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Closed-Small Submodules and Closed-hollow Modules

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ABSTRACT: The aim of this study is to present the concept of closed-small submodule and closed-hollow module as generalization of small and hollow concepts respectively. As evidence, attributes of these ideas. Moreover, we study the concept of the closed-Radical of M as a generalization of Radical module M.

Keywords: Small submodule, closed-small submodule, hollow module, closed-hollow module, not closed-maximal (nc −maximal), closed-Radical of module M.

1. INTRODUCTION

Let R is any ring with identity and M be a unitary left R-module. A proper sub-module N of an R-module M is called small $(N \ll M)$, if for any sub-module K of M that away $M = N + K$ there are $K = M$ [1] .A proper sub-module is referred to be a hollow module if each of its proper sub-modules is small in M [2]. A sub-module N is essential in M $(N \leq_e M)$ if for any $X \leq M, N \cap X = 0$ Implies $X = 0$ [1]. Recall that the sub-module $N$of an R-module M is called closed if N is not properly extended inside M, which is the only solution of the relation $N \leq_e K \leq M$ is $K = M$ [2].

In the study of many subjects has excited the curiosity of many authors of generalizations of small sub-modules See [3], [4], [5], [6] , [7] , [8], [9] . As in [3], we will use closed sub-module to introduce a new generalization of small sub-module namely closed-small sub-module. That allows us to introduce a closed-hollow module as an extension of hollow module.

Which essay, we discuss some basic of these properties in relation of hollow module. In the last section, we introduce and study the concepts of closed-maximal sub-module and closed-radical of M.

2. CLOSED – SMALL SUB MODULE

This section introduces the idea of a closed-small submodule as a generalization for the small sub-module notion. Also, we study an important property of this submodule type.

Definition (2.1): A proper sub-module N of an R-module $M$is called closed-small (c-small) sub-module ($N \ll_c M$), If $N + K = M$ where K is a sub-module of M , then K is a closed sub-module in M.
Or a proper sub-module N is a closed – small of M if for every not closed sub-module K of M, $N + K \neq M$.

Remarks and Examples (2.2):

1- Since M is closed in M hence every small sub-module is closed- small, but
The opposite is untrue , like the example below :
In Z_6 as Z-module, $\{\bar{0},\bar{3}\}$ is closed-small in Z_6 , since $\{\bar{0},\bar{3}\}+\{\bar{0},\bar{2},\bar{4}\}=Z_6$ and $\{\bar{0},\bar{3}\}+Z_6=Z_6$ and $\{\bar{0},\bar{2},\bar{4}\}, Z_6$ are closed in Z_6.

2- If M is semi simple, then every proper submodule of $M$is closed-small.

3- In Z_{12} as Z-module , $\{\bar{0}\}, 2Z_{12}, 4Z_{12}$ are closed – small , since $2Z_{12}+3Z_{12}=Z_{12}$ and $2Z_{12}+Z_{12}=Z_{12}$ such that $3Z_{12,}Z_{12}$ are closed in Z_{12} , similarly $4Z_{12}+3Z_{12}=Z_{12}$ and $4Z_{12}+Z_{12}=Z_{12}$, but $3Z_{12}$ is not closed-small, since $3Z_{12}+2Z_{12}=Z_{12}$ and $2Z_{12}$ is not enclosed in Z_{12}.

4- In Z_8 as Z-module, $\{\bar{0}\}, 2Z_{8\,,}4Z_8$ are closed-small, since $2Z_8+Z_8=Z_8$ and $4Z_8+Z_8=Z_8$ and Z_8 is closed in Z_8.

5- In Z_{24} as Z-module , since $(\bar{0}), (\bar{6}), (\overline{12})$ are small submodules so are closed-small , also $(\bar{2})$ and $(\bar{4})$ are closed-small since $(\bar{2})+(\bar{3})=Z_{24}$, and $(\bar{2})+Z_{24}=Z_{24}$, similarly $(\bar{4})+(\bar{3})=Z_{24}$ and $(\bar{4})+Z_{24}=Z_{24}$ and $(\bar{3})$ is closed in Z_{24}.

In the following propositions we give a property of closed-small submodule.

Proposition (2.3): Let N and K are sub-modules of a module M such that $K \le N \le M$, if $N \ll_c M$,then $K \ll_c M$.

Proof: If U is a sub-module of M such that $K+U=M$**.** since $K \le N$, then $N+U=M$ and since $N \ll_c M$, then $U \le_c M$ and hence $K \ll_c M$.

Corollary (2.4): Let M be an R-module such that $K_1 \ll_c M$ or $K_2 \ll_c M$ then $K_1 \cap K_2 \ll_c M$

Proof: If $K_1 \ll_c M$, and since $K_1 \cap K_2 \le K_1 \le M$ then from Proposition (2.3), we have $K_1 \cap K_2 \ll_c M$.

Corollary (2.5): Let K_1, K_2 are submodules of an R-module M,if $K_1+K_2 \ll_c M$ then $K_1 \ll_c M$ and $K_2 \ll_c M$.
.

Proof: Since $K_1 \le K_1+K_2 \le M$, and $K_1+K_2 \ll_c M$, then from proposition (2.3), we have $K_1 \ll_c M$. similarly, $K_2 \ll_c M$.

Proposition (2.6): If M is an R-module and K and N are its sub-modules, such that $K \le N \le M$ if $N \ll_c M$, then $\frac{N}{K} \ll_c \frac{M}{K}$.

Proof : Let $\frac{W}{K}$ be a sub-module of $\frac{M}{K}$ such that $\frac{N}{K}+\frac{W}{K}=\frac{M}{K}$, and hence $N+w=M$, we have $W \le_c M$, then $\frac{W}{K}$ is closed in $\frac{M}{K}$ by [2].

Proposition (2.7): Let $f: M \to N$ be an isomorphism, if M and N are R-modules, such that $K \ll_c M$, then $f(K) \ll_c N$.

Proof: Let W be any sub-module of N such that $f(K)+W=N$ for some $W \le N$. thus $f^{-1}(f(K))+f^{-1}(W)=M$,so $M=K+f^{-1}(w)$ but $K \ll_c A$, then $f^{-1}(W)$ is closed and since f is isomorphism then by [7] $f(f^{-1}(W))=W$ is closed , so $f(K) \ll_c N$.

In [10] the closed intersection property is a property of an R-module M (CIP), if any two closed submodules of M intersection are once again closed.

Proposition (2.8): Let M be an R-module with property at closed intersections where A and B are sub-modules of M such that $B \le A \le M$, and A is the direct summand of M , if $B \ll_c M$, then $B \ll_c A$.

Proof: Suppose a sub-module of M is U such that $B+U=A$. Since A direct summand of M, then $M=A \oplus V$ where V be a sub-module of M, so $A+V=M$, then $(B+U)+V=M$, hence $B+(U+V)=M$, and since $B \ll_c M$, then $U+V \le_c M$. So $(U+V) \cap A \le_c M \cap A = A$, hence by modular law, we have $U+(V \cap A) \le_c A$, and since $A \cap V=0$, then $U \le_c A$ and hence $B \ll_c A$.

Proposition (2.9): Let M be an R-module and if M has CIP and A_1, A_2 are submodules of M,such that $A_1 \ll_c M$ and $A_2 \ll_c M$ then $A_1 \oplus A_2 \ll_c M$.

Proof: (⇒) Assume U be sub-module of M where that $A_1+A_2+U=M$. Since $A_1 \ll_c M$ and $A_2 \ll_c M$, then $A_2+U \le_c M$ and $A_1+U \le_c M$, so we have $(A_1+U) \cap (A_2+U) \le_c M$ by (CIP) , hence $U=(A_1+U) \cap (A_2+U) \le_c M$, so $U \le_c M$ and hence $A_1 \oplus A_2 \ll_c M$.

(⇐) Since $A_1 \le A_1 \oplus A_2 \le M$ and since $A_1 \oplus A_2 \ll_c M$, then from Proposition (2.3) we have, $A_1 \ll_c M$, similarly we have $A_2 \ll_c M$.

Proposition (2.10): Let M be an R-module ,such that $M=M_1 \oplus M_2$ and $R=Ann(M_1)+Ann(M_2)$, then $A_1 \ll_c M_1$ and $A_2 \ll_c M_2$ iff $A_1 \oplus A_2 \ll_c M_1 \oplus M_2$. .

Proof: (⇒) Assume U be a submodule of M such that $A_1 \oplus A_2+U=M$. Since $R=Ann\, M_1+Ann\, M_2$, then $U=K_1 \oplus K_2$for some $K_1 \le M_1$ and $K_2 \le M_2$ then $A_1 \oplus A_2+K_1 \oplus K_2=M_1 \oplus M_2$. So $(A_1+K_1) \oplus (A_2+K_2)=M_1 \oplus M_2$

and hence $A_1 + K_1 = M_1$ and $A_2 + K_2 = M_2$ and since $A_1 \ll_c M_1$ and $A_2 \ll_c M_2$, then $K_1 \leq_c M_1$ and $K_2 \leq_c M_2$, then $K_1 \oplus K_2 \leq_c M_1 \oplus M_2$, and hence $U = K_1 \oplus K_2 \leq_c M_1 \oplus M_2 = M$, so $U \leq_c M$ and hence $A_1 \oplus A_2 \ll_c M_1 \oplus M_2$.
(⇐) Let $A_1 \oplus A_2 \ll_c M_1 \oplus M_2 = M$, since $A_1 \leq A_1 \oplus A_2 \ll_c M_1 \oplus M_2 = M$ then by Proposition (1.3) we have, $A_1 \ll_c M$ and since $A_1 \leq M_1 \leq M$ and M_1 is direct summand of M then by Proposition (2.8), $A_1 \ll_c M_1$. Similarly, we have $A_2 \ll_c M_2$.

Definition (2.11): [11] An R-module M has the name faithful module if $Ann(M) = 0$.

Definition (2.12): [12] The R-module When N is a submodule of M and there is an ideal I of R such that N=IM, M is referred to as multiplication.

Lemma (2.13): Let M be a faithful and multiplication module, I in R is closed if and only if IM is closed submodule in RM.

Proof: Assume that $IM \leq_e L \leq RM$ since M is multiplication then $L = TM$, T is an ideal in R ,now $IM \leq_e TM \leq RM$, M is faithful multiplication, so $I \leq_e T \leq R$ but I is closed in R, so $I = T$.then $IM = TM$.so IM is closed submodule in RM.
⇐) Assume that $I \leq_e T \leq R$, since M is faithful multiplication [12] ,so $IM \leq_e TM \leq RM$, where $TM = IM$,so $T = I$ then I is closed in R.

Proposition (2.14): Let M be an R-module, If I be an ideal of R and is faithful, finitely generated, and a multiplication module, then $I \ll_c$ R iff $IM \ll_c M$.

Proof: (⇒) Let $I \ll_c$ R we must provide evidence for this $IM \ll_c M$. Let K be a submodule of M such that $IM + K = M$, M being a multiplication, then $K = UM$, U is an ideal of R .for A. then $IM + UM = M$ hence $(I + U)M = M$ and so, $(I + U)M = RM$ and since the A module for multiplication, M is faithful and finitely generated. Then we have $I + U =$ R by [9] , since $I \ll_c$ R then $U \leq_c$ R hence $UM \leq_c M$ by lemma (2.13), so $K \leq_c M$ hence $IM \ll_c M$.
(⇐) Let $IM \ll_c M$, we must prove that $I \ll_c$ R. If J were the ideal for R, then $I + J =$ R, since M is multiplication, then $IM + JM =$ RM and hence $IM + JM = M$, and since $IM \ll_c M$, then $JM \leq_c M$, so $JM \leq_c$ RM and we have $J \leq_c$ R by [12] and hence by (2.13) , $I \ll_c$ R.

Proposition (2.15): Let K be a submodule of an R-module M , Consequently, the following are equivalent.

1- $K \ll_c M$
2- If $K + X = M$, then X is a relative complement for some $A \leq M$
3- If $K + X = M$ then for any sub-module B of M, so that $X \leq B \leq_e M$ then $\frac{B}{X} \leq_e \frac{M}{X}$.

Proof: (1) (⇒) (2), Let $X + K = M$ and $K \ll_c M$, thus $X \leq_c M$ then by [2] X is relative complement for some $A \leq M$.
(2) (⇒) (3), since $B \leq_e M$ then $\frac{B}{X} \leq_e \frac{M}{X}$ by [2].
(3) (⇒) (1), Let $K + X = M$, since $\frac{B}{X} \leq_e \frac{M}{X}$,so by [2] $X \leq_c M$, thus $K \ll_c M$.

Proposition (2.16): Let $f: M \to N$ be an isomorphism where M and N be an R-modules such that $K \ll_c N$, then $f^{-1}(K) \ll_c M$.

Proof: Let A be any submodule of M such that $f^{-1}(K) + A = M$ for some $A \leq M$ thus $f(f^{-1}(K)) + f(A) = f(M)$,since f is isomorphism so $N = K + f(A)$, but $K \ll_c N$, then $f(A)$ is closed in N , so $f^{-1}(f(A)) \leq_c f^{-1}(N)$, implies that $A \leq_c M$ .so $f^{-1}(K) \ll_c M$.

3. CLOSED – HOLLOW MODULES

In [13] , [14] , [15] various writers provide the definition of hollow modules and semi hollow modules, e-Hollow module and large-hollow lifting module. In the same way give the following definition of closed-hollow.

Definition (3.1): The R-module M is called closed-hollow module if every proper sub-module of M is a closed-small in M.

Remarks and examples (3.2):

1- Each hollow module is a closed-hollow module because every small is a closed-small.
The opposite is not true, as shown by the example below.
As a Z-module in Z_6 , $\{\bar{0}, \bar{2}, \bar{4}\}$, $\{\bar{0}, \bar{3}\}$, $\{\bar{0}\}$ are submodules, so Z_6 is closed -hollow, but not hollow.
2- Every simple module is closed-hollow, since every proper submodule is closed-small.
3- If M is semi simple, then M is closed-hollow module.
4- The Z-module Zp^{∞} is closed-hollow module.

Proposition (3.3): Let M be a closed-hollow has CIP , then M has a closed-hollow direct summand.

Proof: Let A and B be appropriate sub-modules of M, so that $B \leq A$ and A is direct summand of M. M being a closed-hollow, then $B \ll_c M$,since A is the direct summand of M. Hence by proposition (2.8), $B \ll_c A$ so, A is closed-hollow.

Proposition (3.4): Let M_1 and M_2 be two R-modules such that $f: M_1 \to M_2$ the isomorphism, so M_2 will also be closed-hollow if M_1 is closed-hollow.
Proof: Assume M_2 has a proper sub-module called A. Thus $f^{-1}(A)$ is a proper sub-module of M_1. If not then $f^{-1}(A) = M_1$, so $f(f^{-1}(A) = M_2$ and this contradiction. Since M_1 is closed-hollow, then $f^{-1}(A) \ll_c M_1$, hence, by Proposition (2.7) we have $f(f^{-1}(A)) \ll_c M_2$, so $A \ll_c M_2$ and hence M_2 is c-hollow.
Definition (3.5): [16] A sub-module N of an R-module M is called invariant if for each $f \in End_R(M)$ and $f(N) \leq N$. Also is called fully invariant sub-module.
Definition (3.6): [17] If each submodule of M is fully invariant, then an R-module is referred to as a duo module.
proposition (3.7): Assume that M_1 and M_2 R-modules and that $M = M_1 \oplus M_2$in which M is a duo module and has (CIP), then M is closed-hollow iff, M_1 and M_2 are closed-hollow, given that $N \cap M_i \neq M_i$ for $i = 1, 2$ and $N \leq M$.
Proof: (⇒)Clearly by Proposition (3.3)
(⇐) Let N be a proper submodule of M and M_1, M_2 are closed- hollow. M being a duo module, then $N = (N \cap M_1) \oplus (N \cap M_2)$, hence $N \cap M_1$ and $N \cap M_2$ are proper submodules of M_1 and M_2, also since M_1 and M_2 are closed-hollow, then $N \cap M_1 \ll_c M_1$ and $N \cap M_2 \ll_c M_2$, so by Proposition(2.10), we have $(N \cap M_1) \oplus (N \cap M_2) \ll_c M_1 \oplus M_2$ and hence $N \ll_c M$.

4. CLOSED – RADICAL OF M

In this section we introduce a closed-maximal submodule and closed-Radical of M with some of its properties.
Definition (4.1): A proper non-closed submodule N of M is called non-closed-maximal (nc-maximal) submodule o M if $N < K \leq M$ where K any submodule of M.Then $K = M$.
Remarks and examples (4.2):
1.Every nc-maximal submodule is a maximal submodule.

Proof: Let N be any nc −maximal submodule of M such that $N < W \leq M$ then $W = M$.
As shown by the example below, the opposite is not true. In Z_6 as Z-module. $(\bar{2}), (\bar{3})$ are maximal submodule of Z_6 but are not nc-maximal submodule of Z_6 since $(\bar{2}), (\bar{3})$ are closed- submodules of Z_6.

1. In Z as Z-module every non-zero proper sub-module is a non-closed then every maximal submodule is nc-maximal, thus every nZ ,where n is a prime integer is nc-maximal.
2. If M is semisimple module, then M has no nc-maximal.
3. Every proper non-zero of $Z_{P\infty}$ is non-closed and $N < W \leq Z_{P\infty}$ where $W \leq M$, so $Z_{P\infty}$ has no nc-maximal submodule.
4. In Z_4 as Z-module: $\{\bar{0}, \bar{2}\}$ is nc-maximal submodule in Z_4.
5. In Z_{24} as Z-module: $(\bar{2})$ is nc-maximal. But $(\bar{3})$ is not since is closed, also $(\bar{4}), (\bar{6}), (\bar{8})$. Since are not maximal.
6. In Z_{36} as Z-module: $(\bar{2}), (\bar{3})$ are nc-maximal.
7. In Z_{48} as Z-module: $(\bar{2}), (\bar{3})$ are nc-maximal.
8. If $\frac{M}{N}$ is simple, then N is maximal sub by [20] and hence N is nc-maximal sub-module by (1).
9. If M is uniform, then not every submodule of M is nc-maximal, for example: Q as Z-module, Q is uniform $Z \leq \frac{1}{2} Z \leq Q$,Then Z is not nc-maximal in Q (since $\frac{1}{2}Z \neq Q$).

Now we introduce the definition of closed -radical of M as a generalization of $Rad(M)$ [2]
In [18], [19] , [20] , [21], [22], many authors study the notions of Some Results on the Jacobson Radical and the M-Radical, additionally, provide the following definition.

Definition (4.3): Assuming M is an R-module, the closed-Radical of M is represented by $Rad_c(M)$ in such a way that it is the sum of all closed-small submodules.
$Rad_c(M) = \sum N \leq M \ ; \ N \ll_c M\}$.
Remarks and examples (4.4):

1. In Z as Z-module: $Rad_c(Z_6) = Z_6$, since all submodule in Z_6 are not small.
2. In Zas Z −module: $Rad_c(Z) = \{\bar{0}\}$,since the only closed-small submodule in Z is $\{\bar{0}\}$.

3. In Z_{12} as Z-module: $Rad_c\ (Z_{12}) = (\bar{2})$.Since the only closed-small of Z_{12} are $(\bar{2}), (\bar{4})$.
4. $Rad_c(Z_8) = (\bar{2})$.
5. Every small is closed -small then every radical of M is closed- radical of M.

As shown by the example below, the opposite is not true.: $Rad_c\ (Z_6) = Z_6$, but $Rad(Z_6) = (\bar{0})$.

Proposition (4.5):

Let M be an R-module. If $f: M \to N$ is an isomorphism, then $Rad_c(N) = f\big(Rad_c(M)\big)$.

Proof: For any $L \ll_c M$, $f(\,L) \ll_c N$, hence $f(L) \le Rad_c(N)$. thus $f(Rad_c(M) = f(\,\sum_{L\ll_c M} L) = \sum_{L\ll_c M} f(L) \le Rad_c(N)$. Thus$f(Rad_c(M)) \le Rad_c(N)$, Now to show $Rad_c(N) \le f(Rad_c(M))$,
Let $L\ \le Rad_c(N)$, Then $L \ll_c N$and $f^{-1}(\,L) \ll_c M$ by proposition (2.16) and so $f^{-1}(\,L) \le Rad_c(M)$.
It follows that $f(f^{-1}(L) \le f\big(Rad_c(M)\big)$. Thus $L \le f(\,Rad_c(M)$. That is for each $L \ll_c N, L \le f(Rad_c(M))$.
Therefor $f\big(Rad_c(M)\big) \le Rad_c(N)$.

Theorem (4.6): $Rad_c = \cap\{N \le M\ ;\ N$ is nc-maximal$\}$.

Proof:((⇒) Suppose that $U = \sum\{\,N \le M | N \ll_c M\}$. Let $L \ll_c M$ and K any not closed submodule then $L \le K$, if not $K + L = M$,since $L \ll_c M$, implice that $K$is closed in M contradiction, thus $L \le K$ and $U \subseteq Rad_c(M)$.

(⇐) Assuming $X \in Rad_c(M)$, we must have to show that Rx is closed-small, if not
Let $\Gamma = \{B | B$ not closed, $B\ \le\ M$ and $Rx + B = M\}$, so $\Gamma \ne \emptyset$ since Rx is not c-small,
According to Zorn`s lemma Γ has a maximal element say B_0 ,claim B_0 is maximal but not closed in M.
If not there exists a submodule $C \le M$ such that $B_0 \lneqq C \le M$ then $Rx + C \ge Rx + B_0 = M$,So B_0 is maximal and not closed in M. Thus $X \in B_0$,since $Rx + B_0 = M$ .Thus $B_0 = M$ contradiction, so $Rx \ll_c M$ and $Rad_c(M) \subseteq U$.

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