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RESEARCH ARTICLE -MATHEMATICS

Mixed Crank-Nicolson and Galerkin Methods for Solving Nonlinear Hyperbolic Partial Differential Equation

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Abstract

In this work, the approximation solution for nonlinear hyperbolic partial differential equation (NLHPDE) is obtained by using the mixed Crank-Nicolson (CN) scheme and the Galerkin Method (GM) and it is symbolized by (MCNGM). At first the CN is utilized for the variable of time to obtain the discrete weak form for the NLHPDE, and then the GM is utilized which reduces the DWF into the Galerkin nonlinear algebraic system (GNLAS) at each step of time. Through utilizing the predictor-corrector techniques which are symbolized by the obtained GNLAS is transformed into Galerkin linear algebraic system (GLAS) which is solved by applying the Cholesky method. The convergence of the method is studied. Some examples are given to illustrate the efficiency and the accuracy for the proposed method.

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Keywords: Nonlinear Hyperbolic Partial Differential Equation, method of Galerkin finite elements, Crank-Nicolson Scheme, Convergent.

1. Introduction

As it is known many natural phenomena usually are described by mathematical problems represent by nonlinear PDEs (NLPDEs) in general and by NLHPDEs in particular. Of course there is a need to solve such problem often; but in fact solving them analytically is difficult if it is not impossible. Therefore, numerical and approximate methods became an urgent need to find the approximate solution of such mathematical problems.

In the previous decades many aproximate and numerical methods were used for solving the PDEs in general; like the finite difference method (FDM) [1][2][3][4], method of lines [5][6], meshless method [7][8], and many other methods. The CN and GM have been used in many applications like: digital image processing [9], reactors [10], power system [11] and in groundwater [12]. In the recent years many other methods have been introduced to solve such problems, more precisely in 2018, GM has been used to solve the hyperbolic partial differential equation [13], in 2019, the MCNGM has been applied to solve NLPDEs of parabolic type [14], while in 2020 [15], used the CN with heredity and its program implementation for solving PDEs, also in 2020 the CN with the FDM analyzed and proposed for a NLPDEs of integral type [16], also in 2020 the CN was applied to a random component heat equation [17], in 2021 the Galerkin-Implicit Methods [18], also in 2021 MCNGM are used for nonlinear time fractional parabolic problems [19], in 2022 the mixed method of homotopy perturbation and the variation iteration method [20] used to solve NLHPDEs, also in 2022 the time fractional telegraph equation is solved numerically based on the CN [21], also in 2022 the MCNGM are used for Maxwell's equations [22] and later in 2022 CN is used to solve uncertain heat equation [23].

In this work, the MCNGM is used to find the approximation solution of the NLHPDE. In the proposed method, at first the CN scheme is utilizedtoobtain the discrete weak form, and then the GM is utilized to get the GNLAS at each step of time. Then the predictor (PrT)- corrector (CoT) techniques (PrCoT) is applied to solve this GNLAS, in the PrCoT, the GNLAS transforms to a GLAS, which is solved by using the cholesky method. The convergence of the method (the PrCoT) is proved. Some examples are given to illustrate the efficiency and the accuracy for the method. Also, the numerical results are given.

2. Basic Concepts

Definition1 [24]: A point $n^* \in Q \subset R^2$ is called a fixed point (FP) of the function $Q \to R^2$if $y(n^*) = n^*$.

Definition2 [25]: A function $y: Q \subset R^2 \to R^2$ is called contractive on P if $\forall\, p_1, p_2 \in Q$: $\|y(p_2) - y(p_1)\| \leq \alpha\|p_1 - p_2\|$, where $\alpha \in (0,1)$.

Theorem1 [26]: A contractive function y on a complete normed space Q has a unique FP n^*in Q.

Theorem2 [27]: Let $\{v_n\}$ be a bounded sequence in the space $L^\infty(Q)$. Then, there exists a subsequence $\{\acute{\theta}\}$ and a function $v_0 \in L^\infty(Q)$ s.t, $v_{\acute{\theta}} \to v_0$ in $L^\infty(Q)$.

3. Description of the NLHPDE

Let $S = [0, T]$, with $0 < T < \infty$, $Q \subset R^2$ be a bounded region with boundary $\partial Q, \vartheta = Q \times S, E = \partial Q \times S$. Then, the NLHPDE is given by:

$$X_{tt} - \Delta X + X = g(\vec{r}, t, X),\ \ in\ \vartheta \tag{1}$$

$$X(\vec{r}, 0) = X^0(\vec{r}), in\ Q \tag{2}$$

$$X_t(\vec{r}, 0) = X^1(\vec{r}), in\ Q \tag{3}$$

$$X(\vec{r}, t) = 0, on\ E \tag{4}$$

where $X = X(\vec{r}, t) \in H_0^2(Q), \Delta X = \sum_{i=1}^{2} \frac{\partial^2 X}{\partial r_i^2}, \vec{r} = (r_1, r_2) \in R^2, 0 < r_1, r_2 < 1$, and $g \in L^2(Q)$.

Now, let $V = H_0^1(Q) = \{\sigma: \sigma \in H^1(Q), \sigma = 0\ on\ \partial Q\}$ and $X_t = p$. Then, the weak form of (1-4) is:

$$\langle X_{tt}, \sigma\rangle + (\nabla X, \nabla\sigma) + (X, \sigma) = (g(X), \sigma), \forall\sigma \in V \tag{5}$$

$$(X(0), \sigma) = (X^0, \sigma)\ \ in\ Q, X^0 \in V \tag{6}$$

$$(p(0), \sigma) = (X^1, \sigma)\ \ in\ Q, X^1 \in L^2(\vartheta) \tag{7}$$

The following assumptions are necessary to study both the existence and the convergence of the solution.

Assumptions (ASM):

(1) Let q_1 and q_2 be a nonnegative constants that satisfy the following:

a) $|(\nabla \mathrm{X}, \nabla\sigma)| \leq q_1\|\nabla \mathrm{X}\|_1\|\nabla\sigma\|_1$, $\forall X, \sigma \in V$

b) $(\nabla \mathrm{X}, \nabla \mathrm{X}) \geq q_2\|\nabla \mathrm{X}\|_1^2$, $\forall$ X $\in V$

(2) g is continuous w.r.t $X_j^n$and defined on $Q \times R$, and satisfies:

a)$\left|g(\vec{r}, t_j^n, X_j^n)\right| \leq e(\vec{r}, t) + \gamma\left|X_j^n\right|$where, $\gamma > 0, X_j^n \in Q$ and $e \in L^2(\vartheta)$.

b)$\left|g\left(\vec{r}, t_j^n, X_j^n\right) - g(\vec{r}, t_j^n, X_i^n)\right| \leq L\left|X_j^n - X_i^n\right|$,where L is a Lipchitz constant and $X_j^n, X_i^n \in Q$.

4. Discretization of the Problem:

In this section, the method of GM is applied for discretizing ((5)-(7)). Let ϑ be divided into sub regions $\vartheta_{ij} = Q_i^n \times S_j^n, and\ let \{Q_i^n\}_{i=1}^{N(n)}$ be a triangulation of $\bar{Q}$ and $\{S_j^n\}_{j=0}$ be a subdivision of the interval S into $Y(n)$ intervals. Then, $S_j = S_j^n \coloneqq [t_j^n, t_{j+1}^n]$ has the same lengh $\Delta t = \frac{T}{Y}$. Also, let $V_n \subset V = H_0^1(Q)$ be the space of piecewise affine functions in Q.

Now, the following weak form is obtained from using CN formula and is given $\forall\sigma \in V_n$as:

$$\left(p_{j+1}^n - p_j^n, \sigma\right) + \Delta t\left(\nabla X_{\frac{1}{2}j}^n, \nabla\sigma\right) + \Delta t\left(X_{\frac{1}{2}j}^n, \sigma\right) = \Delta t\left(g(t_j^n, X_{\frac{1}{2}j}^n), \sigma\right) \tag{8}$$

Since, $\Delta t\left(p_{j+1}^n\right) = X_{j+1}^n - X_j^n$ then, (8) becomes

$$\left(X_{j+1}^n - X_j^n, \sigma\right) + (\Delta t)^2\left(\nabla X_{\frac{1}{2}j}^n, \nabla\sigma\right) + (\Delta t)^2\left(X_{\frac{1}{2}j}^n, \sigma\right) =$$

$$(\Delta t)\left(p_j^n, \sigma\right) + (\Delta t)^2\left(g(t_j^n, X_{\frac{1}{2}j}^n), \sigma\right) \tag{9}$$

Where $X_{\frac{1}{2}j}^n = \frac{1}{2}\left(X_{j+1}^n + X_j^n\right)$

$$(X(0), \sigma) = (X^0, \sigma), \text{ in} Q \tag{10}$$

$$(p(0), \sigma) = (X^1, \sigma), \text{ in} Q \tag{11}$$

where$X^0 \in V, X^1 \in L^2(Q)$,$X_j^n = X^n\left(r, t_j^n\right), and p_j^n = p^n\left(\vec{r}, t_j^n\right) \in V_n,\ \forall j = 0,1,\cdots, Y-1.$

5. The Approximation Solution of the NLHPDE:

In this part, the MCNGM is used to find the APSOL $\bar{X}^n = (X_0^n, X_1^n, \cdots, X_y^n)$ for the DWF (8-11) through the following steps:

1- From the basis of V_n and by using the MCNGM, let $\bar{X}^n\left(\vec{r}, t_j^n\right)$(with$\bar{X}_t^n\left(\vec{r}, t_j^n\right) = \bar{p}\left(\vec{r}, t_j^n\right)$)

be an APSOL of ((8)-(11)) s.t:

$\bar{X}^n\left(\vec{r}, t_j^n\right) = \sum_{k=1}^N f_k^j \sigma_i$and $\bar{p}^n\left(\vec{r}, t_j^n\right) = \sum_{k=1}^N w_k^j \sigma_i$,$\forall \mu_i \in V_n$,

where $f_k^j = f_k(t_j^n)$, and $w_k^j = w_k(t_j^n)$ are unknown to be determined,$\forall j = 0,1,\cdots, Y-1$.

2-Using the APSOL in ((8)-(11)), $\forall j = 0,1,\cdots, Y-1$, given:

$$\left(\left(1 + \frac{1}{2}(\Delta t)^2\right)D + \frac{1}{2}(\Delta t)^2 Z\right)F^{j+1} =$$

$$\left(\left(1 - \frac{1}{2}(\Delta t)^2\right)D - \frac{1}{2}(\Delta t)^2 Z\right)F^j + (\Delta t)DW^j + (\Delta t)^2\vec{U} \tag{12}$$

$$W^{j+1} = \frac{1}{\Delta t}(F^{j+1} - F^j) \tag{13}$$

$$DF^0 = \vec{U}^0 \tag{14}$$

$$DW^0 = \vec{U}^1 \tag{15}$$

Where$D = (d_{ik})_{N\times N},\ d_{ik} = (\sigma_k, \sigma_i),\ Z = (z_{ik})_{N\times N}, z_{ik} = (\nabla\sigma_k, \nabla\sigma_i), U = \left(g\left(t_j^n, X_{\frac{1}{2}j}^n\right), \sigma_i\right)$

$F_{N\times 1}^j = (f_1^j, f_2^j, \cdots, f_N^j)^T, W_{N\times 1}^j = (w_1^j, w_2^j, \cdots, w_N^j)^T, \vec{U}^0 = (u_i^0)_{N\times 1}, u_i^0 = (X^0, \sigma_i)$, and

$\vec{U}^1 = (u_i^1)_{N\times 1}\ with\ u_i^1 = (X^1, \sigma_i),\ \forall\ i, k = 1,2,\cdots, N.$

3-The GNLAS ((12)-(15)) has a unique solution. To solve it, first and from solving (14) and

(15) respectively, the solutions F^0 and $W^0$are found, then (12) is solved by using the PrCoT(for each $j = 0,1, \dots, Y-1$) as:

In the PrT, suppose that $F^{j+1} = F^j$ in the elements of $\vec{U}$(in the RHS), then it convert to a GLAS, which is solved to get F^{j+1}, then resolve (12) with setting $\bar{F}^{j+1} = F^{j+1}$ (in the elements of $\vec{U}$) to get the corrector solution (CoS) F^{j+1}, finally W^{j+1} is obtained from setting F^{j+1} in (13); of course this technique can be repeated for more than one time. This PrCoT can be expressed as:

$$\left(X_{j+1}^{(l+1)}, \sigma_i\right) + (\Delta t)^2 \left(\nabla X_{\frac{1}{2}j}^{(l+1)}, \nabla\sigma_i\right) + (\Delta t)^2 \left(X_{\frac{1}{2}j}^{(l+1)}, \sigma_i\right) =$$

$$\left(X_j^n, \sigma_i\right) + (\Delta t)\left(p_j^n, \sigma_i\right) + (\Delta t)^2 \left(g(t_j^n, X_{\frac{1}{2}j}^{(l)}), \sigma_i\right) \tag{16}$$

$$p_{j+1}^{(l+1)} = \frac{(X_{j+1}^{(l+1)} - X_j^n)}{\Delta t} \tag{17}$$

Equation (17) shows that the iterative method depends only on $X_{j+1}^{(l+1)}$, where l represents the number of iterations.

Theorem: For "sufficiently small"Δt and for any fixed j $(0 \le j \le Y - 1)$, the DWF ((8)-(11)), has a unique solution $X^n = (X_0^n, X_1^n, \cdots, X_N^n)$ and the sequence of the CoS converges in R.

Proof: Let $X^{(l+1)} = (X_0^{(l+1)}, X_1^{(l+1)}, \cdots, X_N^{(l+1)})$, and $\bar{X}^{(l+1)} = (\bar{X}_0^{(l+1)}, \bar{X}_1^{(l+1)}, \cdots, \bar{X}_N^{(l+1)})$, where $X^{(l+1)}$ and $\bar{X}^{(l+1)}$ are the solutions of equation (16). Hence

$$\left(X_{j+1}^{(l+1)}, \sigma_i\right) + (\Delta t)^2 \left(\nabla X_{\frac{1}{2}j}^{(l+1)}, \nabla\sigma_i\right) + (\Delta t)^2 \left(X_{\frac{1}{2}j}^{(l+1)}, \sigma_i\right) = \left(X_j^n, \sigma_i\right) + \Delta t\left(p_j^n, \sigma_i\right) +$$

$$(\Delta t)^2 (g\left(t_j^n, X_{\frac{1}{2}j}^{(l)}\right), \sigma_i) \tag{18}$$

and

$$\left(\bar{X}_{j+1}^{(l+1)}, \sigma_i\right) + (\Delta t)^2 \left(\nabla \bar{X}_{\frac{1}{2}j}^{(l+1)}, \nabla\sigma_i\right) + (\Delta t)^2 \left(\bar{X}_{\frac{1}{2}j}^{(l+1)}, \sigma_i\right) =$$

$$\left(X_j^n, \sigma_i\right) + \Delta t\left(p_j^n, \sigma_i\right) + (\Delta t)^2 (g\left(t_j^n, \bar{X}_{\frac{1}{2}j}^{(l)}\right), \sigma_i) \tag{19}$$

Subtracting (19) from (18) then setting $\sigma_i = (\bar{X}_{j+1}^{(l+1)} - X_{j+1}^{(l+1)})$, it yields to

$$\left(\bar{X}_{j+1}^{(l+1)} - X_{j+1}^{(l+1)}, \bar{X}_{j+1}^{(l+1)} - X_{j+1}^{(l+1)}\right) + (\Delta t)^2 \left(\nabla \bar{X}_{\frac{1}{2}j}^{(l+1)} - \nabla X_{\frac{1}{2}j}^{(l+1)}, \nabla \bar{X}_{j+1}^{(l+1)} - \nabla X_{j+1}^{(l+1)}\right)$$

$$+(\Delta t)^2 \left(\bar{X}_{\frac{1}{2}j}^{(l+1)} - X_{\frac{1}{2}j}^{(l+1)}, \bar{X}_{j+1}^{(l+1)} - X_{j+1}^{(l+1)}\right) =$$

$$(\Delta t)^2 (g\left(t_j^n, \bar{X}_{\frac{1}{2}j}^{(l)}\right) - g\left(t_j^n, X_{\frac{1}{2}j}^{(l)}\right), \bar{X}_{j+1}^{(l+1)} - X_{j+1}^{(l+1)}) \tag{20}$$

From ASM (1 - b) the 2^d and 3^d terms in the LHS of (20) are nonnegative, and from ASM (2- b) on g in RHS of (20), and then by applying the Cauchy Schwarz inequality to obtain

$$\left\|\bar{X}_{j+1}^{(l+1)} - X_{j+1}^{(l+1)}\right\|_0 \le \xi \left\|\bar{X}_{\frac{1}{2}j}^{(l)} - X_{\frac{1}{2}j}^{(l)}\right\|_0, \xi = \frac{(\Delta t)^2 L}{(1+(\Delta t)^2)} \tag{21}$$

But $\xi < 1$, (for "sufficiently small"Δt), Which leads to that g is contractive. Also since $\{X^{(l)}\} \in R$ for each l, that $\left(\bar{X}^{(l+1)}\right) = X^{(l+1)} \in R$ for each $l, i.e$ $g(X) \in R$, and by theorem 2 we get that $\{X^{(l)}\}$ is converged to a point in R.

6. Numerical Examples

In this section, some numerical examples are carried out to show the efficiency and the accuracy for the proposed method.

Example 1: Consider the following NLHPDE:

$X_{tt} - \Delta X + X = g(\vec{r}, t, X), \ \ \vec{r} = (r_1, r_2), \vartheta = Q \times S, Q = (0,1) \times (0,1), S = [0,1]$

$X(\vec{r}, 0) = r_1 r_2 - r_1^2 r_2 - r_1 r_2 (r_2 - r_1 r_2), in\ Q$

$X_t(\vec{r}, 0) = X^1(\vec{r}), in\ Q$

$X(\vec{r}, t) = 0, on\ E = \partial Q \times S$

Where,$g(\vec{r}, t, X) = e^{-2t}\left[\left(r_1 r_2 - r_1^2 r_2 - r_1 r_2 (r_2 - r_1 r_2)\right)\left(5 - \sin\left(\left(r_1 r_2 - r_1^2 r_2 - r_1 r_2 (r_2 - r_1 r_2)\right)e^{-2t}\right)\right) + 2\left(r_1(1 - r_1) + r_2(1 - r_2)\right)\right].$

Where the exact solution of the problem is

$X(\vec{r}, t) = \left(r_1 r_2 - r_1^2 r_2 - r_1 r_2 (r_2 - r_1 r_2)\right)e^{-2t}$

The MCNGM was used to solve this problem with D=9, Y=20 and T=1, the numerical results are given at $\hat{t} = 0.5$ in the Table (1) and are shown in Figure (1).

Table1- Comparison between exact and approximation solutions

r_1	r_2	Exact	Approximation	Absolute error	r_1	r_2	Exact	Approximation	Absolute error
0.1	0.1	0.0030	0.0026	0.0004	0.5	0.5	0.0230	0.0192	0.0038
0.2	0.1	0.0053	0.0046	0.0007	0.6	0.5	0.0221	0.0185	0.0036
0.3	0.1	0.0070	0.0061	0.0009	0.7	0.5	0.0193	0.0163	0.0030
0.4	0.1	0.0079	0.0069	0.0010	0.8	0.5	0.0147	0.0125	0.0022
0.5	0.1	0.0083	0.0071	0.0012	0.9	0.5	0.0083	0.0071	0.0012
0.6	0.1	0.0079	0.0067	0.0012	0.1	0.6	0.0079	0.0067	0.0012
0.7	0.1	0.0070	0.0059	0.0011	0.2	0.6	0.0141	0.0119	0.0022
0.8	0.1	0.0053	0.0045	0.0008	0.3	0.6	0.0185	0.0156	0.0029
0.9	0.1	0.0030	0.0026	0.0004	0.4	0.6	0.0212	0.0177	0.0035
0.1	0.2	0.0053	0.0046	0.0007	0.5	0.6	0.0221	0.0185	0.0036

0.2	0.2	0.0094	0.0082	0.0012	0.6	0.6	0.0212	0.0178	0.0034
0.3	0.2	0.0124	0.0107	0.0017	0.7	0.6	0.0185	0.0158	0.0027
0.4	0.2	0.0141	0.0121	0.0020	0.8	0.6	0.0141	0.0121	0.0020
0.5	0.2	0.0147	0.0125	0.0022	0.9	0.6	0.0079	0.0069	0.0010
0.6	0.2	0.0141	0.0119	0.0022	0.1	0.7	0.0070	0.0059	0.0011
0.7	0.2	0.0124	0.0105	0.0019	0.2	0.7	0.0124	0.0105	0.0019
0.8	0.2	0.0094	0.0080	0.0014	0.3	0.7	0.0162	0.0137	0.0025
0.9	0.2	0.0053	0.0045	0.0008	0.4	0.7	0.0185	0.0156	0.0029
0.1	0.3	0.0070	0.0061	0.0009	0.5	0.7	0.0193	0.0163	0.0030
0.2	0.3	0.0124	0.0107	0.0017	0.6	0.7	0.0185	0.0158	0.0027
0.3	0.3	0.0162	0.0139	0.0023	0.7	0.7	0.0162	0.0139	0.0023
0.4	0.3	0.0185	0.0158	0.0027	0.8	0.7	0.0124	0.0107	0.0017
0.5	0.3	0.0193	0.0163	0.0030	0.9	0.7	0.0070	0.0061	0.0009
0.6	0.3	0.0185	0.0156	0.0029	0.1	0.8	0.0053	0.0045	0.0008
0.7	0.3	0.0162	0.0137	0.0025	0.2	0.8	0.0094	0.0080	0.0014
0.8	0.3	0.0124	0.0105	0.0019	0.3	0.8	0.0124	0.0105	0.0019
0.9	0.3	0.0070	0.0059	0.0011	0.4	0.8	0.0141	0.0119	0.0022
0.1	0.4	0.0079	0.0069	0.0010	0.5	0.8	0.0147	0.0125	0.0022
0.2	0.4	0.0141	0.0121	0.0020	0.6	0.8	0.0141	0.0121	0.0020

0.3	0.4	0.0185	0.0158	0.0027	0.7	0.8	0.0124	0.0107	0.0017
0.4	0.4	0.0212	0.0178	0.0034	0.8	0.8	0.0094	0.0082	0.0012
0.5	0.4	0.0221	0.0185	0.0036	0.9	0.8	0.0053	0.0046	0.0007
0.6	0.4	0.0212	0.0177	0.0035	0.1	0.9	0.0030	0.0026	0.0004
0.7	0.4	0.0185	0.0156	0.0029	0.2	0.9	0.0053	0.0045	0.0008
0.8	0.4	0.0141	0.0119	0.0022	0.3	0.9	0.0070	0.0059	0.0011
0.9	0.4	0.0079	0.0067	0.0012	0.4	0.9	0.0079	0.0067	0.0012
0.1	0.5	0.0083	0.0071	0.0012	0.5	0.9	0.0083	0.0071	0.0012
0.2	0.5	0.0147	0.0125	0.0022	0.6	0.9	0.0079	0.0069	0.0010
0.3	0.5	0.0193	0.0163	0.0030	0.7	0.9	0.0070	0.0061	0.0009
0.4	0.5	0.0221	0.0185	0.0036	0.8	0.9	0.0053	0.0046	0.0007
					0. 9	0.9	0.0030	0.0026	0.0004

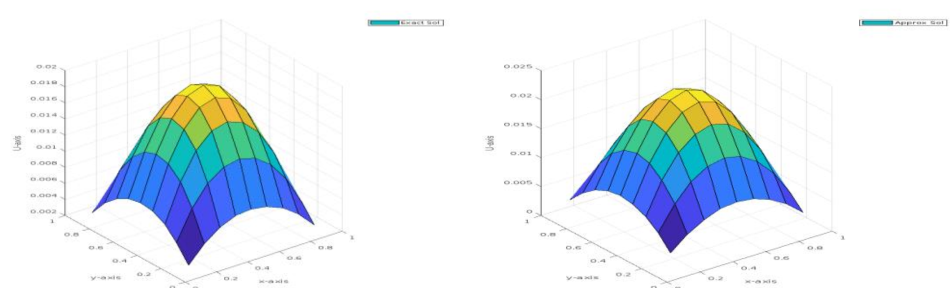

Figure1: shows the exact and the approximation solution

Example 2: Consider the following NLHPDE:

$$X_{tt} - \Delta X + X = g(\vec{r}, t, X), \;\; \vec{r} = (r_1, r_2), \vartheta = Q \times S, Q = (0,1) \times (0,1), S = [0,1]$$

$$X(\vec{r}, 0) = r_1^2 r_2 \sin(r_1 r_2 - r_1 - r_2 + 1)\,, in\ Q$$

$X_t(\vec{r},0) = X^1(\vec{r}), in\ Q$

$X(\vec{r},t) = 0, on\ E = \partial Q \times S$

Where, $g(\vec{r},t,X) = e^{-2t}[r_2 \sin(r_1 r_2 - r_1 - r_2 + 1)\,(5r_1^2 - 2 - r_1^2 \sin(r_1^2 r_2 \sin(r_1 r_2 - r_1 - r_2 + 1)\,e^{-2t}) + (r_1^4 - 2r_1^3 + r_1^2) + (r_1^2 r_2^2 - 2r_1^2 r_2 + r_1^2)) - 2r_1 \cos(r_1 r_2 - r_1 - r_2 + 1)(r_1(r_1 - 1) + 2r_2(r_2 - 1))]$.

Where the exact solution of this problem is $X(\vec{r},t) = r_1^2 r_2 \sin(r_1 r_2 - r_1 - r_2 + 1)\,e^{-2t}$

The MCNGM was used to solve this problem with D=9, Y=20 and T=1, the numerical results are given at $\hat{t} = 0.5$ in the Table (2) and are shown in Figure (2).

Table2: Comparison between exact and approximation solutions

r_1	r_2	exact	approximation	Absolute error	r_1	r_2	exact	approximation	Absolute error
0.1	0.1	0.0003	0.0002	0.0001	0.5	0.5	0.0114	0.0095	0.0019
0.2	0.1	0.0010	0.0009	0.0001	0.6	0.5	0.0132	0.0109	0.0023
0.3	0.1	0.0020	0.0017	0.0003	0.7	0.5	0.0135	0.0112	0.0023
0.4	0.1	0.0030	0.0026	0.0004	0.8	0.5	0.0118	0.0099	0.0019
0.5	0.1	0.0040	0.0034	0.0006	0.9	0.5	0.0074	0.0063	0.0011
0.6	0.1	0.0047	0.0039	0.0008	0.1	0.6	0.0008	0.0007	0.0001
0.7	0.1	0.0048	0.0040	0.0008	0.2	0.6	0.0028	0.0025	0.0003
0.8	0.1	0.0042	0.0036	0.0006	0.3	0.6	0.0055	0.0048	0.0007
0.9	0.1	0.0027	0.0023	0.0004	0.4	0.6	0.0084	0.0071	0.0013
0.1	0.2	0.0005	0.0004	0.0001	0.5	0.6	0.0110	0.0092	0.0018
0.2	0.2	0.0018	0.0015	0.0003	0.6	0.6	0.0127	0.0106	0.0021
0.3	0.2	0.0035	0.0031	0.0004	0.7	0.6	0.0129	0.0109	0.0020
0.4	0.2	0.0054	0.0047	0.0007	0.8	0.6	0.0113	0.0096	0.0017
0.5	0.2	0.0072	0.0061	0.0011	0.9	0.6	0.0071	0.0062	0.0009
0.6	0.2	0.0083	0.0070	0.0013	0.1	0.7	0.0007	0.0006	0.0001
0.7	0.2	0.0086	0.0072	0.0014	0.2	0.7	0.0024	0.0022	0.0002

0.8	0.2	0.0075	0.0063	0.0012	0.3	0.7	0.0048	0.0042	0.0006
0.9	0.2	0.0048	0.0040	0.0008	0.4	0.7	0.0074	0.0063	0.0011
0.1	0.3	0.0007	0.0006	0.0001	0.5	0.7	0.0096	0.0081	0.0015
0.2	0.3	0.0023	0.0021	0.0002	0.6	0.7	0.0111	0.0094	0.0017
0.3	0.3	0.0047	0.0041	0.0006	0.7	0.7	0.0113	0.0097	0.0016
0.4	0.3	0.0072	0.0062	0.0010	0.8	0.7	0.0099	0.0085	0.0014
0.5	0.3	0.0095	0.0080	0.0015	0.9	0.7	0.0063	0.0054	0.0009
0.6	0.3	0.0110	0.0092	0.0018	0.1	0.8	0.0005	0.0005	0.0000
0.7	0.3	0.0113	0.0094	0.0019	0.2	0.8	0.0019	0.0017	0.0002
0.8	0.3	0.0099	0.0083	0.0016	0.3	0.8	0.0037	0.0033	0.0004
0.9	0.3	0.0063	0.0053	0.0010	0.4	0.8	0.0056	0.0048	0.0008
0.1	0.4	0.0008	0.0007	0.0001	0.5	0.8	0.0073	0.0062	0.0011
0.2	0.4	0.0027	0.0024	0.0003	0.6	0.8	0.0085	0.0072	0.0013
0.3	0.4	0.0054	0.0047	0.0007	0.7	0.8	0.0086	0.0075	0.0011
0.4	0.4	0.0083	0.0071	0.0012	0.8	0.8	0.0075	0.0066	0.0009
0.5	0.4	0.0109	0.0091	0.0018	0.9	0.8	0.0048	0.0042	0.0006
0.6	0.4	0.0126	0.0105	0.0021	0.1	0.9	0.0003	0.0003	0.0000
0.7	0.4	0.0129	0.0107	0.0022	0.2	0.9	0.0011	0.0010	0.0001
0.8	0.4	0.0113	0.0094	0.0019	0.3	0.9	0.0021	0.0018	0.0003
0.9	0.4	0.0071	0.0060	0.0011	0.4	0.9	0.0032	0.0027	0.0005
0.1	0.5	0.0008	0.0007	0.0001	0.5	0.9	0.0041	0.0035	0.0006
0.2	0.5	0.0029	0.0025	0.0004	0.6	0.9	0.0048	0.0041	0.0007
0.3	0.5	0.0057	0.0049	0.0008	0.7	0.9	0.0049	0.0042	0.0007
0.4	0.5	0.0087	0.0074	0.0013	0.8	0.9	0.0042	0.0037	0.0006
					0.9	0.9	0.0027	0.0024	0.0003

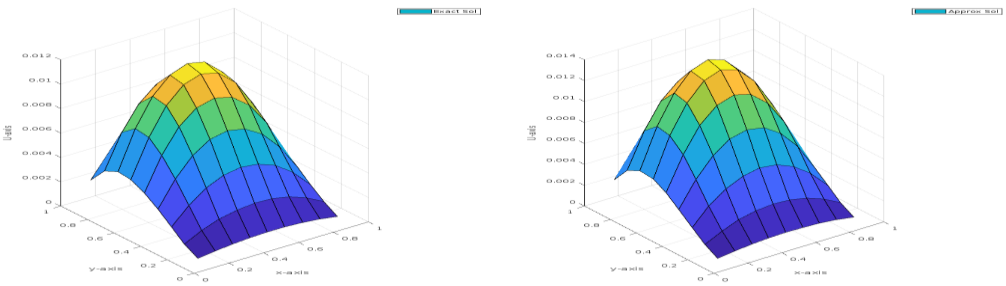

Figure2: shows the exact and the approximation solutions

7. Conclusions

From the solutions for the above given examples, one can conclude that:

The MCNGM was used successfully to find the APSOL of the NLHPDES. The uniqueness of the APSOL for the weak form (which was obtained from the PrCrT) was proved. The GFEM was applied easily and the elements in the GNAS are in analytic form (exact) comparing with other methods that the elements are in approximate or in a full discrete form. The cholesky method which used inside the PrCrT was very efficient and fast to solve the GLAS because it saves a lot of calculations. The approximation solution for the two examples illustrate the accuracy and the efficiency of the proposed method. It is important to mention here that the approximate vector solution are given at the value of $\hat{t} = 0.5$ to brief the size of the paper, in fact same results with same accuracy were obtained at any value of $\hat{t}$ provided this value belong to S.

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