Stability Analysis for Fluid Flow Between Two Infinite Parallel Plates I

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الملخـــــص

تمت في هذا البحث دراسة الاستقرارية لنظام يمثل مائعا بين لوحين متوازيين لا نهائيين يسخن من الأسفل والاعتماد على انتقال الحرارة بالتوصيل والحمل والإشعاع.

وقد نوقش الموضوع تحت تأثير عدد من المعلمات ذات دلالة فيزيائية مستخدمين طريقة عددية ، إذ تبين أن استقرارية النظام تعتمد على نسبة الحرارة بين اللوحين، وكذلك على معامل التمدد الحراري كما استخدمت طريقة تحليلية تم من خلالها الحصول على صورة واضحة للجريان بتأثير التسخين.

ABSTRACT

A model of fluid flow with heat transfer by conduction, convection and radiation has been discussed for stability with respect to restricted parameters (k,α,r,T^*) which are proportional to: wave numbers, thermal expansion coefficient, combination of many numbers (Re,Pr,Ec,Bo,W,g) and the ratio of walls temperatures, respectively using numerical technique which illustrate that the stability of the system depends on the parameters T* and α.

A clear picture of the flow is shown by using an analytical method.

Introduction:

Heat transfer in fluid flow is an important field in the mathematical modeling because of its wide range of applications such as designing of the cooling system for nuclear reactor, pressure measurements, turbomachinery and other engineering applications.

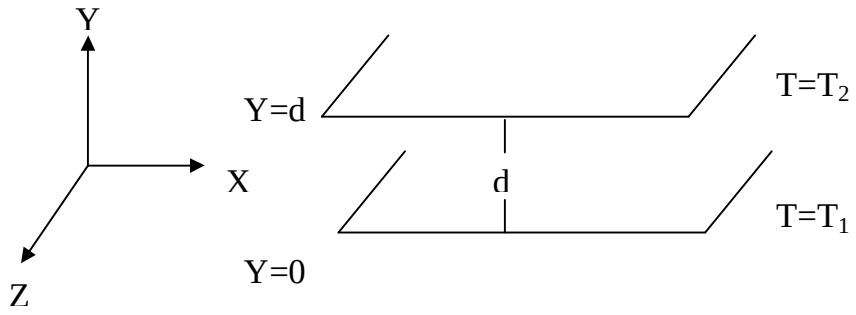
The principle of stability of fluid flow with heat transfer and its applications has been investigated by many authors such as Lorenz [1] and Yorke [3] and others.

In this paper a model of heat transfer by conduction, convection and radiation in a fluid flow between two infinite parallel flat plates has been considered.

The first like model without heating from below was investigated by Logan [2].

Model and Governing differential Equations:

Consider an ideal fluid confined between two infinite parallel plates Y=0, Y=d in X Z Y space separated by a distance d, and heated from below, which is under the influence of a constant gravitation field g acting in the negative Y direction as shown in the following figure:



Using Boussinisque approximation and the optical thick limit and if the temperature differences between the walls and the fluid is small [4] the non- dimensional differential equations (in the new plane x z y , where x=X/d ,z=Z/d, and y=Y/d) govern the problem are :

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad \text{.....1}\\
&\frac{Du}{Dt}=-\frac{\partial p}{\partial x} \quad \text{.....2}\\
&\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}-\{1-a(q-1)\} \quad \text{.....3}\\
&\frac{Dw}{Dt}=-\frac{\partial p}{\partial z} \quad \text{.....4}\\
&\frac{Dq}{Dt}=R\nabla^2 q \quad \text{.....5}
\end{aligned}\right\} \quad (1)$$

where R= ,

$$a = T_1 b \qquad , \quad \frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$$

(γ: specific heat ratio ,Pr: prandtl number ,Re: reynold number, Ec: Eckert number ,Bo: Boltzman number , w: Bouguer number,and b :thermal expansion coefficient) . With the boundary conditions (non-dimensional):

u = v = w = 0 when y = 0, 1

$\theta = 1$ when y = 0

$\theta = \frac{T_2}{T_1}$ when y = 1

Stability Analysis:

Any system whatsoever is bound to be disturbed, a basic question therefore is "will the disturbance gradually die down or will it grow in amplitude in such a way that the system dose not return to its original state ?"

In the field of fluid mechanics the broad study of stability is concerned , in part , with the determination of the critical values , if any , of the flow parameters which distinguish the two different regimes associated with the answer to the above question .

The functions u, v , w , p , and θ are written as:

$$\left.\begin{aligned} u &= u_1+u_2\ (x, y, z, t) \\ v &= v_1+v_2\ (x, y, z, t) \\ w &= w_1+w_2\ (x, y, z, t) \\ p &= p_1+p_2\ (x, y, z, t) \\ \theta &= \theta_1+\theta_2\ (x, y, z, t) \end{aligned}\right\}$$

where :

u_1, v_1, w_1, p_1, and θ_1 represent the steady state

and

u_2, v_2, w_2, p_2, and θ_2 represent the disturbance.

Steady state:

In this case the functions u_1 ,v_1 ,..... , θ_1 , are independent of the time variable t.

Here we take the motionless ($u_1 = v_1 = w_1 = o$) then we can show that:

$P_1 = P_1(y)$ and $\theta_1 = \theta_1(y)$

Unsteady state:

The equations of the three dimensional disturbance are:

$$\left.\begin{aligned}
&\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial w_2}{\partial z}=0 \quad \ldots\ldots 1\\
&\frac{Du_2}{Dt}=-\frac{\partial p_2}{\partial x} \quad \ldots\ldots 2\\
&\frac{Dv_2}{Dt}=-\frac{\partial p_2}{\partial y}+aq_2 \ldots\ldots 3\\
&\frac{Dw_2}{Dt}=-\frac{\partial p_2}{\partial z} \quad \ldots\ldots 4\\
&\frac{Dq_2}{Dt}=R\nabla^2 q_2 \ldots\ldots 5
\end{aligned}\right\} \quad (2)$$

with the boundary conditions : $u_2 = v_2 = w_2 = \theta_2 = 0$, $y = 0,1$

We attempt to find a solution of the form:

$$\left.\begin{aligned}
&u_2 = U(y)\, e^{at}\, e^{i(k_1 x + k_2 z)}\\
&v_2 = V(y)\, e^{at}\, e^{i(k_1 x + k_2 z)}\\
&- \quad - \quad -\\
&- \quad - \quad -\\
&\theta_2 = \theta(y)\, e^{at}\, e^{i(k_1 x + k_2 z)}
\end{aligned}\right\} \quad (3)$$

Where k_1, k_2 are the wave numbers and "a" is the speed number($a = a_1 + ia_2$).

Substituting (3) in system (2) and using the linearized theory we have:

$$\left.\begin{aligned}
&ik_1 U + V' + ik_2 W = 0 \quad \ldots\ldots 1\\
&aV = -ik_1 P \quad \ldots\ldots 2\\
&aV = -P' + aq \quad \ldots\ldots 3\\
&aW = -ik_2 P \quad \ldots\ldots 4\\
&q'' - \frac{(a + Rk^2)}{R} q - T^*/_R V = 0 \quad \ldots\ldots 5
\end{aligned}\right\} \quad (4)$$

where $T^*=(T_2-T_1)/T_1$, (T_1: temperature of the lower wall , T_2: temperature of the upper wall)

Numerical technique:

A numerical technique known as Galerkin method is used . From system (4) and by eliminating the pressure function in equation (3) using equations (1), (2)and (4) we get:

$$\left.\begin{aligned} q'' - \frac{(a+Rk^2)}{R}q - \frac{T^*}{R}V = 0 \qquad (1) \\ V'' - k^2 V + \frac{ak^2}{a}q = 0 \qquad (2) \end{aligned}\right\} \qquad (5)$$

Applying Galerkin method on (5) and using the boundary conditions we get:

$\theta(y) = \sum_{n=0}^{\infty} B_n \sin(n\pi)y.$, B_n: constants.

With the residual $R_1 = \sum_{n=0}^{p}\left\{\left[\left(n^2p^2 + \frac{a+Rk^2}{R}\right)B_n + \frac{T^*}{R}D_n\right].\sin(np)y\right\}$

And:

$V(y) = \sum_{m=0}^{\infty}\left(D_n \sin(mp)y\right)$, D_n: constants.

With the residual $R_2 = \sum_{n=0}^{p}\left\{\left[\frac{ak^2}{a}B_n - (n^2p^2 + k^2)D_n + \right].\sin(np)y\right\}$.

Take the residuals orthogonal to sin ($m\pi$)y Then:

$$\int_0^1 \{R_1\}\sin(m\pi)ydy = 0$$

$$\int_0^1 \{R_2\}\sin(mp)ydy = 0$$

In more convenient we write:

$$X_1(n,m)=\int_0^1\left(n^2\pi^2+\frac{a+Rk^2}{R}\right)\sin(n\pi)y\sin(m\pi)ydy$$

$$X_2(n,m)=\int_0^1\left(\frac{ak^2}{a}\right)\sin(np)y\sin(mp)ydy$$

$$Y_1(n,m)=\int_0^1\left(T^*/_R\right)\sin(np)y\sin(mp)ydy$$

$$Y_2(n,m)=-\int_0^1\left((n^2p^2+k^2\right)\sin(np)y\sin(mp)ydy \quad , m=1,2,\ldots,n$$

Which are square matrices of order n x n .

The constants B_n and D_n satisfy the homogenous algebraic equations:

x_1 Bn + y_1 Dn = 0

x_2 Bn + y_2 Dn = 0

or in a matrices from:

$$\left.\begin{bmatrix}x_1 & y_1\\ x_2 & y_2\end{bmatrix}\begin{bmatrix}B_n\\ D_n\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\right\} \qquad (7)$$

system (7) has a non-trivial solution if

$\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}=0$, using "Matlab" soft were (with n = 2) to evaluate:

1) $X=\begin{bmatrix}x_1 & y_1\\ x_2 & y_2\end{bmatrix}$

2) |X|

(3) Solution of the equation |X| = 0 with respect to (a).

$$a=\frac{-r\left(16p^4+k^4+8p^2k^2\right)\pm\left[\left(16p^4+k^4+8p^2k^2\right)^2r^2-16p^2k^2aT^*-4k^4aT^*\right]^{\frac{1}{2}}}{8p^2+2k^2}$$

Or:

$$a = \frac{-r\left(p^4+k^4+2p^2k^2\right)\pm\left[\left(p^4+k^4+2p^2k^2\right)^2 r^2-4p^2k^2aT^*-4k^4aT^*\right]^{\frac{1}{2}}}{2p^2+2k^2}$$

We conclude the following:

(1) The system is stable if α and T* have the same signs.

(2) The system is unstable if α and T* have different signs.

(3) The system is in neutral stability if T* = 0, or $\alpha = 0$.

Analytical Technique:

In this section the unsteady state " perturbation" in three dimensions is discussed and tested for satisfying Boundary conditions, and cases which gives us a clear picture of the flow are shown.

By eliminating q in system (5) we can write:

$$V^{(4)} + L_1 V'' + L_2 V = 0 \qquad (8)$$

Where: $L_1 = -\left(\frac{a}{R}+2k^2\right)$, $L_2 = \left(k^4+\frac{a}{R}k^2-\frac{aT^*}{aR}k^2\right)$

With the boundary conditions:

$$V(0)=V(1)=V'(0)=V'(1)=0 \qquad (9)$$

In order to find a solution satisfying the boundary condition, we must investigate all cases we get from the characteristic equation.

$m^4+L_1m^2+L_2=0$

which is algebraic equation of degree four, it has four roots

Thus:

$$m^2 = \frac{1}{2}\left[-L_1 \pm \sqrt{L_1^2-4L_2}\right] \qquad (10)$$

For D^2 there are three cases (Here "a" is real):

1. $L_1^2-4L_2<0$ 2. $L_1^2-4L_2=0$ 3. $L_1^2-4L>0$

each one has many cases.

By investigating those cases and applying the boundary conditions (9)

the nontrivial solution holds at the following:

i) $L_1 - 4L_2 > 0, -L_1 < 0$ and: $|-L_1| = \sqrt{L_1^2 - 4L_2}$

ii) $L_1^2 - 4L_2 > 0, -L_1 < 0$ and: $|-L_1| > \sqrt{L_1^2 - 4L_2}$

In this paper we illustrate the case i only.

The general solution here can be written as:

$V(y)=c_1+c_2y+c_3 \cos. d\ y+c_4 \sin d\ y$, where the roots of the characteristic equation (10) are :

$\lambda_1= \lambda_2 = 0$, $\lambda_3= i\alpha$, and $\lambda_4 = -i$

The boundary condition (9) gives the particular solution:

$V_n = C_n(1-\cos(2np)y), \quad where\ a = 2np$

from relations in system (4) we can find the other functions. U_n, P_n, W_n, and θ_n which establish the perturbation functions u_2, v_2, w_2,p2,and$\theta_2$as:

$$u_2 = i\frac{k_1}{k^2}[2np]\sin(2np)e^{at}e^{i(k_1x=k_2z)}$$

$$v_2 = c(1-\cos(2np)y)e^{at}e^{i(k_1x+k_2z)}$$

$$w_2 = i\frac{k_2}{k^2}[2np]\sin(2np)ye^{at}e^{i(k_1x+k_2z)}$$

$$P_2 = -\frac{a}{k^2}(2np)\sin(2np)ye^{at}e^{i(k_1x+k_2z)}$$

$$q_2 = -\frac{a}{a}\left[1-\left(1+\frac{(2np)}{k^2}\right)\cos(2np)y\right]e^{at}e^{i(k_1x+k_2z)}$$

To get a clear picture of the flow by examining two dimensional disturbance, and the case n=1, k=1, (x,y) plane.Therefore $k_1=1$, $k_2=0$, $w_2=0$.

Then: $u_2 = i(2p)\sin(2p)ye^{ix}e^{at}$ and $v_2 = (1-\cos(2p)y)e^{ix}e^{at}$

,and the real solution is: $u_2 = -(2p)\sin(2p)ye^{at}\sin x$ and $v_2 = (1-\cos(2y))e^{at}\cos x$

We study the velocity direction field

$<u_2, v_2>$ for: $0 \le x \le 2p \qquad 0 \le y \le 1$

by noting the following facts:

u_2	v_2
$u_2<0$ when: 1) $0<x<\pi$, $0<y<1/2$ 2) $\pi<x<2\pi$, ½<y<1	$v_2<0$ when: $\frac{p}{2} < x < \frac{3p}{2}$
$u_2=0$ when: y=0,1 x=0, π	$V_2=0$ when: x=$\frac{p}{2}, \frac{3p}{2}$ 2) y=0, 1
$u_2 > 0$ when 1) $0<x<\pi$, $1/2<y<1$ 2) $\pi<x<2\pi$, $0<y<1/2$	$v_2 > 0$ when 1) $0<x<\frac{p}{2}$ and 2) $\frac{3p}{2} < x < 2p$

The above figure is a diagram indicating the sign of u_2 and the directions of the velocity filed $<u_1, v_2>$ are shown in the following figure:

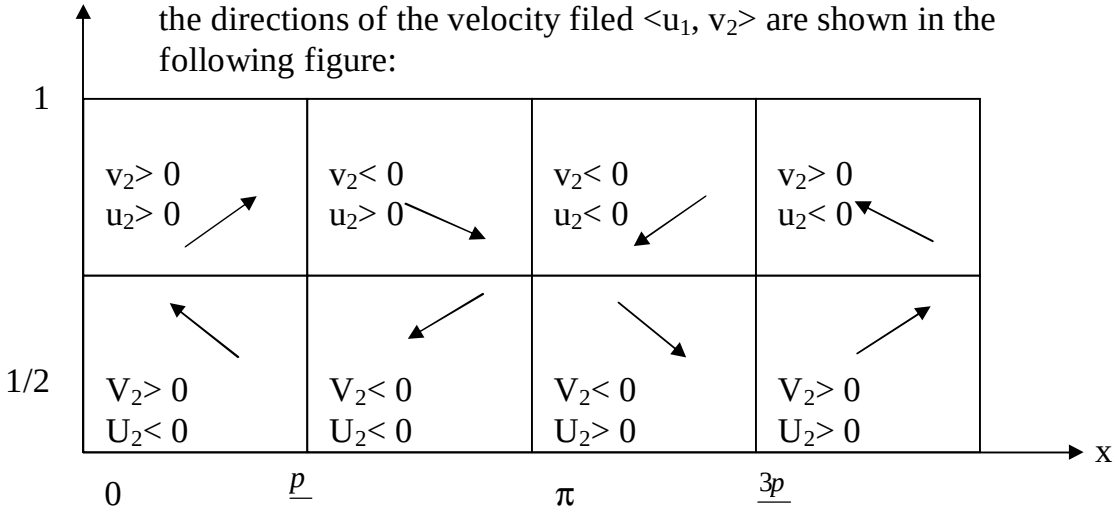


Figure (1) the directions of the velocity vector.

The following figure depicts schematically typical streamlines (obtained from the above figure) viewed from the positive z direction:

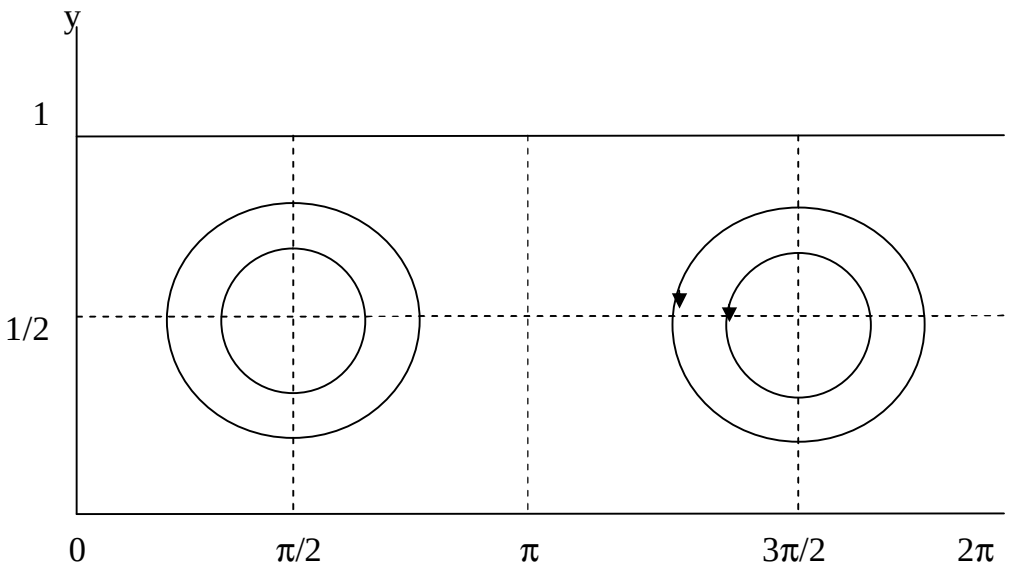


Figure (2): the streamlines.

Conclusion: -

The effect of heating from below was clear, and the ratio of the temperatures of the two plates plays significant parameter to the stability.

The analytical method gives us a clear picture of the flow which is a result of heating effect.

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