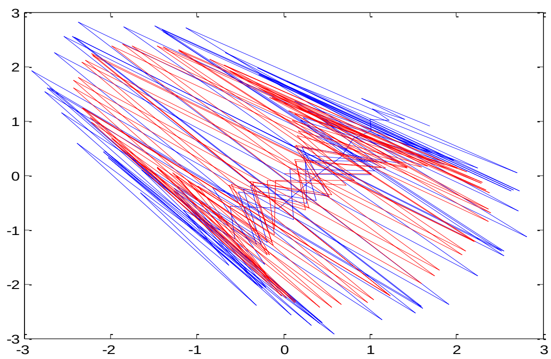
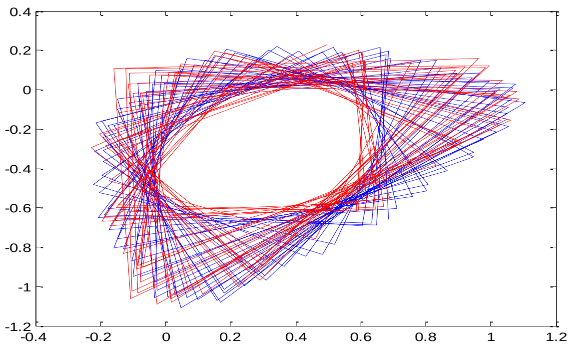


Figure (1.1) :- No sdic x1 (1) = 0.01 ;y1 (1) = 0.02;a1 =0.8;b1=-1.002; x2 (1) = 0.02;y2 (1) = 0.03;a2 =0.9;b2=-1.003;



Figure(1.2) :- No sdic x1 (1) = 0.01 ;y1 (1) = 0.02; a1 =0.8;b1=-1.002; x2 (1) = 0.02;y2 (1) = 0.03; a2 =0.9;b2=-1.003;

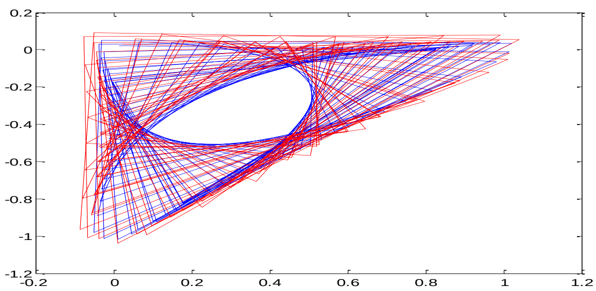
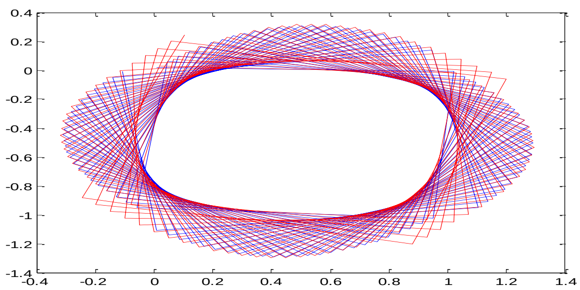


Figure (1.3):- No sdic with x1 (1) = 0.01 ;y1 (1) = 0.02;a1 =1.008;b1=-1.002; x2 (1) = 0.02;y2 (1) = 0.03;a2 =1.009;b2=-1.003;



Figure(1.4):- No sdic with x1 (1) = 0.01 ;y1 (1) = 0.02;a1 =0.04;b1=-1.002; x2 (1) = 0.02;y2 (1) = 0.03;a2 =0.05;b2=-1.003;

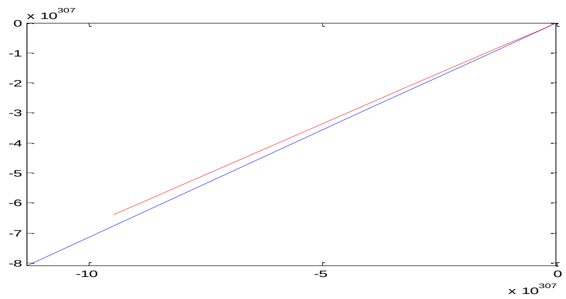


Figure (1.5):- No sdic with x1 (1) = 0.01 ;y1 (1) = 0.02;a1 =2;b1=2.002; x2 (1) = 0.02;y2 (1) = 0.03;a2 =2.009;b2=2.01;

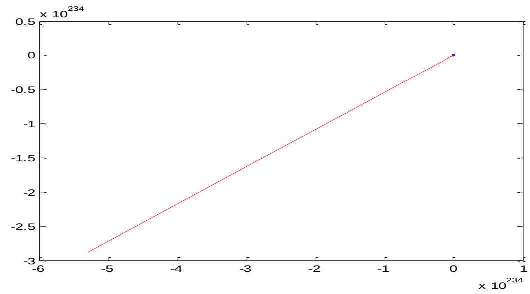
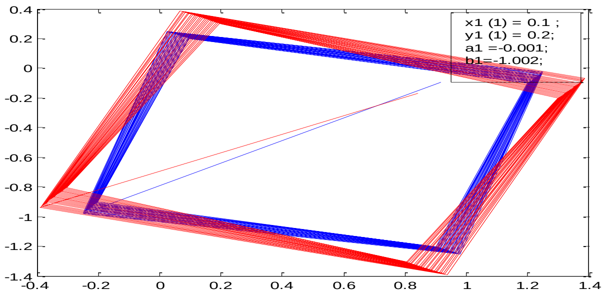
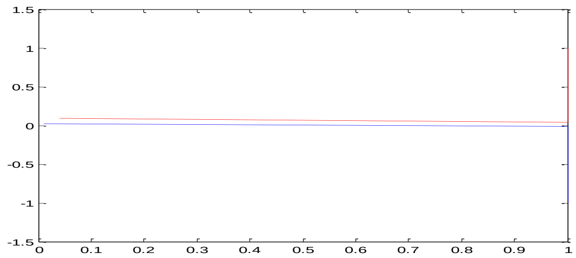


Figure (1.6):- No sdic with x1 (1) = 0.01 ;y1 (1) = 0.02; a1 =0.1;b1=1.0002;; x2 (1) = 0.02;y2 (1) = 0.03;a2 =0.009;b2=1.01;

If b≠0 then (sdic) satisfy

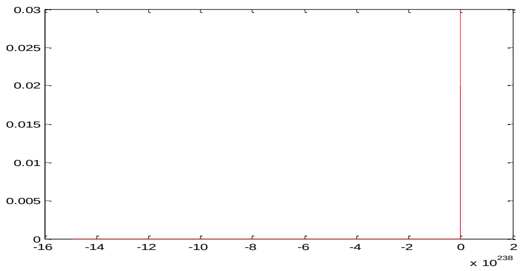


Figure(2.1):-sdic with x1 (1) = 0.1 ;y1 (1) = 0.2;a1 =-0.001;b1=-1.002; x2 (1) = 0.2;y2 (1) = 0.3;a2 =-0.005;b2=-0.003;

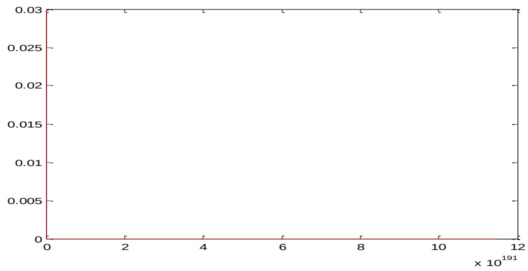


Figure(2.2):- sdic with x1 (1) = 0.01 ;y1 (1) = 0.02; a1 =0;b1=-1.004; x2 (1) = 0.04;y2 (1) = 0.09;a2 =0; b2=1.001;

Figure (2):- If b=0 then (sdic) is not satisfy.



x1 (1) = 0.01 ;y1 (1) = 0.02; a1 =2.01;b1=0; x2 (1) = 0.02 ; y2 (1) = 0.03;a2 =2.02;b2=0;



x1 (1) = 0.01 ;y1 (1) = 0.02;a1 =-1.01; b1=0; x2 (1) = 0.01 ;y2 (1) = 0.02; a2 =-1.01;b2=0;

3.The transitivity :-

One of the most chaotic characteristics on which many definitions are based is the property of transitive, which depends on the dense of orbits

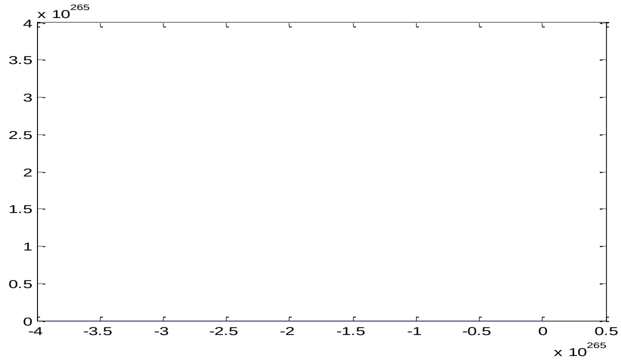
Definition (3.1)[3]:-

Let f:X→X be a dynamical system . If for every pair of nonempty open sets U and V in X , there is a n ∈ N such that , $f^n(U) \cap V \neq \emptyset$, then f has topologically transitive.

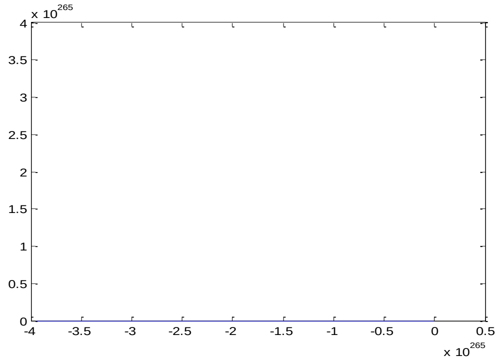
Many times, the system is used to be transitive if there is an $x_0 \in X$ such that $\overline{o(x_0)}$ =X (i.e) f has a dense orbit).

Both of as these definitions of transitivity are equivalent, in a wide class of spaces, including all connected compact metric spaces.

Figure (3):- if b≠0 transitive is no satisfy



Figure(3.1):- No Transitive with x1 (1) = 0.01 ;y1 (1) = 0.02;a1 =0.8;b1=1.002;



Figure(3.2):-No Transitive with x1 (1) = 0.01 ;y1 (1) = 0.02;a1 =0.8;b1=-1.002;

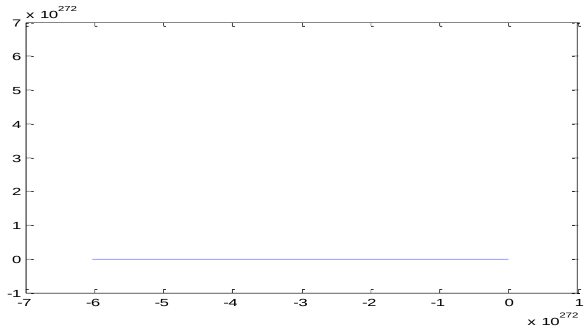


Figure (3.3):- No Transitive with x1 (1) = 0.01 ;y1 (1) = 0.02;a1 =1.008;b1=-1.002;

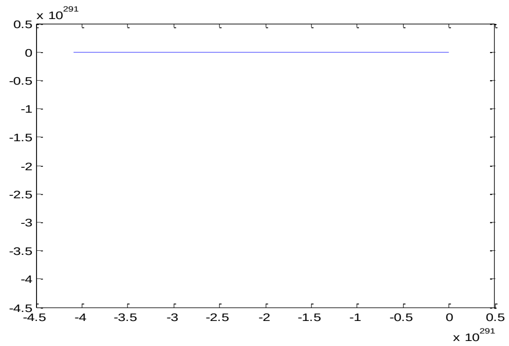


Figure (3.4):- No Transitive with x1 (1) = 0.01 ;y1 (1) = 0.02;a1 =0.04;b1=1.002;

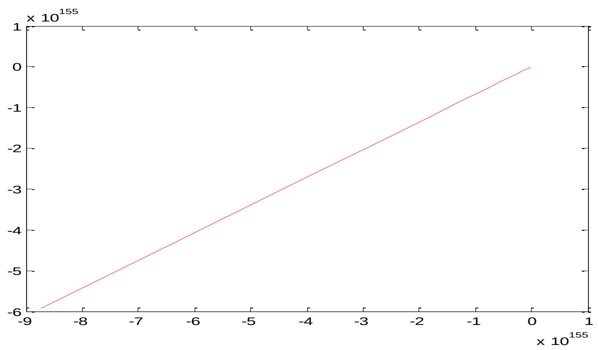


Figure (3.5):- No Transitive with x1 (1) = 0.01 ;y1 (1) = 0.02;
a1 =0.8;b1=1.002;

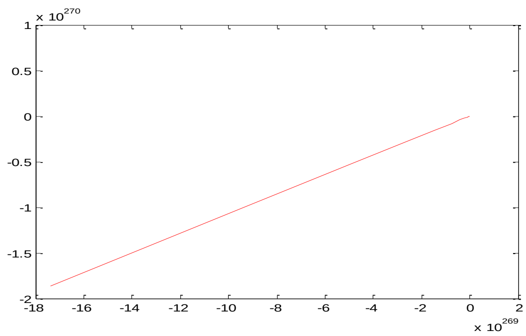


Figure (3.6):- No Transitive with x1 (1) = 0.01 ;y1 (1) = 0.02;
a1 =0.8;b1=2.002;

If b≠0 The Transitive is satisfy

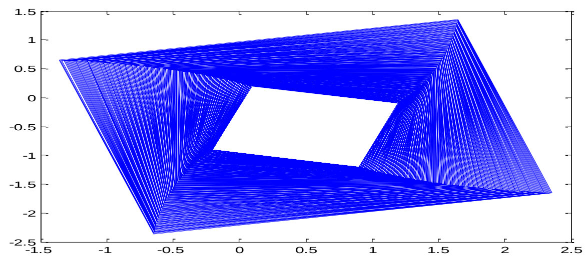
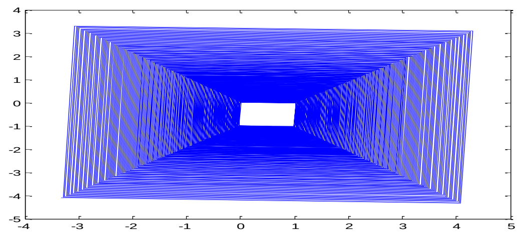


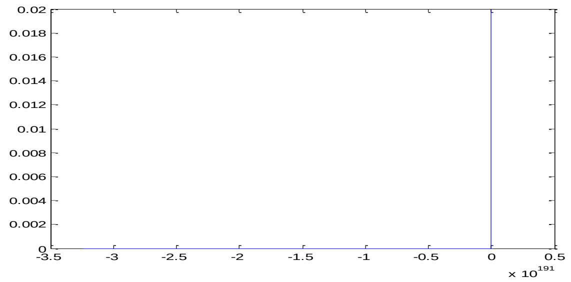
Figure (4.1):- transitive with x1 (1) = 0.1 ;y1 (1) = 0.2;a1 =-0.001;b1=-1.002



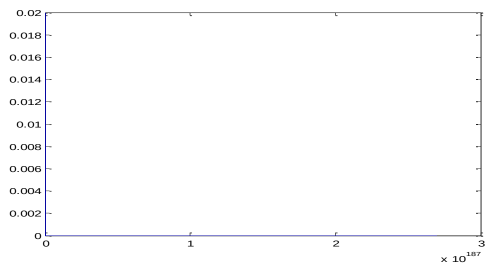
x2 (1) = 0.04;y2 (1) = 0.09;a2 =0; b2=1.001;

Figure(4):

If b=0 no transitive



Figure(4.3):- no transitive with x1(1)= 0.01;y1 (1) = 0.02;a1 =2.01;b1=0;



x1 (1) = 0.01 ;y1 (1)= 0.02;a1 =-1.01; b1=0;

Definition(3-2)(1):-

Let f:X→X be a continuous map and X be a metric space. Then the map f is said to be chaotic according to Wiggins or W-chaotic if :

(1) f is topologically transitive.

(2) f is exhibits sensitive dependent on initial condition

By the figure of sensitive dependent on initial condition and topologically transitive are verified if b≠0 then La is chaotic

4. Lyapunov exponent:-

The Lyapunov exponents give the average exponential rate of divergence or convergence of nearly orbits in the phase- space. In systems exhibiting exponential orbital divergence[4],

Definition (4.1)[6]:-

Let F: X→X be continuous differential map, where X is any metric space. Then all x in X in direction V the Lyapunov exponent was defined of a map F at X by L(x,v)= $\lim_{n\to\infty}\frac{1}{n}\, ln\|\, DF_x^n v\|$ whenever the limit exists in higher dimensions for example in R^n the map F will have n Lyapunov exponents, say $L_1^{\pm}(x,v_1)$,$L_2^{\pm}(x,v_2),\dots,L_n^{\pm}(x,v_n)$, for a maximum Lyapunov exponent that is

$L_{\pm}(x,v)=Max\left\{L_1^{\pm}(x,v_1),L_2^{\pm}(x,v_2),L_3^{\pm}(x,v_3),\dots,L_n^{\pm}(x,v_n)\right\}$, where v=(v_1 ,v_2,…,v_n)

Proposition (4.2):-

3. if b $\neq 0$ then La $\begin{pmatrix}x\\y\end{pmatrix}$ has positive Lyapunov exponent

proof:-

let $x=\begin{pmatrix}x\\y\end{pmatrix}\epsilon R^2$, the Lyapunov exponent of $L_{a,b}$is given by the formula $L_1\left(\begin{pmatrix}x\\y\end{pmatrix},v_1\right)=\lim_{n\to\infty}\frac{1}{n}\ln\left\|DL_{a,b}\begin{pmatrix}x\\y\end{pmatrix},v_1\right\|$by proposition (1-9), we have La has two eigenvalues such that $|\lambda_1|=\frac{1}{|\lambda_2|}$ and since $if\ |\lambda_1|<1\ then\ L_1\left(\begin{pmatrix}x\\y\end{pmatrix},v_1\right)=$ $\lim_{n\to\infty}\frac{1}{n}\ln\left\|\left(DL_{a,b}\begin{pmatrix}x\\y\end{pmatrix},v_1\right)^n\right\|>\ln\left\|\frac{-a+\sqrt{a^2+4b}}{2}\right\|$

By hypothesis L_1>0 , so if $|\lambda_1|<1$ then $L_2\left(\begin{pmatrix}x\\y\end{pmatrix},v_2\right)=\lim_{n\to\infty}\frac{1}{n}\ln\left\|\left(DL_{a,b}\begin{pmatrix}x\\y\end{pmatrix},v_2\right)^n\right\|<\ln\left\|\frac{-a-\sqrt{a^2+4b}}{2}\right\|$

Thus the Lyapunove exponent , $L_1(x,y)=max\{L_1(x,y),L_2(x,y)\}$ hence the Lyapunov exponent map is positive .

Remark (4.3):-

1. If b=0 the lozi map is negative.

Definition (4.4)(2):-

A map La is chaotic in sense of Gulick if it satisfies at least one of the following conditions:-

1. L_a has a positive lyapunove exponent at each point in its domain that is not eventually periodic.
2. L_a has S.D.I on its domain.

By draw of sensitive dependence on initial condition figure (1) and Proposition (4.2) then the lozi map is chaotic in sense of Gulick.

5.Lyapunov dimension:-

Lyapunov dimension is used to compute the dimension for irregular shapes and it will be much accurate than Box dimension for irregular shapes [1].

Definition (5.1)[2]:-

Let V be a subset of R^2 , and H: V→ R^2 has continuous partial derivatives. Assume that v_0 in V , with orbit $\{v_n\}_n^\infty = 0$. For each n=1,2,...we define $D_nF(v_0)$ by the formula $D_nH(v_0) = [D\,H(v_{n-1})][D\,H(v_{n-2})] \dots [D\,H(v_0)]$ Where $D\,H(v_k)$ denotes the 2× 2 matrix identified with the differential of H at v_k. Then $D\,F(v_0)$ is 2× 2matrix (depending on n). If $D_nH(v_0)$ has nonzero real eigenvalues , we denote their absolute values of eigenvalues by $d_{n1}(v_0)$and $d_{n2}(v_0)$. For convenience we will assume that $d_{n1}(v_0) \geq d_{n2}(v_0)$. The Lyapunov numbers $r_1(v_0)$and $r_2(v_0)$ of V at v_0:−

$r_1(v_0) = \lim_{n\to\infty}[d_{n1}(v_0)]^{\frac{1}{n}}$, $r_2(v_0) = \lim_{n\to\infty}[d_{n2}(v_0)]^{\frac{1}{n}}$provided that the limits exit

Definition (5.2)[2]:-

Let v be a subset of R^2 and let map L_a:V→R^2 have coordinate maps with continuous partial derivatives. Also assume that L_a has an attractor A_F and v_0 is in A L_a . Finally assume that $r_1(v_0)>r_3(v_0)>r_2(v_0)$ then the Lyapunov dimension of A L_a at v_0, denoted $\dim_L A_F(v_0)$. is given by $\dim_L A_F(v_0) = 1 - \frac{\ln r_1(v_0)}{\ln r_2(v_0)}$. In the event that $r_1(v_0)$, $r_2(v_0)$ and $r_3(v_0)$ are independent of v_0. We write r_1 , r_2 and r_3 for $r_1(v_0)$, $r_2(v_0)$ and $r_3(v_0)$, respectively.

In that case we define the Lyapunov dimension of A L_a by the formula $\dim_L A_{La} = 1 - \frac{\ln r_1}{\ln r_2}$

Definition(5.3)[2]:-

We say that a map has a strange attractor has a non-integer Lyapunov dimension

Definition(5.4)[7]:-

An attractor is said to be strange if it contains a dense orbit (transitive point) with positive Lyapunov exponent

Proposition(5.5):-

1. If $b = 0$ then The La is Lyapunov dimension attractor .
2. If b=0 and $|\lambda_2| < |\lambda_1|$ then L_a is infinite.

Proof:-

1. Since $b = 0$ so if $|\lambda_2| > |\lambda_1|$ then by definition (5.1)

Dn_1=max eigenvalues of $D_nL_{a,b}\,n$

Dn_2=min eigenvalues of $D_nL_{a,b}\,n$

Then

$$r_1 = \lim_{n\to\infty}(dn_1)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(\frac{-a-\sqrt{a^2+4b}}{2}\right)^n\right)^{\frac{1}{n}} = \frac{-a-\sqrt{a^2+4b}}{2}$$

$$r_2 = \lim_{n\to\infty}(dn_2)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(\frac{-a+\sqrt{a^2+4b}}{2}\right)^n\right)^{\frac{1}{n}} = \frac{-a+\sqrt{a^2+4b}}{2}$$

Therefore $\dim A_F = 1 - \frac{\ln r_1}{\ln r_2} = 1 - \frac{\ln|\lambda_1|}{\ln|\lambda_2|} = 1 - \left(\frac{\frac{-a-\sqrt{a^2+4b}}{2}}{\frac{-a+\sqrt{a^2+4b}}{2}}\right)$

If b=0 then $\dim A_F = \frac{-2\sqrt{a^2+4b}}{-a-\sqrt{a^2+4b}} = 0$

Proposition (5.6):-
If $b \neq 0$ then no Lyapunov dimension attractor is .

$$Dim_{l_{a,b}} A = 1 - \frac{-a + \sqrt{a^2 + 4b}}{-a - \sqrt{a^2 + 4b}}$$

Proof:-
Since $b \neq 0$ so if $|\lambda_1| > |\lambda_2|$ then by definition (5.1)
Dn_1=max eigenvalues of $D_n L_{a,b}\, n$
Dn_2=min eigenvalues of $D_n L_{a,b}\, n$
Then

$$r_{1=} \lim_{n\to\infty} (dn_1)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(\frac{-a + \sqrt{a^2 + 4b}}{2} \right)^n \right)^{\frac{1}{n}} = \frac{-a + \sqrt{a^2 + 4b}}{2}$$

$$r_{2=} \lim_{n\to\infty} (dn_2)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(\frac{-a - \sqrt{a^2 + 4b}}{2} \right)^n \right)^{\frac{1}{n}} = \frac{-a - \sqrt{a^2 + 4b}}{2}$$

Therefore $\dim A_F = 1 - \frac{\ln r_1}{\ln r_1} = 1 - \frac{\ln|\lambda_1|}{\ln|\lambda_1|} = 1 - (\frac{\frac{-a+\sqrt{a^2+4b}}{2}}{\frac{-a-\sqrt{a^2+4b}}{2}}) =$

$$1 - \left(\frac{-a + \sqrt{a^2 + 4b}}{-a - \sqrt{a^2 + 4b}} \right) = \left(\frac{-a - \sqrt{a^2 + 4b} + a - \sqrt{a^2 + 4b}}{-a - \sqrt{a^2 + 4b}} \right) = (\frac{2\sqrt{a^2 + 4b}}{a + \sqrt{a^2 + 4b}})$$

Remark (5.7):-
1. If b=0 then L_a has no attractor
2. If b≠0 then by proposition (4.2) and transitive point (Figure (3))then by Definition(5.4) the maps has strange attractor

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