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RESEARCH ARTICLE - MATHEMATICS

Numerical solution of singular Poisson equations using sinc approximation

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Abstract

In this article, we use the Sinc approximation to solve the single Poisson problem (where the first derivative or higher order does not have an exact answer on the border of the boundary). We have examined the Sinc Galerkin approximation to solve the single Poisson problem, and finally, to solve the Poisson problem, We will use the sinc collocation method and in this method, we will reach a linear system. By carefully choosing the length of the steps and the number of nodal points, we will solve this system with two methods, with the orthogonalization technique; a numerical approximation will be obtained, Its accuracy can be exponential and of exponential order. In the final part, we will give some numerical examples of single Poisson problems

Keywords: *Sinc approximation, Poisson problem, numerical approximation, Galerkin method*

1. Introduction

Boundary value problems have received considerable attention in various fields of science and engineering. These problems are very hard to solve because of the presence of singularities, very methods has been explained to conquest this difficulty. Therefore, among the existing techniques, sinc methods is said for boundary equations for treatment singularities [1]. Singular boundary value equations are discussed and solved by several methods such as modified a domain decomposition for higher-order ordinary differential equations [2-4], variational method [5], Bernstein's iterative method and polynomial decomposition scheme [6], cubic B-spline approximations [7], quartic B-spline numerical method [8,9], quantitative B-spline polynomial [10] and a novel differential transform approach [11].

Frank Stenger introduced Sinc methods in [12]. An applied problem in physiology that is singular boundary value problems is discussed and analyzed in [13] by non-classical sinc method. Numerical method for solving third-order boundary equations using sinc-collocation method expressed in [14]. Numerical solution methods by sinc nystrom formula explained and analyzed for some kind of Fredholm integral equations over infinite certain intervals [15].

2. The sinc function preliminaries

Sinc methods based on the cardinal (Whittaker) function.
From the symbol

$$S(k.h)(x) = \frac{\sin(\pi(x-kh)/h]}{\pi(x-kh)/h} \tag{1}$$

It used to introduce the sink function, where h is a positive number and k is an integer. *S(k.h)(x)* is called the eminence of the sinc function with step length. For the defined and bounded function $f\in(-\infty.\infty)$, the cardinal function of function f is defined as follows:

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)S(k.h)(x). \tag{2}$$

From the aspect of derivation, integration, etc., the cardinal function plays important role for sinc procedure as polynomials for most classical schemes. Polynomials, many operators on $C(f,h)$ implicitly transferred and each of these results leads to one of the explicit approximation methods. The function $C(f,h)$ is equivalent to many approximation formulas such as the trapezoid rule, many techniques obtained from the field of signal processing, etc.

2.1 Interpolation and exact quadrature for Paley wiener class functions

Definition 1. Suppose the function f is defined in real numbers and $h>0$ is given. We define the following series,

$$C(f,h)(x) \equiv \sum_{k=-\infty}^{\infty} f(kh) sinc(\frac{x-kh}{h})$$

Which we have from relation (1).

$$sinc\left(\frac{x-kh}{h}\right) \equiv \begin{cases} \frac{\sin(\pi(x-kh)/h)}{\pi(x-kh)/h}, & x \neq kh \\ 1, & x = kh \end{cases}$$

Wherever the series (2) converges, it called cardinal function f.

Theorem 1. Suppose $\phi' F \in B(D)$ and $h > 0$. Let ϕ be a one-to-one cosine mapping from the domain D to Ds. Suppose $\Gamma = (\mathbb{R})$, $w_k = \psi(kh)$, $\psi = \phi^{-1}$ then for each $\xi \in \Gamma$ the limit of the relation

$$\epsilon(\phi' F)(\xi) \equiv F(\xi) - \sum_{k=-\infty}^{\infty} F(w_k) sinc(\frac{\phi(\xi) - kh}{h})$$

As follows:

$$\|\epsilon(\phi' F)\|_\infty \leq \frac{N(\phi' F, D)}{2\pi d sinh(\pi d/h)}$$

In addition, suppose there are constant values β.α and C that

$$|F(\xi)| \leq C \begin{cases} \exp{(-\alpha|\phi(\xi)|}, & \xi \in \Gamma_\alpha \\ \exp{(-\beta|\phi(\xi)|}, & \xi \in \Gamma_b \end{cases}$$

If we consider the following choices.

$$N = \left[\left|\frac{\alpha}{\beta} M + 1\right|\right],$$

$$h = \left(\frac{\pi d}{\alpha M}\right)^{1/2} \leq \frac{2\pi d}{\ln{(2)}}$$

Then for every ξ∈Γ we have:

$$\epsilon_{M,N}(\phi' F)(\xi) \equiv F(\xi) - C_{M.N}(F, h, \phi)(\xi)$$

$$C_{M,N}(F, h, \phi)(\xi) \equiv \sum\nolimits_{k=-\infty}^{\infty} F(\omega_k) sinc(\frac{\phi(\xi) - kh}{h})$$

And also

$$\|\epsilon_{M.N}(\phi' F)\|_\infty \leq K_5 M^{1/2} exp\left(-(\pi d \alpha M)^{1/2}\right)$$

Which K_5 is arbitrary fixed number dependent on d, ϕ and D.

3. Solving Poisson problem

The sinc approximation method for solving the single Poisson equation was first used by F. Stenger and J. Lund. Stenger used the Sinc-Galerkin method for this problem and arrived at a linear system. Lund also achieved a symmetric linear system by using the same method and choosing appropriate weight functions. Both of them reached the same exponential convergence speed in their works. Before we solve the single Poisson problem using the collocation method of the sinc, it will be very useful if Let us examine Sinc Galerkin method.

3.1 Solving the Poisson problem using the Sinc-Galerkin method

First, for a non-negative integer *k*, we put:

$$\delta_{jl}^{(k)} = h^k \frac{d^k}{d\phi^k}\left[S(j,h)o\,\phi(w)\right]\Big|_{w=x_l}. \tag{3}$$

Using the following relation,

$$S(k,h)o\phi(x) = \frac{1}{2}\int_{-1}^{1} e^{t\frac{\pi}{h}[\phi(x)-kh]t}\,dt, \tag{4}$$

We have:

$$\delta_{jl}^{(0)} = \begin{cases} 1 & if \quad j = l, \\ 0 & if \quad j \neq l, \end{cases}$$

$$\delta_{jl}^{(1)} = \begin{cases} 0 & if \quad j = l \\ -\dfrac{(-1)^{j-l}}{j-l} & if \quad j \neq l \end{cases}$$

$$\delta_{jl}^{(2)} = \begin{cases} -\dfrac{\pi^2}{3} & if \quad j = l \\ -\dfrac{2(-1)^{j-l}}{(j-l)^2} & if \quad j \neq l \end{cases}$$

$$\delta_{jl}^{(3)} = \begin{cases} 0 & if \quad j = l \\ -\dfrac{(-1)^{j-l}}{(j-l)^3}\left[6-\pi^2(j-l)^2\right] & if \quad j \neq l \end{cases}$$

$$\delta_{jl}^{(4)} = \begin{cases} \dfrac{\pi^4}{5} & if \quad j = l \\ -\dfrac{(-1)^{j-l}}{(j-l)^4}\left[24-4\pi^2(j-l)^2\right] & if \quad j \neq l \end{cases}$$

The original Galerkin method for solving the desired problem on $S=(0,1)\times(0,1)$ is as follows:

$$\begin{aligned} &\nabla^2 u(x,y) = f(x,y), && (x,y)\in S \\ &u(x,y) = 0, && (x,y)\in \partial S \end{aligned} \tag{5}$$

A set of basic functions as,

$$\left\{S_k(x), S_q(y)\right\}_{-N\le k,q\le N}$$

And we can define an approximate solution for (5) as follows:

$$u_m(x,y)=\sum_{k=-N}^{N}\sum_{q=-N}^{N}u_k^q S_k(x)S_q(y), \qquad m=2N+1 \tag{6}$$

We know that u_k^q in (6) are determined with the assumption $\nabla^2 u_m - f$ must be orthogonal to the basic functions.

$$\left(\nabla^2 u_m - f, S_k S_q\right)=0, \qquad -N\le k,q\le N \tag{7}$$

The above inner multiplication in (7) also defines the basic functions of our method. In order to obtain basic functions on $(0,1)$, we define the following isometric mapping

$$w=\phi(z)=\ln\left(\frac{z}{1-z}\right) \tag{8}$$

Basic functions in Sinc Galerkin method defined as follows:

$$S_k(x)=S(k,h)o\phi(x), \qquad S_q(x)=S(q,h)o\phi(y) \tag{9}$$

This inner multiplication in (7) for the Sinc Galerkin method defined as follows:

$$\left(u,v\right)=\iint_s u(x,y)v(x,y)\frac{dxdy}{\phi'(x)\phi'(y)} \tag{10}$$

3.2. Two-dimensional Galerkin sinc methods:

In this section, to solve the Poisson problem in two dimensions, we first define a group of one-dimensional problems as follows:

$$\begin{aligned} u_{xx}(x,y_j)&=f_j(x)\equiv f(x,y_j)-u_{yy}(x,y_j),\\ u_{yy}(x_i,y)&=g_i(y)\equiv f(x_i,y)-u_{xx}(x_i,y)\end{aligned} \tag{12}$$

Which has zero boundary conditions for every *i* and *j*.
For each of the problems (12), we define the following approximate solution:

$$\begin{aligned} u_y^{(j)}(x)&=\sum_{k=-N_x}^{N_x}u_k^{(j)}S(k.h_x)o\phi(x),\\ u_x^{(i)}(y)&=\sum_{q=-N_y}^{N_y}v_q^{(i)}S(q,h_y)o\phi(y).\end{aligned} \tag{13}$$

The coefficients in (12) obtained from the following systems:

$$\begin{aligned} D\left(\phi'\left(x_p\right)^2\right)B_m\vec{u}_j&=h_x^2\overrightarrow{f^{\,j}}, \qquad -N_x\le j\le N_x\\ D\left(\phi'\left(y_p\right)^2\right)B_m\vec{v}_i&=h_y^2\overrightarrow{g^{\,i}}, \qquad -N_y\le i\le N_y\end{aligned} \tag{14}$$

$$\overrightarrow{f^{\,j}}=\left(\left(f-u_{yy}\right)\left(x_{-N_x},y_j\right),\ldots,\left(f-u_{yy}\right)\left(x_{N_x},y_j\right)\right)^t \text{ و } \overrightarrow{u^{j}}=\left(u_{-N_x}^j,u_{-N_{x+1}}^j,\ldots,u_{N_x}^j\right)^t$$

Where $\overrightarrow{v^i}$ and $\overrightarrow{g^i}$ are similarly. If $u(z, y_j)$, $u(x_i, z)$ apply in the conditions of theorem 1 and $h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}$ then with regards to the law of interpolation we can write:

$$\begin{aligned} \left|u_y^{(j)}(x) - u(x, y_j)\right| \le C_y e^{-\sqrt{\pi d \delta_j N_x}} \\ \left|u_x^{(i)}(y) - u(x_i, y)\right| \le C_x e^{-\sqrt{\pi d \gamma_i N_y}} \end{aligned} \quad (15)$$

If the exact answer (5) applies in the following relation

$$|u(x, y)| \le K\left(x(1-x)y(1-y)\right)^{\delta}$$

And if γ_i, δ_j on the right side (15) is replaced by δ and we put $h_x = h_y = h$ and $N_x = N_y = N$, we get the following matrix systems:

$$\begin{aligned} D(\varphi'(x_p)^2)B_m U = h^2\left[F - U^{yy}\right], \\ D(\varphi'(y_p)^2)B_m U^t = h^2\left[F - U^{xx}\right]^t \end{aligned} \quad (16)$$

.($p = xx$ یا yy) که $U^p = \left(u_p(x_i, y_j)\right)_{-N \le i, j \le N}$ و $F = \left(f(x_i, y_j)\right)_{-N \le i, j \le N}$

By transposing the second equation (16) and $U^{xx} + U^{yy} = F$, we get the following relation:

$$D(\phi'(x_p)^2)B_m U + U B_m^t D(\phi'(y_p)^2) = h^2 F \quad (17)$$

4. Scheme of solving and selecting parameters

The method of finding the answer of the system (17) and the method of parameter selection that is effective in reducing the program execution time are discussed; For simplicity, if we consider the following relation as the Sinc-Galerkin focal approximation

$$u_c(x, y) = \sum_{k=-N_y}^{N_y} \sum_{g=-N_x}^{N_x} u_k^g S(k, h) o \phi(x) S(g, h) o \phi(y) \quad (18)$$

The following approximate solution represents the non-focal Sinc-Galerkin approximation

$$u_{nc}(x, y) = \sum_{k=-M_y}^{N_y} \sum_{g=-M_x}^{N_x} u_k^g S(k, h) o \phi(x) S(g, h) o \phi(y) \quad (19)$$

First, we solve the system (17) for the focal state ($N_x = N_y$), because all the variables in (17) are matrices with dimension $m \times m$ or $m = 2N_x + 1$, we skip writing the dimension of the matrix in this part, we put $C = D\left((\phi')^2\right)B$, then there exists the matrix E^{-1} such that

$$C = E^{-1}\Lambda E$$

Which $\Lambda = diag(\lambda_{-N_x}, \lambda_{-N_x+1}, ..., \lambda_{N_x})$ are eigenvalues of *C*. By replacing $Y = E^{-1}UE$ in the system (17), we get the following equations to determine *Y*:

$$\Lambda Y + Y\Lambda = h^2 W \equiv h^2\left(EFE^t\right)$$

Therefore, the answer *U* is as follows:

$$U = E^{-1}Y\left(E^{-1}\right)^t \quad (20)$$

4.1. Results of the numerical solution of the single Poisson problem

In this section, we use the sinc approximation method to numerically solve single Poisson problems, in the examples of this section, the exact solution has undetermined second derivatives on the boundary.

The parameter $\alpha > 0$ should be in such a way that the exact solution *u(x,y)* is applied in the following conditions:

$$|u(x,y)| \le M\left((x-a)(b-y)\right)^{\alpha+\frac{1}{2}} \qquad (21)$$

We take the parameter *h* equal to $\left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}$, where *d* is chosen equal to $\frac{\pi}{2}$ in the investigated problems.

The absolute value of the maximum error between the numerical approximation $(v_N(x,y))$ obtained from the above method and the analytical solution $(u(x,y))$ of the problem at nodal points sinc with $\|E\|_g$ and the absolute value of the maximum error in 100 equidistant points with $\|E\|_u$, which are equidistant points (x_m, y_n) and $x_m = a_1 + \frac{m}{100}(b_1 - a_1)$, $y_n = a_2 + \frac{n}{100}(b_2 - a_2)$ are chosen.

Example 1. Consider the following Poisson problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \qquad u(x,y) = (1-x^2)^{\frac{7}{4}}(y-1)^2(4-y)^2,$$

That $\Omega = (-1,1)\times(1,4)$ and $\alpha = \frac{5}{4}$. The results shown in Table 1.

Table 1

N	h	$\|E\|_g$	$\|E\|_u$
2	1.404963	1.0802D-1	1.1046D-1
4	0.993459	1.1282D-1	1.1887D-2
8	0.702481	1.0286D-3	1.0048D-3
16	0.496729	2.4315D-5	2.3147D-5
32	0.352113	4.5137D-8	8.3172D-8

Example 2. Consider the following Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \qquad u(x,y) = (x^{\frac{3}{2}} - x)(y^{\frac{3}{2}} - y),$$

Where . $\Omega = (0,1)\times(0,1)$

In this example, u has indefinite derivatives in $x = 0$ and $y = 0$. Problems of difference method for elliptic equations presented in the works of Wieser in [16] and Kaufman and Warner in [18] and Dyksen in [17].

In all these researches, the numerical approximation error of this problem is significantly higher than similar analytical problems. Parameter $\alpha = \frac{1}{2}$, and the results are displayed in the Table 2.

Table 2

N	H	$\|E\|_g$	$\|E\|_u$
2	2.221441	2.5315D-4	7.4670D-4
4	1.570796	7.0055D-5	1.2511D-4
8	1.110721	7.7647D-6	1.3159D-5
16	0.785398	2.5963D-7	5.1589D-7
32	0.555360	1.7498D-9	8.6613D-9

Conclusion

These methods are effective in solving and removing the singularity and work well in confronting with this type of problem, considering that the singularity of the equation occurs at the end point of the interval. Since most of the sink grid points clustered in the neighborhood of the endpoints of the interval, it helps us to control the singularity well. Numerical examples also confirm the efficiency of the method.

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