Comparing Three Different Algorithms to Estimate Parameters of new Generated Marshal – Olkin Uniform Distribution

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***Abstract*—**

This paper deals with constructing a new generated Marshal – Olkin Uniform family distribution which include finding the probability density function ($p.d.f$), cumulative distribution function ($C.D.F$), this new compound $p.d.f$ is necessary to represent the time to failure data of complex system, so we insist to extend Marshal – Olkin distribution to another family called Marshal – Olkin Weibull, we deriving the ($p.d.f$), ($C.D.F$), and reliability function, then applying simulation procedure taking different sample size ($n = 20,40,80,100$) and different set of initial values of (α,θ), each experiment repeated ($R = 1000$), all results explained in tables.

Index Terms*—*MOEU(Marshall-Olkin extended uniform , MLE,Maximum likelihood estimator,MOM,Moment estimators,REM,Regresion estimators

I. Introduction

The $p.d.f$ of Marshal – Olkin extended uniform (MOEU) is defined by;

$$f(x;\alpha,\theta) = \frac{\alpha\theta}{[\alpha\theta+(1-\alpha)x]^2} \qquad 0 < x < \theta,\ \ \alpha > 0$$
(1)

Where; (α) is the shape parameter and (θ) is the scale parameter, while the corresponding cumulative distribution function is; (16)

$$F(x;\alpha,\theta) = \frac{x}{[\alpha\theta+(1-\alpha)x]} \qquad 0 < x < \theta,\ \ \alpha > 0$$
(2)

The reliability function of (MOEU) is;

$$R(x;\alpha,\theta) = \frac{\alpha(\theta-x)}{[\alpha\theta+(1-\alpha)x]} \qquad 0 < x < \theta,\ \ \alpha > 0$$
(3)

For the $p.d.f$ in equation (1) we observe that the shape of this $p.d.f$, $f(x;\alpha,\theta)$ depend on parameter (α), when [$\alpha \in (0,1)$], the $p.d.f$ is decreasing function ($0,\theta$) with;

$$f(0;\alpha,\theta) = \frac{1}{\alpha\theta} \quad and \quad f(\theta;\alpha,\theta) = \frac{\alpha}{\theta}$$
(4)

But when ($\alpha > 1$), the $p.d.f$ is an increasing function on ($0,\theta$) with (4). Also we can find hazard rate of the random variable (x), which has $p.d.f$ in (1);

$$h(x;\alpha,\theta) = \frac{\theta}{[\alpha\theta+(1-\alpha)x](\theta-x)}$$
(5)

The r^{th} moment about origin of this distribution [MOEU(α,θ)] is;

$$\mu'_r = E(x^r)\int_0^\theta x^r f(x;\alpha,\theta)\,dx$$
$$= \frac{\alpha\theta}{(1-\alpha)^{r+1}}\sum_{s=0}^{r}\frac{r!\,(-\alpha\theta)^s}{(r-s)!\,s!\,(r-s-1)}[(\theta)^{r-s-1}-(\alpha\theta)^{r-s-1}]$$
$\dots.(6)$

According to this formula, the mean and variance of $r.v\ x$ with [MOEU(α,θ)] is;

$$\mu'_1 = \frac{\alpha\theta}{(1-\alpha)^2}(\alpha - \log\alpha - 1)$$
(7)

$$\mu'_2 = \frac{\alpha\theta^2}{(1-\alpha)^4}[(1-\alpha)^2 - \alpha(\log\alpha)^2]$$
(8)

Also we can show that the coefficient of variation is;

$$c.v = \frac{\sqrt{[(1-\alpha)^2-\alpha(\log\alpha)^2]}}{\sqrt{\alpha}(\alpha-\log\alpha-1)} \qquad \alpha > 0$$
(9)

Which depend on shape parameter (α), and the coefficient of Skewness is;

$$S_k = \frac{\mu_x-\mu_0}{\sigma_x}$$

Where;

$$\mu_0 = x = \frac{\alpha\theta}{\alpha-1}$$

We can prove that;

$$S_k = \frac{-\sqrt{\alpha}\,\ln\alpha}{\sqrt{[(1-\alpha)^2-\alpha(\ln\alpha)^2]}}$$
(10)

II. Estimation of the Parameters

This section deals with introducing some methods to estimate two parameters (α,θ), then applying simulation procedure to compare the results, these methods are;

II.1 Maximum Likelihood Method

The estimators by this method obtained from maximizing;

$$L(\alpha,\theta) = \frac{(\alpha\theta)^n}{\prod_{i=1}^{n}[\alpha\theta+(1-\alpha)x_i]^2}$$

$$\ln L = nL(\alpha,\theta) - 2\sum_{i=1}^{n}\ln[\alpha\theta+(1-\alpha)x_i] \quad (11)$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} - 2\sum_{i=1}^{n}\frac{(\theta - x_i)}{[\alpha\theta+(1-\alpha)x_i]} \quad (12)$$

If we consider $[\hat{\theta}_{MLE} = x_{(n)}]$, then;

$$\hat{\alpha}_{MLE} = \frac{n}{2}\left[\sum_{i=1}^{n-1}\frac{(x_{(n)} - x_i)}{\hat{\alpha}_{MLE}\ x_{(n)} + (1-\hat{\alpha}_{MLE})x_i}\right]^{-1} \quad (13)$$

Which is an implicit function can be solved numerically.

II.2 Moment Estimator

Let $x\ be\ r.v \sim MOEU(\alpha,\theta)$, then the moment estimators of (α,θ) obtained from equating sample moment $[\mu_r = \frac{\sum_{i=1}^{n} x_i^r}{n}]$ with population moment (μ_r') defined in equation (6), we use coefficient of variation $(c.v)$;

$$c.v = \frac{\sqrt{v(x)}}{E(x)} = \frac{\sqrt{[(1-\alpha)^2-\alpha(\ln\alpha)^2]}}{\sqrt{\alpha}(-\ln\alpha-1+\alpha)}$$

(14)

Which is a function of shape parameter (α). Therefore equating $(c.v)$ with sample $(c.v)$;

$$\frac{s}{\bar{x}} = \frac{\sqrt{[(1-\alpha)^2-\alpha(\ln\alpha)^2]}}{\sqrt{\alpha}(-\ln\alpha-1+\alpha)}$$

This yield $(\hat{\alpha}_{MOME})$ and then from $[E(x) = \mu_1']$, we can find $(\hat{\alpha}_{MOME})$:

$$\hat{\alpha}_{MOME} = \frac{\bar{x}(1-\hat{\alpha}_{MOME})^2(\hat{\alpha}_{MOME})^{-1}}{[\hat{\alpha}_{MOME}-1-\ln\hat{\alpha}_{MOME}]}$$

(15)

II.3 Moment Estimator

This method used to estimate (α,θ) for $[MOEU(\alpha,\theta)]$ is regression estimator which explained as follows;
Let $(x_1,x_2,\dots,x_n)$ be a random sample from $[MOEU(\alpha,\theta)]$, defined in equation (1), then;

$$\sqrt{F(x_i)} = \frac{\sqrt{\alpha\theta}}{\alpha\theta+(1-\alpha)x_i} = \frac{1}{\sqrt{\alpha\theta}+\frac{(1-\alpha)}{\sqrt{\alpha\theta}}x_i}$$

$$\sqrt{\frac{1}{F(x_i)}} = \sqrt{\alpha\theta} + \frac{(1-\alpha)}{\sqrt{\alpha\theta}}x_i$$

Let $y_i \Rightarrow \sqrt{\frac{1}{F(x_i)}}$ $\quad \beta_0 = \sqrt{\alpha\theta}$ $\quad \beta_1 = \frac{(1-\alpha)}{\sqrt{\alpha\theta}}$

$y_i = \beta_0 + \beta_1 x_i + u_i$ simple linear regression model, therefore;

$$\hat{\alpha}_{RE} = 1 - \hat{\beta}_0\hat{\beta}_1$$

(16)
And

$$\hat{\theta}_{RE} = \frac{\hat{\beta}_0^2}{1-\hat{\beta}_0\hat{\beta}_1}$$

(16)

III. Application

The application has been done through simulation procedure to compare between the estimators, for the values of scale parameter (θ) and shape parameter (α) with $(n = 20,40,80,100)$, the comparison for scale parameter and shape parameter was done by MSE.

θ	3	3.5	4
α	1	1.5	2

Table (1): $\alpha = 1 \quad \theta = 3$

n	Method	α	$MSE(\alpha)$	θ	$MSE(\theta)$
20	MLE	1.031278	0.113673	2.851806	0.039166
	Moment	1.068291	0.276983	2.996657	0.001013
	Regression	1.031082	0.11297	2.847869	0.040374
BEST		MOM			MOM
40	MLE	1.00589	0.059886	2.937422	0.007572
	Moment	1.019005	0.106815	2.999201	0.00036
	Regression	1.005839	0.059516	2.933136	0.008074
BEST		REG			MOM
80	MLE	1.068839	0.080575	2.949082	0.005078
	Moment	1.107214	0.134356	3.000366	0.000264
	Regression	1.068609	0.080022	2.944403	0.00561
BEST		REG			MOM
100	MLE	0.992154	0.026968	2.971291	0.001815
	Moment	0.984331	0.036687	2.999123	5.07E-05
	Regression	0.992206	0.026804	2.966234	0.002137
BEST		REG			MOM

n	Method	α	$MSE(\alpha)$	θ	$MSE(\theta)$
20	MLE	1.100314	0.238803	3.32512	0.059525
	Moment	1.229697	0.510707	3.502232	2.10E-03
	Regression	1.099771	0.236671	3.318926	0.061781
BEST		REG			MOM
40	MLE	1.066186	0.062155	3.443084	0.00695
	Moment	1.084501	0.083461	3.499059	1.49E-04
	Regression	1.065977	0.061891	3.438127	0.007579
BEST		REG			MOM
80	MLE	1.012072	0.034551	3.457482	0.003955
	Moment	1.00564	0.069503	3.500705	1.36E-04
	Regression	1.012036	0.034326	3.451485	0.004553
BEST		REG			MOM
100	MLE	0.996889	0.024841	3.465704	0.002261
	Moment	1.005863	0.045934	3.500502	9.70E-05
	Regression	0.996893	0.024648	3.459205	0.002801
BEST		REG			MOM

Table (2): $\alpha = 1 \quad \theta = 3.5$

Table (3): $\alpha = 1 \quad \theta = 4$ Prepare Your PAPER

n	Method	α	$MSE(\alpha)$	θ	$MSE(\theta)$
20	MLE	1.042908	0.138847	3.811649	0.063222
	Moment	1.156049	0.287677	4.007926	1.98E-03
	Regression	1.042368	0.137414	3.804282	0.065918
BEST		REG		MOM	
40	MLE	1.029217	0.078406	3.911941	0.014818
	Moment	1.043457	0.159298	4.002348	5.13E-04
	Regression	1.029042	0.07787	3.904754	0.01623
BEST		REG		MOM	
80	MLE	1.034246	0.034645	3.958564	0.004069
	Moment	1.054223	0.060447	3.999109	1.51E-04
	Regression	1.034102	0.034334	3.950485	0.004775
BEST		REG		MOM	
100	MLE	1.029207	0.041988	3.969557	0.001767
	Moment	1.006764	0.050583	3.999158	9.19E-05
	Regression	1.029003	0.041595	3.961514	0.002371
BEST		REG		MOM	

Table (4): $\alpha = 1.5 \quad \theta = 3$

n	Method	α	$MSE(\alpha)$	θ	$MSE(\theta)$
20	MLE	1.603319	0.470726	2.898675	0.020577
	Moment	1.677129	1.051682	3.000486	1.98E-03
	Regression	1.600803	0.465825	2.895817	0.021229
BEST		REG		MOM	
40	MLE	1.650908	0.240887	2.946342	0.005478
	Moment	1.768623	0.535315	3.008162	6.91E-04
	Regression	1.648319	0.238308	2.943197	0.005813
BEST		REG		MOM	
80	MLE	1.519302	0.10133	2.972898	0.001614
	Moment	1.544359	0.182681	3.001927	3.03E-04
	Regression	1.51761	0.100462	2.970086	0.001766
BEST		REG		MOM	
100	MLE	1.510354	0.064359	2.981114	0.00074
	Moment	1.546363	0.168409	3.002502	2.96E-04
	Regression	1.50853	0.063907	2.977822	0.000893
BEST		REG		MOM	

Table (5): $\alpha = 1.5 \quad \theta = 3.5$

n	Method	α	$MSE(\alpha)$	θ	$MSE(\theta)$
20	MLE	1.662229	0.482339	3.395761	0.01763
	Moment	1.75725	0.731479	3.500527	2.40E-03
	Regression	1.659424	0.477065	3.391316	0.018693
BEST		REG		MOM	
40	MLE	1.50306	0.184362	3.436737	0.007136
	Moment	1.556405	0.355055	3.502946	8.79E-04
	Regression	1.500762	0.182108	3.43237	0.007696
BEST		REG		MOM	
80	MLE	1.531236	0.065936	3.47274	0.001643
	Moment	1.565229	0.134749	3.501977	4.93E-04
	Regression	1.529203	0.065282	3.468705	0.001913
BEST		REG		MOM	
100	MLE	1.50272	0.050678	3.476985	0.000941
	Moment	1.533148	0.159062	3.502192	3.75E-04
	Regression	1.500694	0.05031	3.472647	0.001158
BEST		REG		MOM	

Table (6): $\alpha = 1.5 \quad \theta = 4$

n	Method	α	$MSE(\alpha)$	θ	$MSE(\theta)$
20	MLE	1.602384	0.368624	3.869891	0.032305
	Moment	1.825823	1.436017	4.015709	3.21E-03
	Regression	1.598837	0.362828	3.863818	0.034078
BEST		REG		MOM	
40	MLE	1.551459	0.185092	3.93784	0.007377
	Moment	1.605304	0.440727	4.002972	1.27E-03
	Regression	1.548679	0.183124	3.932044	0.00812
BEST		REG		MOM	
80	MLE	1.549335	0.101966	3.970651	0.001636
	Moment	1.619259	0.251243	4.005788	7.38E-04
	Regression	1.546586	0.100513	3.964991	0.002042
BEST		REG		MOM	
100	MLE	1.511176	0.076053	3.971001	0.001802
	Moment	1.52876	0.151724	4.002049	3.62E-04
	Regression	1.508754	0.075092	3.965391	0.002171
BEST		REG		MOM	

Table (7): $\alpha = 2 \quad \theta = 3$

n	Method	α	$MSE(\alpha)$	θ	$MSE(\theta)$
20	MLE	2.203633	0.877015	2.936145	0.008075
	Moment	2.510654	2.760199	3.012835	2.48E-03
	Regression	2.199787	0.869804	2.93423	0.00837
BEST		REG		MOM	
40	MLE	2.141178	0.345645	2.9709	0.00167
	Moment	2.313537	1.099686	3.006987	1.55E-03
	Regression	2.136453	0.341346	2.968616	0.001801
BEST		REG		MOM	
80	MLE	2.034213	0.137878	2.982707	0.000516
	Moment	2.139557	0.554344	3.004001	9.57E-04
	Regression	2.030249	0.136308	2.980478	0.000607
BEST		REG		MOM	
100	MLE	1.982068	0.133995	2.985069	0.000551
	Moment	1.969524	0.263091	2.99837	6.01E-04
	Regression	1.978385	0.132882	2.98279	0.000631
BEST		REG		MOM	

Table (8): $\alpha = 2 \quad \theta = 4$

n	Method	α	$MSE(\alpha)$	θ	$MSE(\theta)$
20	MLE	2.213872	0.653475	3.898859	0.01965
	Moment	2.798112	6.377826	4.022145	7.83E-03
	Regression	2.207895	0.644301	3.895201	0.020476
BEST		REG		MOM	
40	MLE	2.109576	0.409909	3.942162	0.007259
	Moment	2.266288	0.997309	4.010185	3.08E-03
	Regression	2.103151	0.401116	3.938412	0.007763
BEST		REG		MOM	
80	MLE	2.038608	0.150793	3.977039	0.001124
	Moment	2.114785	0.365654	4.004357	1.43E-03
	Regression	2.033227	0.148042	3.973238	0.001312
BEST		REG		MOM	
100	MLE	2.081594	0.142793	3.984962	0.00041
	Moment	2.095828	0.26858	3.999847	9.30E-04
	Regression	2.076102	0.140141	3.981166	0.000536
BEST		REG		MOM	

IV. Conclusion

From table one the best estimator is α (where MSE(α)=0.026804) REG and the best estimator for θ is $\hat{\theta}_{MOME}$,since MSE(θ)=0.002137 also from table two , we find $\hat{\alpha}_{RE}$ is best and $\hat{\theta}_{MOM}$ is best , where MSE($\hat{\alpha}_{RE}$)=0.024648 MSE($\hat{\theta}_{MOM}$)=0.002801

Also from table three, we find best estimator is $\hat{\alpha}_{RE}$ and $\hat{\theta}_{MOM}$, similarly from table four we find best estimator for α is $\hat{\alpha}_{RE}$ and for θ is $\hat{\theta}_{MOM}$

V. Recommendation

Since this constructed model represent a new generated Marshal – Olkin uniform family , and it have two parameters(α) shape parameter , θ is scale parameter so we use simulation taking sample size (n=20,40,80,100) and each experiment is Repeated 1000 times to estimate these two parameters , and the results are compared by statistical measure (Mean Square error) and the results is explained in tables .

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