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A Novel Spectral Modified Pell Polynomials for Solving Singular Differential Equations

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ABSTRACT

This paper studies the modified Pell polynomials. Some important properties of modified Pell polynomials are presented. An exact formula of modified Pell polynomials derivative in terms of modified Pell themselves is first derived with the proof and then a new relationship is constructed which relates the modified Pell polynomials expansion coefficients of a derivative in terms of their original expansion coefficients. An interesting new formula for product operational matrix of modified Pell polynomials is also derived in this work. With modified Pell polynomials expansion scheme, the powers $\{1, x, \dots, x^n\}$ are expressed in terms of such polynomials. The main goal of all presented formulas is to simplify the original equations and the determination of the coefficients of expansion based on modified Pell polynomials will be easy. Spectral techniques together with all the derived formulas of modified Pell polynomials are utilized to solve some singular initial value problems. Three test examples are solved in this work to illustrate the validity of the proposed method. The computational method is replaced by exact and explicit formulas. More accurate results are obtained than those presented by other existing methods in the literature.

KEYWORDS: Modified Pell polynomials; singular initial value problem; operation matrix; spectral method; product matrix of two modified Pell polynomials.

الخلاصة

هذا البحث يدرس متعددات حدود بيل المعدلة. تم عرض بعض الخصائص المهمة لمتعددات حدود بيل المعدلة. تم اولا اشتقاق الصيغة الدقيقة لمشتقات بيل متعددة الحدود المعدلة بدلاله بيل نفسها مع البرهان وبعدها تم إنشاء علاقة جديدة تربط معاملات التوسع بيل متعددة الحدود المعدلة للمشتقات بدلاله معاملات التوسع الأصلية. تم أيضًا اشتقاق صيغة جديدة مثيرة للاهتمام لمصفوفة الضرب لمتعددات حدود بيل المعدلة في هذا العمل. باستخدام توسيع بيل متعددة الحدود المعدلة، تم التعبير عن القوى $\{1, x, \dots, x^n\}$ بدلاله متعددات الحدود. الهدف الرئيسي لجميع الصيغ المقدمة هو تبسيط المعادلات الأصلية وسيكون من السهل تحديد معاملات التمدد بناءً على متعددات حدود بيل المعدلة. تم استخدام التقنية الطيفية مع جميع الصيغ المشتقة من متعددات حدود بيل المعدلة لحل بعض مسائل القيمة الابتدائية الفردية. تم حل ثلاثة أمثلة اختبار في هذا العمل لتوضيح صحة الطريقة المقترحة. تم استبدال الطريقة الحسابية بصيغ دقيقة وصريحة. تم الحصول على نتائج أكثر دقة من تلك المقدمة من خلال طرق أخرى موجودة في الأدبيات.

INTRODUCTION

The mathematical formulations of many physical problems occurs in the various disciplines such as quantum mechanics, fluid dynamic and elasticity [1-5], often lead to either nonlinear ordinary differential equations or singular ordinary differential equations [1-3]. Accurate and efficient approximate methods are often required for the solution of such types of ordinary differential equations. Several techniques are available in the literature. A basis function is needed with interesting properties. Polynomials and wavelets are basis functions, which obtain a suitable place for many problems [4-10]. One of the important families of polynomials is modified Pell polynomials. Due to it's implicitly, modified Pell polynomials has become an effective tool for solving differential equations. In this study, an approximate solution based on such polynomials is developed to solve special singular initial value problems. Two different approaches are suitable for solving differential equations. The first approach based on reducing the differential equations into integral equations through integration. Approximating different singles contained in the equation by truncated orthogonal series, and using their operational matrix of integration to estimate the integral operations [11-13]. The second approach is based on applying the operation matrix of derivative to reduce the original problem into system of equations [14-18]. The purpose of this paper is to introduce modified Pell spectral method for solving singular initial value problems (I.V.P.) with this method, the given D.E. and its related initial conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of Pell series solution. This method is useful to obtain the approximate solution of I.V.P., no need to linearization or discretization and large computational work. It has been used to solve effectively, easily and accurately a large class of problems with approximations.

The goal of this work is to derive new properties concerning modified Pell polynomials and then utilized them to evaluate the unknown coefficients and find an approximate solution.

The article is organized as follows: in section 2, the definition of modified Pell polynomials is given. The main results of this paper are described in sections 3-6. Lemma 1 it greatly simplifies the calculation. Lemma 2 gives the relationship between the power and modified Pell polynomials. Lemma 3 reveals the product of two modified Pell polynomials in an explicit form. The derivative of modified Pell polynomials and the relationship between the coefficients is included in section 4. This is of great significance in numerical analysis and it has made new contributions to the study of spectral method. Section 7 concerns with the implementation of modified Pell polynomials spectral method for solving some singular initial value problems. The conclusion is listed in section 8.

Definitions of Modified Pell Polynomials

First-level sub-title

For $n \geq 1$, modified Pell polynomials $q_n(x)$, is defined by the following recurrence relations

$$q_{n+1}(x) = 2xq_n(x) + q_{n-1}(x) \quad (1)$$

with the initial conditions

$$q_0(x) = 1, q_1(x) = x. \quad (2)$$

The characteristic equation of recurrence relation Eq. 1 is

$t^2 - xt - 1 = 0$, this equation has two real roots

$a_1 = \frac{x+\sqrt{x^2+4}}{2}, a_2 = \frac{x-\sqrt{x^2+4}}{2}$

From Eqns. 1 and 2, the following modified Pell polynomials are obtained

$q_0(x) = 1$

$q_1(x) = x$

$q_2(x) = 2x^2 + 1$

$q_3(x) = 4x^3 + 3x$

$q_4(x) = 8x^4 + 8x^2 + 1$

$q_5(x) = 16x^5 + 20x^3 + 5x$

$q_6(x) = 32x^6 + 48x^4 + 18x^2 + 1$

$q_7(x) = 64x^7 + 112x^5 + 56x^3 + 7x$

The first eight coefficients of $q_n(x)$ are arranged in Table 1. Let q_{nj} denote the element in row n and column j, where $j \geq 1$, $n \geq 1$, it is obvious form Table 1 that

$$q_{n1}(x) = \begin{cases} 1 & \text{if n even} \\ 0 & \text{otherwise} \end{cases}$$

In general, $q_{nm} = 2q_{n-1,m-1} + 2q_{n-2,m}$, $q_{nn} = 2^{n-1}$, $n > m$

Table 1. The first eight coefficients of $q_n(x)$

n/j	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7
0	1							
1	0	1						
2	1	0	2					
3	0	3	0	4				
4	1	0	8	0	8			
5	0	5	0	20	0	16		
6	1	0	18	0	48	0	32	
7	0	7	0	56	0	112	0	64

Representing Derivatives of $q_n(x)$ by Modified Pell Polynomials

The objective of this subsection is to state and prove a new analytical formula representing explicitly the first derivative of modified Pell polynomials in terms of modified Pell polynomials themselves. As a direct consequence, a novel spectral algorithm based on the derivative for modified Pell polynomials is proposed.

Lemma (1):

The explicit expression for modified Pell polynomials derivative is given by

$$\dot{q}_n(x) = 2n \sum_{i=1\,(n+i)\,odd}^{n} (-1)^{\left[\frac{i}{2}\right]} q_{n-i}(x)$$

for n even (3)

$$\dot{q}_n(x) = 2n \sum_{i=0\,(n+i)even}^{n-1} (-1)^{\left[\frac{i}{2}\right]} q_{n-i}(x) + nq_0$$

for n odd (4)

Proof:

The mathematical induction principle is used to prove Eqns. 3 and 4.

Take n is even: Eq. 3 is true when $n = 2$ by direct calculation, since

$$\dot{q}_2(x) = 2x2\, q_1 = 4x.$$

Now, assume that Eq. 3 is true for a particular positive integer $n = 2k$, that is

$$\dot{q}_{2k}(x) = 4k \sum_{i=1\ (2k+i)odd}^{2k} (-1)^{\left[\frac{i}{2}\right]} q_{2k-i}(x) \quad (5)$$

To see that Eq. 3 is still true for $n = 2(k+1)$, consider the basic recurrence for modified Pell polynomials

$$q_n(x) = 2xq_{n-1} + q_{n-2}(x)$$

or

$$\dot{q}_n(x) = 2x\dot{q}_{n-1} + 2q_{n-1}(x) + \dot{q}_{n-2}(x)$$

$$\dot{q}_{2(k+1)}(x) = 2q_1(x).\dot{q}_{2(k+1)-1} + 2q_{2(k+1)-1}(x) + \dot{q}_{2(k+1)-2}$$

$$\dot{q}_{2(k+1)}(x) = 2q_1(x)\left\{4(k-1)\sum_{i=1\ 2(k+1)-1+i\ odd}^{2(k-1)} (-1)^{\left[\frac{i}{2}\right]} q_{2(k-1)-i}(x)\right\} + 2q_{2(k+1)-1}(x) + 4k\sum_{i=1}^{2k}(-1)^{\left[\frac{i}{2}\right]} q_{2k-i}(x)$$

$$= 4(k+1)[q_{2k+1}(x) - q_{2k-1}(x) + q_{2k-3}(x) - q_{2x-5} + \cdots + q_1(x)].$$

That is

$$\dot{q}_{2(k+1)}(x) = 4(k+1)\sum_{i=1\ 2(k+1)-1+i\ odd}^{2(k+1)} (-1)^{\left[\frac{i}{2}\right]} q_{2(k+1)-i}(x) \quad (6)$$

Since $n = 2(k+1) \Rightarrow k = \frac{n}{2} - 1$, hence

$$\dot{q}_n(x) = 2n \sum_{i=1\ n+i\ odd}^{n} (-1)^{\left[\frac{i}{2}\right]} q_{n-i}(x)$$

which is the same result in Eq. 3.

The Derivative of Modified Pell Polynomials and The Relationship Between the Coefficients

If a function $f(x)$ can be approximated by a modified Pell series of length n as

$$f(x) = \sum_{i=0}^{n} a_i q_i(x)$$

Then the derivative of $f(x)$ is given by

$$\dot{f}(x) = \sum_{i=0}^{n-1} c_i q_i(x).$$

The relationship between the coefficients a_i and c_i is given by

$c_n = 2na_n$

$c_{n-1} = 2(n-1)a_{n-1}$

$c_r = -c_{r+1} + 2ra_r, \qquad r = n-2, n-3, \dots, 2$

with $c_1 = -c_3 + 2a_1.$

The Derivative of Modified Pell Polynomials and The Relationship Between the Coefficients

Lemma (2):

The product of two modified Pell Polynomials is given by the following formula

$$q_n(x)q_m(x) = \frac{1}{2}\begin{cases} q_{n+m}(x) - q_{|n-m|}(x), & n\ odd \\ q_{n+m}(x) + q_{|n-m|}(x), & n\ even \end{cases} \text{when } m \geq n \quad (7)$$

Proof:

Eq. 7 is an identity for $n = 0$, since $q_0 = 1$, it then follows that

$$q_0 q_m(x) = \frac{1}{2}\left(q_m(x) + q_m(x)\right) = \frac{1}{2}\left(2q_m(x)\right) \quad (8)$$

Multiply both sides of Eq. 8 by 2x yield

$$q_m(x)(2xq_0) = \frac{1}{2}\left(2xq_m(x)\right) \quad (9)$$

By applying the basic recurrence relationship for modified Pell polynomials

$$q_m(x) = 2xq_{m-1}(x) + q_{m-2}(x)$$

or $2xq_m(x) = q_m(x) - q_{m-2}(x) \quad (10)$

With the aid of Eq. 10, one can get

$$q_m(x)q_1(x) = \frac{1}{2}\left(q_{m+1}(x) - q_{m-1}(x)\right).$$

That is Eq. 7 is true for $n = 1$, therefore; it is true for n since it is true for $n = 0$ and $n = 1$.

Now, Eq. 7 must be valid for $n + 1$

$$q_n(x)q_m(x) = \frac{1}{2}\left(q_{n+m}(x) - q_{|n-m|}(x)\right).$$

Multiply both sides by 2x

$$q_m(x)(2xq_n(x)) = \frac{1}{2}\left(2xq_{n+m}(x) - 2xq_{n-m}(x)\right)$$

$$q_m(x)\left(q_{n+1}(x) - q_{n-1}(x)\right) = \frac{1}{2}\left(q_{n+m+1}(x) - q_{n-m-1}(x) - q_{n-m+1}(x) + q_{n-m-1}(x)\right) + q_m(x)q_{n+1}(x) + q_{m+n}(x) - q_{n-m+1}(x)$$

$$q_n(x)q_{m+1}(x) = \frac{1}{2}[q_{n+m+1}(x) - q_{n-m-1}(x)]$$

The relationship Between The Power $\{1, x, x^2, \dots, x^n\}$ and Modified Pell Polynomials The Derivative of Modified Pell Polynomials

The relationship between the power $\{1, x, x^2, \dots, x^n\}$ and modified Pell polynomials is illustrated in this subsection.

Lemma (3):

For $n \geq 1$,

$$x^n = 2^{1-n}\sum_{k=0}^{[\frac{n}{2}]}(-1)^{\frac{k}{2}}\binom{n}{k} q_{n-2k}(x), \text{ n odd} \quad (11)$$

and

$$x^n = 2^{1-n}\sum_{k=0}^{[\frac{n}{2}]}(-1)^{\frac{k-1}{2}}\binom{n}{k} q_{n-2k}(x) + (-1)^{\frac{n}{2}}2^{-n}\binom{n}{\frac{n}{2}} q_0(x) \quad (12)$$

for even n, where $1 = q_0$

Proof:

The mathematical induction is used to prove the Lemma. First, Eq. 11 is true for $n = 1$, by direct calculation, since $x^n = x = \frac{1}{2^0}(q_1) = q_1$

Assume that Eq. 11 is true for a particular positive integer $n = 2m + 1$, that is

$$x^{2m+1} = 2^{-2m}\sum_{k=0}^{[\frac{2m+1}{2}]}(-1)^{\frac{k}{2}}\binom{2m+1}{k} q_{2m-2k+1}(x)$$

It must be valid for $n = 2m + 3$

$$\begin{aligned} x^{2m+3} &= \frac{1}{2^2 2^{2m}}\Big[q_{2m+1}(x) - \binom{2m+1}{1} q_{2m-1}(x) + \binom{2m+1}{2} q_{2m-3}(x) + \cdots\Big] q_2(x) \\ &= \frac{1}{2^{2m+2}}\Big[q_{2m+1}(x)q_2(x) - \binom{2m+1}{1} q_{2m-1}(x)q_2(x) + \binom{2m+1}{2} q_{2m-3}(x)q_2(x) \dots\Big] \\ &= \frac{1}{2^{2m+2}}\Big[q_{2m+3}(x) - q_{|2m-1|}(x) - \binom{2m+1}{1}(q_{2m+1}(x) - q_{|2m-3|}(x)) + \binom{2m+3}{1}[q_{|2m-1|}(x) - q_{2m-5}(x)] \\ &= \frac{1}{2^{2m+2}}\Big[q_{2m+3}(x) - \Big[\binom{2m+1}{0} q_{2m+1}(x) + \binom{2m+1}{1} q_{2m+1}\Big] + \Big[\binom{2m+1}{1} + \binom{2m+1}{2}\Big] q_{2m-1} - \Big[\binom{2m+1}{2} + \binom{2m+1}{3}\Big] q_{2m-3}\Big] \end{aligned}$$

$$\binom{2m+1}{i} + \binom{2m+1}{i+1} = \binom{2m+3}{i+1}, \quad i = 0, 1, \dots$$

Therefore;

$$x^{2m+3} = \frac{1}{2^{2m+3}}\Big[q_{2m+3}(x) - \binom{2m+3}{1} q_{2m+1}(x) + \binom{2m+3}{2} q_{2m-1}(x) \dots\Big]$$

which is the required result. By the same way, one can prove when n is even.

Implementation of Modified Pell Spectral Method on Some Problems

A special class of singular initial value problem is considered in this section.

$$\ddot{f}(x) + \frac{m}{x}\dot{f}(x) + g(x)f(x) = h(x) \quad (13)$$

subject to $f(0) + \dot{f}(0) = \alpha$ (14)

which arising in physiology: The unknown function $f(x)$ is approximated by

$$f(x) = c^T q(x) \quad (15)$$

where: c and $q(x)$ are defined by $c^T = [c_0 \quad c_1 \quad \dots \quad c_n]^T$ and $q(x) = [q_0(x) \quad q_1(x) \quad \dots \quad q_n(x)]$

one can obtain

$$\dot{f}(x) = c^T\dot{q}(x) = b^T q(x), \quad (16)$$

$$\ddot{f}(x) = c^T\ddot{q}(x) = d^T q(x) \quad (17)$$

rewrite $g(x)f(x)$ and $h(x)$ as

$$g(x)f(x) = h^T q(x) \quad (18)$$

and $h(x) = h^T q(x)$ (19)

by substituting Eqns. 15-19 into Eq. 13, yields

$$b^T q(x) + \frac{m}{x}d^T q(x) + g^T q(x) = h^T q(x) \quad (20)$$

Rewrite Eq. 20 as follows

$xb^T q(x) + mb^T q(x) + xg^T q(x) = xh^T q(x)$ or

$$d_1^T q(x) + mb^T q(x) + g_1^T q(x) = h_1^T q(x) \quad (21)$$

In addition, using Eqns. 14-15 to get

$$c^T q(0) + b^T q(0) = \alpha \quad (22)$$

Now, the resulting Eq. 21 generates n linear equations together with Eq. 22, which can be solved to obtain the vector, c^T and hence the approximate solution $f_n(x)$ can be obtained.

Test Example 1

Consider the following singular initial value problem

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$$\ddot{f}(x) + \frac{2}{x}\dot{f}(x) - 10f(x) = 12 - 10x^4 \quad (23)$$

subject to the condition $f(0) + \dot{f}(0) = 0$ (24)

This example is solved by modified Pell polynomials spectral method with $n = 4$

$f(x) = cq^T(x)$

$\dot{f}(x) = bq_4^T(x)$

$\ddot{f}(x) = dq_4^T(x)$

$x\ddot{f}(x) = d_1q^T(x), xf(x) = c_1q^T(x)$

$12x - 10x^5 = h_1q_5^T(x)$

where

$c = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4]$

$b = [(a_1 - 3a_3) \quad (4a_2 - 8a_4) \quad 6a_3 \quad 8a_4 \quad 0]$

$d_1 = [-12a_3 \quad (4a_2 - 56a_4) \quad 12a_3 \quad 24a_4 \quad 0]$

$d = [(4a_2 - 32a_4) \quad 24a_3 \quad 48a_4 \quad 0 \quad 0]$

$h_1 = \left[0 \quad \frac{23}{4} \quad 0 \quad \frac{50}{16} \quad 0 \quad \frac{-5}{8}\right]$

$q_4^T = [q_0 \quad q_1 \quad q_2 \quad q_3 \quad q_4]^T$

$q_5^T = [q_0 \quad q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5]^T$ and Eq. 24 can be approximated to obtain the following equation

$$[a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4]q_4^T(x) = 0 \quad (25)$$

Eq. 23 leads to the equation:

$$d_1d_4^T(x) + 2bd_4^T(x) - 10c_1d_4^T(x) = h_1d_5^T(x) \quad (26)$$

From Eqns. 25-26, the following five linear equations are obtained:

$a_0 + a_1 + a_2 + 3a_3 + a_4 = 0$

$-5a_4 = \frac{-5}{8}, \qquad 5a_3 = 0, \qquad 45a_4 - -5a_2 = \frac{50}{16},$

$29a_3 - 5a_1 = 0$

The modified Pell polynomials coefficients are given by the vector $c = \left[\frac{-5}{8} \quad 0 \quad \frac{1}{2} \quad 0 \quad \frac{1}{8}\right]$. The approximate solution of Eq. 23 in comparison with the exact solution is listed in Figure 1.

The results section is where you tell the reader the basic descriptive information about the scales you used (report the mean and standard deviation for each scale). If you have more than 3 or 4 variables in your paper, you might want to put this descriptive information in a table to keep the text from being too choppy and bogged down (see the APA manual for ideas on creating good tables). In the results section, you also tell the reader what statistics you conducted to test your hypothesis (-ses) and what the results indicated. In this paper, you conducted bivariate correlation (s) to test your hypothesis.

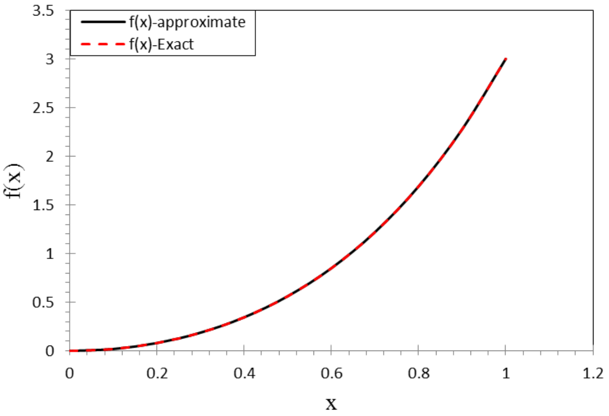

Figure 1. Comparison between exact solution and approximate solution for test example 1.

Test Example 2

Consider the following linear Lane-Emden equation

$$\ddot{f}(x) + \frac{1}{x}\dot{f}(x) + f(x) = x^2 - x^3 - 9x + 4,$$

$$f(0) + \dot{f}(0) = 0 \quad (27)$$

The approximate solution is given for this equation by the proposed modified spectral Pell method with $n = 4$. The following six linear equations are obtained

$a_0 + a_1 + a_2 + 3a_3 + a_4 = 0,$

$\frac{1}{2}a_3 = \frac{-1}{8}, \ \frac{35}{2}a_3 + \frac{1}{2}a_1 = -4, \ \frac{1}{2}a_4 = 0, \ \frac{15}{2}a_2 - 64a_4 + a_0 = \frac{13}{4}, -15a_3 + \frac{1}{2}a_1 = \frac{33}{8},$

Solving this system will give the vector c

$$c = \left[\frac{-1}{2} \quad \frac{3}{4} \quad \frac{1}{2} \quad \frac{-1}{4} \quad 0\right]$$

and the approximate solution is

$$f(x) = \frac{-1}{2}q_0(x) + \frac{3}{4}q_1(x) + \frac{1}{2}q_2(x) - \frac{1}{4}q_3(x)$$

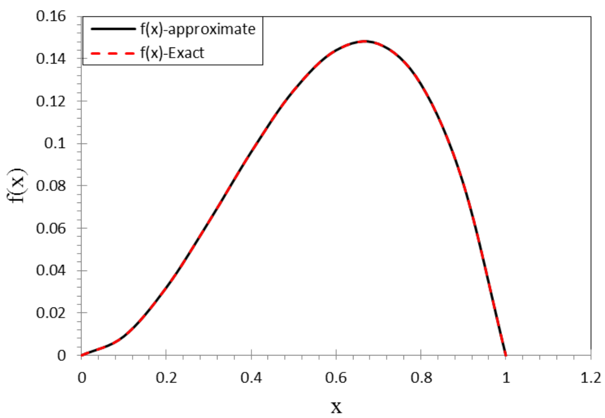

Figure 2. Comparison between exact solution and approximate solution for test example 2.

The obtained approximate solution of Eq. 27 in comparison with the exact solution is listed in Figure 2.

Test Example 3

Consider the nonhomogeneous singular initial value problem [18]

$$\ddot{f}(x) + \frac{8}{x}\dot{f}(x) + xf(x) = x^5 - x^4 + 44x^2 - 30x, \quad (28)$$

together with $f(0) = 0, \dot{f}(0) = 0$ applying the modified Pell method presented in section 4 for this problem. The modified Pell coefficients c_i's are obtained,

$$c = \begin{bmatrix} \frac{3}{8} & \frac{3}{4} & \frac{-1}{2} & \frac{-1}{4} & \frac{1}{8} \end{bmatrix}.$$

The obtained approximate solution is listed in comparison with other methods and the exact solution $f(x) = x^4 - x^3$ in Table 2. A comparison of ADM and HWCM $(N = 16)$ solutions listed in [18] with our method $(N = 4)$ is also included in Table 2 and Figure 3 against the exact solution.

Table 2. Comparison of ADM and HWCM $(N = 16)$ solutions with our method $(N = 4)$ against the exact solution.

x=1/32	ADM	HWCM	Exact	Proposed Method
1	0	-0.000048	-0.000029	-0.000029
3	0	-0.0000640	-0.000746	-0.000746
5	0	-0.003104	-0.003218	-0.003218
7	0	-0.008034	-0.008177	-0.008177
9	0	-0.015832	-0.015990	-0.015990
11	0	-0.026497	-0.026656	-0.026656
13	0	-0.039660	-0.039809	-0.039809
15	0	-0.054590	-0.054717	-0.054717
17	0	-0.070188	-0.070281	-0.070281
19	0	-0.084990	-0.085036	-0.085036
21	0	-0.097163	-0.097151	-0.097151
23	0	-0.0104511	-0.104430	-0.104430
25	0	-0.104470	-0.104308	-0.104308
27	0	-0.094110	-0.093855	-0.093855
29	0	-0.070136	-0.069777	-0.069777
31	-0.028411	-0.028885	-0.028410	-0.028410

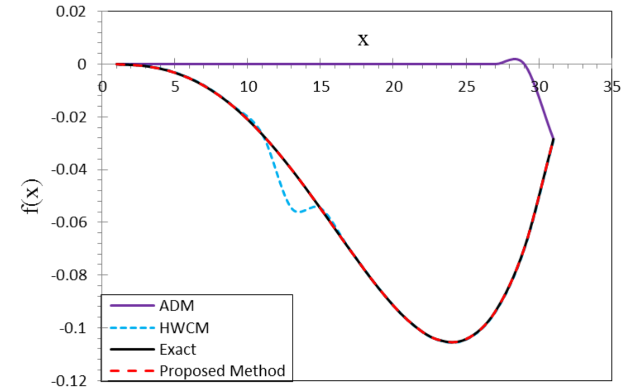

Figure 3. Comparison of approximate solutions in [18] with our method $(N = 4)$ against the exact solution.

The error analysis for different values of $\boldsymbol{N}$ is given in Table 3.

Table 3. The error analysis for different values of N

N	ADM L_∞	HWCM L_∞	HWAGM L_∞	Proposed Method
8	$1015e^{-01}$	$1.6292e^{-03}$	$4.0839e^{-04}$	0
16	$1.0443e^{-01}$	$4.7476e^{-04}$	$1.0293e^{-04}$	-
32	$1.0520e^{-01}$	$1.2686e^{-04}$	$3.0278e^{-05}$	-
64	$1.054e^{-01}$	$3.2718e^{-05}$	$8.0095e^{-06}$	-
128	$1.045e^{-01}$	$8.3037e^{-06}$	$2.0090e^{-06}$	-
256	$1.0546e^{-01}$	$2.0904e_{-06}$	$9.0098e^{-07}$	-

CONCLUSIONS

This work concerns with modified Pell polynomials. The analytical formula associated with the first derivative of modified Pell polynomials is constructed in terms of modified Pell polynomials themselves. Then, the relationship between the coefficients in the differentiated expansions of modified Pell polynomials and those of the original expansion is formulated. Other important properties of modified Pell polynomials are derived in this article. As an application and with the help of the obtained formulas, spectral solutions of some singular initial value problems are proposed.

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