BLOCK Methods for Solving Nth Order Linear Differential Difference Equations

Shymaa Hussain Salih Al-khaylani
Department of Applied Sciences
University of Technology

الخلاصـة: يقـدم هـذا البحـث طريقـة مطـورة جديـدة مـع خوارزميـة جديـدة لتطـوير الحلـول العدديـة للمعـادلات الفـروق الخطيـة مـن الرتـب العليـا باسـتخدام طـرق البلـوك .مـن الممكـن ملاحظـة كفـاءة هـذة الطريقـة وسـهولة الحـسابات فيهـا حيـث تمـت مقارنـة نتـائج هـذه الخوارزميات مع نتائج متسلسلة تيلر من خلال بعض الامثلة العددية .

Abstract:-

The paper presents anewly-developed method with new algorithm to modify numerical solutions for nth-order linear differential difference equations using Block method. Moreover, this method can be used comparatively greater computational ease and efficiency. The results of these algorithms are compared with the Taylor series method .Three numerical examples are given for conciliated the results of this method.

Keywords: Differential Difference Equation; Numerical solution, Block Methods.

1.Introduction:-

Differential Difference Equations are differential equations in which function appears with difference values of the argument and which has been developed over twenty years , only in the last few years has much effort been devoted to study differential equation s of the form

$$F(t, y(t), y(t-t_1),..., y(t-t_k), y'(t), y'(t-t_1),..., y'(t-t_k), \\ y^n(t),..., y^n(t-t_k)) = 0 \quad ... (1)$$

Where F is a given function and $t_1, t_2, ..., t_k$ difference order k [1].

The differential equation (1) is categorized into three types: - Equation (1) is called a Retarded type, if the highest- order derivative of unknown function appears for just one value of the argument. Equation (1)is called a Neutral type, if the highest- order derivative of unknown function appears both with and without difference argument. All other differential equation (1) with a advanced types [2,3] .

The main difficulty between differential difference equations and ordinary differential equations is the kind of initial condition that should be used in differential difference equation differs from ordinary differential equation so that one should specify in differential difference equations an initial function on some interval of length t , say $[t_0 - t, t_0]$ and then try to find the solution of equation (1) for all $t \geq t_0$.

2. Block Method: -

A Block method provides easy and efficient mean for the solution of the many problems arising in various fields of science and engineering. The concept of Block method is essentially an extrapolation procedure and has the advantage of being self-starting. Block method was described for differential equation by Shampin and for integral equations was given by Sumaya [4, 5].

In this research we employ Block method for finding numerical solution for Nth order differential-difference equations.

Consider the following first order differential equation

$$\frac{dy}{dt} = f(t, y) \quad \text{With} \quad y(t_0) = y_0 \qquad \dots (2)$$

A four-order Block method for equ. (2) Computed by: -

$$y_{n+2} = y_n + 2hy'_n \qquad \text{Order 2}$$

$y_{n+1} = (y_n + y_{n+2})/2$ Order 2

$y_{n+1} = y_n + \frac{h}{2}[y'_n + y'_{n+1}]$ Order 3

$y_{n+2} = y_n + \frac{h}{2}[y'_n + y'_{n+2}]$ Order 3

$y_{n+1} = y_n + \frac{h}{12}[5y'_n + 8y'_{n+1} - y'_{n+2}]$ Order 4

$y_{n+2} = y_n + \frac{h}{3}[y'_n + 4y'_{n+1} + y'_{n+2}]$ Order 4

Block method of second and third order have been little used for ordinary differential equations, in general, and differential – difference equations, in particular, because they required more evaluation of f .

However the following fourth order Block method which is most popular and more efficient for dealling with differential equations.

Let

$$
\begin{aligned}
B_1 &= f(t_n, y(t_n)) \\
B_2 &= f(t_n + h, y(t_n) + hB_1) \\
B_3 &= f(t_n + h, y(t_n) + h/2B_1 + h/2B_2) \\
B_4 &= f(t_n + 2h, y(t_n) + 2hB_3) \\
B_5 &= f(t_n + h, y(t_n) + h/12(5B_1 + 8B_3 - B_4)) \\
B_6 &= f(t_n + 2h, y(t_n) + h/3(B_1 + B_4 + 4B_5))
\end{aligned} \quad \dots (3)
$$

Then fourth order Block method may be written in the form.

$$y_{n+1} = y_n + \frac{h}{12}(5B_1 + 8B_3 - B_4) \quad \dots (4)$$

$$y_{n+2} = y_n + \frac{h}{3}(B_1 + 4B_5 + B_6) \quad \dots (5)$$

2.1 Solution of 1st Order Linear Differential –Difference Equation Using Fourth order Block Method :

Consider the following first order differential difference equation.

$$y'(t) = f(t, y(t), y(t-t), y'(t-t)), t \in [t_{0,\infty}) \qquad \ldots (6)$$

With initial function

$$y(t)=f(t) \quad \text{for} \quad t_0 - t \le t \le t_0$$

equ. (6) may be solved by Block method if we use the initial function .

$$y'(t)=f(t,y(t),f(t-t),f'(t-t)) \qquad \ldots (7)$$

With initial condition

$$y(t_0)=f(t_0)$$

For applying the Block method equation (7) by using. (3), (4) and (5) we get the following formula.

$$y_{j+1} = y_j + \frac{h}{12}(5B_1 + 8B_3 - B_4)$$

$$y_{j+2} = y_j + \frac{h}{3}(B_1 + 4B_5 + B_6)$$

Where

$$B_1 = f(t_j, f(t_j), f(t_j - t), f'(t_j - t))$$

$$B_2 = f(t_j + h, f(t_j) + hB_1, f(t_j - t) + hB_1, f'(t_j - t) + hB_1)$$

$$B_3 = f(t_j + h, f(t_j) + h/2B_1 + h/2B_2, f(t_j - t) + h/2B_1 + h/2B_2, f'(t_j - t) + h/2B_1 + h/2B_2)$$

$$B_4 = f(t_j + 2h, f(t_j) + 2hB_3, f(t_j - t) + 2hB_3, f'(t_j - t) + 2hB_3)$$

$$B_5 = f(t_j + h, f(t_j) + h/12(5B_1 + 8B_3 - B_4), f(t_j - t) + h/12(5B_1 + 8B_3 - B_4), f'(t_j - t) + h/12(5B_1 + 8B_3 - B_4))$$

$$B_6 = f(t_j + 2h, f(t_j) + h/3(B_1 + B_4 + 4B_5), f(t_j - t) + h/3(B_1 + B_4 + 4B_5), f'(t_j - t) + h/3(B_1 + B_4 + 4B_5))$$

for j=0, 1, 2… n

Algorithm (Blo4SN)

The numerical solution of (1^{st} order linear differential-difference equation) using fourth order Block method is computed as follows: -

Step1: set h is step size

Step 2: Compute $B_1 = f(t_j, f(t_j), f(t_j - t), f'(t_j - t))$.

Step 3: Compute $B_2 = f(t_j + h, f(t_j) + hB_1, f(t_j - t) + hB_1, f'(t_j - t) + hB_1)$

Step4: Compute

$$B_3 = f(t_j + h, f(t_j) + (h/2B_1 + h/2B_2), f(t_j - t) + (h/2B_1 + h/2B_2), f'(t_j - t) + (h/2B_1 + h/2B_2))$$

Step 5: Compute

$$B_4 = f(t_j + 2h, f(t_j) + 2hB_3, f(t_j - t) + 2hB_3, f'(t_j - t) + 2hB_3)$$

Step6: Compute

$$B_5 = f(t_j + 2h, f(t_j) + h/12(5B_1 + 8B_3 - B_4), f(t_j - t) + h/12(5B_1 + 8B_3 - B_4), f'(t_j - t) + h/12(5B_1 + 8B_3 - B_4))$$

Step7: Compute

$$B_6 = f(t_j + 2h, f(t_j) + h/3(B_1 + B_4 + 4B_5), f(t_j - t) + h/3(B_1 + B_4 + 4B_5), f'(t_j - t) + h/3(B_1 + B_4 + 4B_5))$$

Step8: Compute $y_{j+1} = y_j + \frac{h}{12}(5B_1 + 8B_3 - B_4)$

Step 9: Compute $y_{j+2} = y_j + \frac{h}{3}(B_1 + 4B_5 + B_6)$

Step 10: Set t_j = jh for each $j = 0,1,....,n$.

2.2 Solution of System 1^{st} Order Linear Differential-Difference Equations Using Fourth Order Block Method

Consider a system of first order differential-difference equation such as:

$$y'(t)=f_i(t,y_1(t),...,y_n(t),y_1(t-t_1),...,y_n(t-t_n),y_1'(t-t_1),...,y_n'(t-t_n)),$$

$$t\in[-t,\infty)$$

(8)

With initial function

$$y_1(t)=f_1(t) \quad for \quad t_0-t_1\le t\le t_0$$

$$\vdots \qquad\qquad \vdots$$

$$y_n(t)=f_n(t) \quad for \quad t_0-t_n\le t\le t_0$$

Equation (7) may be solved by fourth order Block method if

$$y'(t)=f(t,y_1(t),...,y_n(t),f_1(t-t_1),...,f_n(t-t_n),f_1'(t-t_1),...,f_n'(t-t_n))$$

With initial condition

$$y_1(t_0)=f_1(t_0),....,y_n(t_0)=f_n(t_0)$$

For treating system of differential-difference equation equ.(3),(4)and(5)we have

$$y_{ij+1}=y_{ij}+\frac{h}{12}(5B_{1i}+8B_{3i}-B_{4i})$$

$$y_{ij+2}=y_{ij}+\frac{h}{3}(B_{1i}+4B_{5i}+B_{6i})$$

Where

$$B_{1i}=f_i(t_j,f_1(t_j),\ldots,f_m(t_j),f_1(t_j-t_1),\ldots,f_m(t_j-t_m),f_1'(t_j-t_1),\ldots,f_m'(t_j-t_m))$$

$$B_{2i}=f_i(t_j+h,f_1(t_j)+hB_{11},\ldots,f_m(t_j)+hB_{1m},f_1(t_j-t_1)+hB_{11},\ldots,$$
$$f_m(t_j-t_m)+hB_{1m},f_1'(t_j-t)+hB_{11},\ldots,f_m'(t_j-t_m)+hB_{1m})$$

$$B_{3i}=f_i(t_j+h,f(t_j)+(h/2B_{11}+h/2B_{21}),\ldots,f_m(t_j)+(h/2B_{1m}+h/2B_{2m}),f_1(t_j-t_1)+(h/2B_{11}+h/2B_{21}),\ldots,$$
$$f_m(t_j-t_m)+(h/2B_{1m}+h/2B_{2m}),f_1'(t_j-t_1)+(h/2B_{11}+h/2B_{21}),\ldots,f_m'(t_j-t_m)+(h/2B_{1m}+h/2B_{2m}))$$

$$B_{4i}=f_i(t_j+2h,f_1(t_j)+2hB_{31},\ldots,f_m(t_j)+2hB_{3m},f_1(t_j-t_1)+2hB_{31},\ldots,f_m(t_j-t_m)+2hB_{3m},$$
$$f_1'(t_j-t_1)+2hB_{31},\ldots,f_m'(t_j-t_m)+2hB_{3m})$$

$$B_{5i}=f_i(t_j+h,f(t_j)+h/12(5B_{11}+8B_{31}-B_{41}),\ldots,f_m(t_j)+h/12(5B_{1m}+8B_{3m}-B_{4m}),f_1(t_j-t_1)+h/12(5B_{11}+8B_{31}-B_{41}),\ldots,$$
$$f_m(t_j-t_m)+h/12(5B_{1m}+8B_{3m}-B_{4m}),f_1'(t_j-t_1)+h/12(5B_{11}+8B_{31}-B_{41}),\ldots,f_m'(t_j-t_m)+h/12(5B_{1m}+8B_{3m}-B_{4m}))$$

$$B_{6i}=f_i(t_j+2h,f(t_j)+h/3(B_{11}+B_{41}+4B_{51}),\ldots,f_m(t_j)+h/3(B_{1m}+B_{4m}+4B_{5m}),f_1(t_j-t_1)+h/3(B_{11}+B_{41}+4B_{51}),\ldots,$$
$$f_m(t_j-t_m)+h/3(B_{1m}+B_{4m}+4B_{5m}),f_1'(t_j-t_1)+h/3(B_{11}+B_{41}+4B_{51}),\ldots,f_m'(t_j-t_m)+h/3(B_{1m}+B_{4m}+4B_{5m}))$$

For j=0,1,…,n , i=0,1,…,m

Algorithm (Blo4SY)

The numerical solution of (System1st order linear differential-difference equation) using fourth order Block method is computed as follows: -

Step1: set h is step size

Step 2: Compute

$$B_{1i}=f_i(t_j,f_1(t_j),\ldots,f_m(t_j),f_1(t_j-t_1),\ldots,f_m(t_j-t_m),f'_1(t_j-t_1),\ldots,f'_m(t_j-t_m)).$$

Step 3: Compute

$$B_{2i}=f_i(t_j+h,f_1(t_j)+hB_{11},\ldots,f_m(t_j)+hB_{1m},f_1(t_j-t_1)+hB_{11},\ldots,$$
$$f_m(t_j-t_m)+hB_{1m},f'_1(t_j-t)+hB_{11},\ldots,f'_m(t_j-t_m)+h_{1m})$$

Step 4: Compute

$$B_{3i}=f_i(t_j+h,f(t_j)+(h/2B_{11}+h/2B_{21}),\ldots f_m(t_j)+(h/2B_{1m}+h/2B_{2m}),f_1(t_j-t_1)+(h/2B_{11}+h/2B_{21}),\ldots,$$
$$f_m(t_j-t_m)+(h/2B_{1m}+h/2B_{2m}),f'_1(t_j-t_1)+(h/2B_{11}+h/2B_{21}),\ldots f'_m(t_j-t_m)+(h/2B_{1m}+h/2B_{2m}))$$

Step 5: Compute

$$B_{4i}=f_i(t_j+2h,f_1(t_j)+2hB_{31},\ldots,f_m(t_j),+2hB_{3m},f_1(t_j-t_1)+2hB_{31},\ldots,f_m(t_j-t_m),+2hB_{3m},$$
$$f'_1(t_j-t_1)+2hB_{31},\ldots,f'_m(t_j-t_m)+2hB_{3m})$$

Step 6: Compute

$$B_{5i}=f_i(t_j+h,f(t_j)+h/12(5B_{11}+8B_{31}-B_{41}),\ldots,f_m(t_j)+h/12(5B_{1m}+8B_{3m}-B_{4m}),f_1(t_j-t_1)+h/12(5B_{11}+8B_{31}-B_{41}),\ldots$$
$$f_m(t_j-t_m)+h/12(5B_{1m}+8B_{3m}-B_{4m}),f'_1(t_j-t_1)+h/12(5B_{11}+8B_{31}-B_{41}),\ldots f'_m(t_j-t_m)+h/12(5B_{1m}+8B_{3m}-B_{4m}))$$

Step 7: Compute

$$B_{6i}=f_i(t_j+2h,f(t_j)+h/3(B_{11}+B_{41}+4B_{51}),\ldots f_m(t_j)+h/3(B_{1m}+B_{4m}+4B_{5m}),f_1(t_j-t_1)+h/3(B_{11}+B_{41}+4B_{51}),\ldots$$
$$f_m(t_j-t_m)+h/3(B_{1m}+B_{4m}+4B_{5m}),f'_1(t_j-t_1)+h/3(B_{11}+B_{41}+4B_{51}),\ldots f'_m(t_j-t_m)+h/3(B_{1m}+B_{4m}+4B_{5m}))$$

Step 8:Compute

$$y_{ij+1}=y_{ij}+\frac{h}{12}(5B_{1i}+8B_{3i}-B_{4i})$$

Step 9:Compute

$$y_{ij+2} = y_{ij} + \frac{h}{3}(B_{1i} + 4B_{5i} + B_{6i})$$

Step 10:

Set t_j = jh for each $j = 0,1,....,n$.

2.3 Solution of Nth-Order Linear Differential-Difference Equations Using Fourth Order Block Method

The general form of nth–order differential difference equations

$$y^n(t) = f(t, y(t), y'(t),...., y^{n-1}(t), y(t-t), y'(t-t),...., y^{n-1}(t-t)), \quad t \geq t_0$$

With initial function

$$y(t) = f(t) \quad \text{For} \quad t_0 - t \leq t \leq t_0$$

Where $f(t)$ and its first n-1 derivatives $f'(t),....,\ f^{n-1}(t)$ are continuous on interval $[t_0 - t, t_0]$

Obviously, the nth order equation with difference argument may be replaced for equations without difference argument, by a system of nth-equation of first order differential difference equation as follows .

Let

$$x_1(t) = y(t)$$
$$x_2(t) = y'(t)$$
$$\vdots$$
$$x_{n-1}(t) = y^{n-2}(t)$$

Therefore, we get the following system of the first order equation

$$x_1'(t) = x_2(t)$$
$$x_2'(t) = x_3(t)$$
$$\vdots$$
$$x_{n-1}'(t) = x_n(t)$$
$$x_n'(t) = f(t, x_1(t), \ldots., x_n(t), x(t-t), \ldots., x_n(t-t))$$

The above system can be treated using method prescribed in previous sections of this search.

3. Test Example:

Example (1):

Consider the problem, which is 1st order, linear differential

$$y'(t) = y(t) - y'(t-1) + 2y(t-1) + e^{2t} \qquad t \geq 0$$

difference equation: -

With initial function

$y(t) = e^{2t} \qquad 0 \leq t \leq 1$

While the exact solution is

$y(t) = e^{2t} \qquad 0 \leq t \leq 1$

Take n=10, h=0.1 and t_i = ih i=0, 1…n . Fourth order Block method used to solve this problem. Their results are obtained by running program Blo4SN

Table (1) presents the comparison between the exact and numerical solution using: Blo4SN and Taylor series [6] , depending on least square error.

t	Exact	Blo4SN	Taylor series [6]
0	1.0000	1.000	1.000
0.1	1.2213	1.2214	1.2207

0.2	1.4916	1.4918	1.4902
0.3	1.8218	1.8221	1.8194
0.4	2.2251	2.2255	2.2216
0.5	2.7177	2.7183	2.7127
0.6	3.3193	3.3201	3.3126
0.7	4.0541	4.0552	4.0453
0.8	4.9516	4.9530	4.9402
0.9	6.0478	6.0496	6.0332
1	7.3877	7.3891	7.3683
L.S.E.		9.85e-006	0.0011

Table (1) Solution of Example (1)

Example (2): -

Consider the problem, which are systems of 1^{st} order linear differential difference equation: -

$$y_1'(t) = y_5(t-1) + y_3(t-1), \qquad t \geq 0$$

$$y_2'(t) = y_1(t-1) + y_2(t-\frac{1}{2}), \qquad t \geq 0$$

$$y_3'(t) = y_3(t-1) + y_1(t-\frac{1}{2}), \qquad t \geq 0$$

$$y_4'(t) = y_5(t-1)y_4(t-1), \qquad t \geq 0$$

$$y_5'(t) = y_1(t-1), \qquad t \geq 0$$

With initial function

$$y_1(t) = \exp(t+1), \qquad t \le 0$$

$$y_2(t) = \exp(t+\frac{1}{2}), \qquad t \le 0$$

$$y_3(t) = \sin(t+1), \qquad t \le 0$$

$$y_4(t) = \exp(t+1), \qquad t \le 0$$

$$y_5(t) = \exp(t+1), \qquad t \le 0$$

With analytical solution

$$y_1(t) = e^t - \cos t + e, \qquad 0 \le t \le \frac{1}{2}$$

$$y_2(t) = 2e^t + \exp(\frac{1}{2}) - 2, \qquad 0 \le t \le \frac{1}{2}$$

$$y_3(t) = \exp(t+\frac{1}{2}) - \cos t + 1 - \exp(\frac{1}{2}) + \sin(1), \qquad 0 \le t \le \frac{1}{2}$$

$$y_4(t) = \frac{1}{2}\exp(2t) - \frac{1}{2} + e, \qquad 0 \le t \le \frac{1}{2}$$

$$y_5(t) = e^t + e - 1, \qquad 0 \le t \le \frac{1}{2}$$

Take n=10, h=0.1 and t_i = ih i=0, 1…n . Fourth order Block method used to solve this problem. Their results are obtained by running program Blo4SY.

Table (2) presents the comparison between the exact and numerical solution using: Blo4SY and Taylor series [6] ,depending on least square error.

Example (3): -

Consider the differential difference equation of the second order:

$$y''(t) + \frac{1}{2}y(t) - \frac{1}{2}y(t-3p) = 0, \qquad t \ge t_0$$

with initial functions

$$y(t) = \cos t \qquad -3p \le t \le 0$$

can be replaced in system of two first order equations.

$$x_1'(t) = x_2(t)\,, \qquad t \geq 0$$

$$x_2'(t) = -\frac{1}{2}x_1(t) + \frac{1}{2}x_1(t-3p)\,, \qquad t \geq 0$$

with initial conditions

$x(0)=1$, $x'(0)=0$ $-3p \leq t \leq 0$

which has the exact solution

$$x_1(t) = \cos t \qquad 0 \leq t \leq 3p$$

$$x_2(t) = -\sin t \qquad 0 \leq t \leq 3p$$

For the numerical results, least square error is obtained by running programs for the algorithms (Blo4SY) and Taylor series [6] for this problem.

t	$Exat_1$	$Blo4SY_1$	Taylor series [6]	$Exact_2$	$Blo4SY_2$	Taylor series[6]
0	1	1	1.000	0	0	0
0.1	0.9950	0.9950	0.9948	-0.0998	-0.0998	-0.1000
0.2	0.9801	0.9801	0.9797	-0.1987	-0.1987	-0.1990
0.3	0.9553	0.9553	0.9548	-0.2955	-0.2955	-0.2960
0.4	0.9211	0.9210	0.9203	-0.3894	-0.3894	-0.3900
0.5	0.8776	0.8776	0.8766	-0.4794	-0.4794	-0.4801
0.6	0.8253	0.8253	0.8241	-0.5646	-0.5646	-0.5653
0.7	0.7648	0.7648	0.7634	-0.6442	-0.6442	-0.6450
0.8	0.6967	0.6967	0.6950	-0.7174	-0.7173	-0.7181
0.9	0.6216	0.6216	0.6196	-0.7833	-0.7833	-0.7841
1	0.5403	0.5403	0.5381	-0.8415	-0.8415	-0.8422

L.S.E	4.88e-9	1.76e-4	L.S.E	5.745e-10	3.67e-6

Table (3) Solution of Example (3)

4. Conclusion:

Fourth order Block method has been presented for solving differential difference equations. The results show a marked improvement in the least square errors. For some test examples the following points are included

1-Fourth order Block method gives a better accuracy and consistent than Taylor series method for solving nth-order differential difference equations

2-The good approximation depends on the size of h, if h is decreased then the number of division points increases and L.S.E. approaches zero.

3- Fourth order Block method solves linear differential difference equations of any order by reducing the equation to system of first order.

5. References:

[1] Bellman, R. and Cooke, K.L.; Differential, Difference Equations, Academic press, Inc, New York 1963.

[2] Bassim ,N.A.; On The Numerical Solution of Delay Differential Equations, Ph.D. Thesis college of Science ,Al-Mustansiriah Unviersity ,2004

[3] Julian , L.and Simon ; Report On Introductory Talk On DDEs , School of computer science , McGill University , Montreal , Quebec ,Canada , May 2003

[4] Shampin L.and Watts H.; The Stability of the Block Methods, IMJ J.Numer.Anal, 13(2001)101-114

[5] Sumaya A.A.; Numerical Methods for A system of Linear Volterra Integral Equations M.SC. Thesis College of Science, Technology University 2005

[6] Al-Daynee ,K.K. , On Solution of Delay Boundary Value Problems ,College of Science ,Saddom University 2003