On ERT And MERT-Rings

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في الحلقات من النوع ERT وMERT

الملخــص

الهدف الرئيس من البحــث هــو دراســة الحلقــات مــن الــنمط ERT وMERT لكي ندرس العلاقة بين هذه الحلقات والحلقات المنتظمة من النمط-II .

ABSTRACT

The main purpose of this paper is to study ERT and MERT rings, in order to study the connection between such rings and II-regular rings.

1- Introduction:

Throughout this paper, R denotes an associative ring with identity, and all modules are unitary right R-module. Recall that; 1- An ideal I of the ring R is essential if I has a non-zero intersection with every non-zero ideal of R; 2- A ring R is said to be $\prod$-regular if for every a in R there exist a positive integer n and b in R such that $a^n = a^n b a^n$ 3- A right R-module M is said to be GP- injective if, for any $0 ¹ a \in R$, there exists a positive integer n such that $a^n ¹ 0$ and any right R-homomorphism *of* a^nR into M extends to one of R into M. 4- For any element a in R, $r(a)$, $I(a)$ denote the right annihilator of a and the left annihilator of a, respectively.

2- ERT-R1NGS:

Following [3J, a ring R is said to be ERT-ring if every essential right ideal of R is a two-sided ideal.

Definition 2-1:

A ring R is said to be right weakly regular if for all a in R, there exists b in RaR such that $a = ab$, or equivalently every right ideal of R is idempotent.

We begin this section with the following main result:

Theorem 2.2:

If R is ERT-ring with every essential right ideal is idempotent, then R is weakly regular.

Proof:

For any $a \in R$, if RaR not essential, then there exists an ideal I, such that $K = RaR \oplus I$ is essential then $K = K^2$.

In order to prove that R is weakly regular, we need to prove $RaR = (RaR)^2$.

For a $\in K$, we have $a \in K^2$ *,that is* $a \in (RaR \oplus I)^2$

Thus $a = (rar' + i)(sas' + i')$ for some r, r' $s, s' \in R$ and i, $i' \in$ I.

This implies that $a = (rar' + i)sas' + (rar' + i)i'$

$$= rar'sas' + isas' + (rar^1 + i)\, i'$$

but $isas' \in I \cap RaR = 0$, also we have $(rar' + i)i' \in RaR \cap I = 0$.
Therefore $a = (rar')(sas') \in (RaR)^2$, this implies that RaR Í $(RaR)^2$ Thus $RaR = (RaR)^2$, this proves that R is weakly regular ring.
Following [2], the singular submodule *of* R is
$Y(R) = \{y \in R, r(y)$ *is essential right ideal of* $R\}$.

Theorem 2.3:
Let R be a semi-prime ERT right GP-injective ring. Then R is a right non singular.

Proof:
Let E be an essential right ideal of R. Then E is a two-sided ideal, and hence $l(E)$ is a two-sided ideal *of* R.
Now $(l(E) \cap E)^2$ Í $(E)E = 0$.
Since R is semi-prime, then $l(E) \cap E = 0$, whence $l(E) = 0$. This proves that R is right non singular.

3- MERT-R1NGS:
Following [3], a ring R is said to be MERT-ring if every maximal essential right ideal of/? is a two-sided ideal.

Theorem 3.1:
Let R be an MERT-ring, if for any maximal right ideal A/of R, and for any b e M, bR/bM is GP-injective, then R is strongly Pi-regular ring.

Proof:
Let b be a non-zero element in R, we claim that $b^n r+r(b'')=R$.
If $b^n r + r(b^n)$ ¹ R, let M be a maximal right ideal containing $b^n r + r(b^n)$. Then M is essential right ideal of R.
If $bR = bM$, then b = bc, for some c in M, this implies $(1-c) \in r(b)$ Ì $r(b^n)$ Ì M, therefore $1 \in M$, this contradics M ¹ R.

Now, since *R/ M @ bR/bM.* Then R */ M* is GP-injective.
Now, define *f : b* ^{n} *R ® R / M by f(b*^{n}*r)=r* + M, note that f is a well-defined R-homomorphism.
Since R/M is GP-injective, then there exists $c \in R$, such that:
1+M=f(b")=*cb"*+M and so (1-c b^n) $\in$ M, since $b^n \in$ M, and R is MERT-ring, this implies that M is a two-sided ideal, and hence $\in$ c *b"* $\in$ M.
Thus I$\in$ M, a contradiction.
Therefore $b^n R + r(b^n) = R$.
In particular $l=b^n u+v; v \in r(b^n)$, $u \in R$.
Thus $b^n = b^{2n} u$ and therefore R is strongly $\prod$-regular ring.

Theorem 3.2:

If *R* is MERT-ring with every simple singular right ideal is GP-injective, then *Y(R)=0.*

Proof:

If *Y(R) ¹0,* by Lemma (7) of [6], there exists *0 ¹ y* $\in$ *Y(R)* with $y^2=0$. Let *L* be a maximal right ideal of *R,* set L = *y* R+ r(y) , we claim that *L* is essential right ideal of R. Suppose this is not true, then there exists a non-zero ideal *T* of R such that L $\cap$ T =*(0).* Then yRT $\subseteq$ *LT* $\subseteq$*L* $\cap$ *T* =*0* implies T $\subseteq$ r(y) $\subseteq$L, so
L Ç T=(0). This contradiction proves that *L* is an essential right ideal, that is *R/L* is
simple singular and hence *R/L* is GP-injective.
Now; Let *f;yR —> R/L* be defined by *f(yr)=r+L,* then f is a well-defined *R*-
homomorphism.
Since *R/L* is GP-injective, so *\$ c Î R,* such that *l+L=f(y)=cy+L.*
Hence *l+L=cy+L*, implies that *1-cy Î L.*
Since *R* is MERT, then *eye L* and thus *1 Î L,* a contradiction.
Therefore *Y(R)=[0].*

Following [1], a ring *R* is zero insertive (briefly ZI) if for *a, b Î R, ab=0* implies *aRb=0.*

Theorem 3.3:

Let R be a ZT ring. If every simple singular rights-modules is GP-injective which is left self-injective, then R is strongly H-regular ring.

Proof:

Since R is simple singular GP-injective, then R is semi-prime, by Lemma (4)
of [5].
Thus for any left ideal I, $L(I) Ç l = 0$.
Since R is simple singular GP-injective and ZI, then R is reduced and hence $r(a)=l(a)$ for any element a in R.
Thus $l(r(a)) Ç l(a)=l(l(a)) Ç l(a)=0$.
Since R is left self-injective ring, then aR is a right annihilator, by Proposition (4)of[4].
Since $r(a)$ Í $r(a^n)$, then $a^nR = r(a^{fl})$.
Now, since $R= r(l(r(a)))+ r(l(a))_9$ then we have $R = r(l(r(a^n)) + r(l((a^n)) = r(a^n) +a^nR$
In particular, for some b in R, and d in $r(a^n)$.
Thus $a'' = a^{n2} b$.
Therefore R is strongly ∏-regular.

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