On Dual Rings

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الملخص

يقال للحلقة R بأنها اثنينية يمنـــى إذا كـــان rl(T)=T لكـــل مثـــالي أيمـــن T فـــي R. الهدف من هذا البحث هو تطوير بعض الخواص الأساســية للحلقـــات الاثنينيـــة، وإيجـــاد بعض العلاقات التي تـــربط الحلقـــات الاثنينيـــة والحلقـــات المنتظمـــة والحلقـــات المنتظمـــة بقوة.

ABSTRACT

A ring R is called a right dual ring if rl(T) = T for all right ideals T of R. The main purpose of this paper is to develop some basic properties of dual rings and to give the connection between dual rings, regular rings and strongly regular rings.

1. INTRODUCTION

Throughout this paper, R represents an associative ring with identity and all R-modules are unitary. Recall that: (1) A ring R is reduced if R contains no non–zero nilpotent element ; (2) R is said to be von Neumann regular (or just regular) ring if $a \in aRa$ for every a in R; (3) A right R-module M is called P-injective if, for any principal right ideal I of R, every right R-homomorphism of I into M extends to R. we say that, R is a right P-injective ring if R_R is P-injective; (4) R is called right duo-ring if every right ideal of R is a two- sided ideal; (5) For every $a \in R$, r(a) and l(a) will stand respectively for right and left annihilators of a; (6) Y(R) will denote the right singular ideal of R.

2. DUAL RINGS (BASIC PROPERTIES).

Following [7], a ring R is said to be a right dual ring if rl(T)=T, for all right ideals T of R. A left dual ring is similarly defined .

A ring R is called dual ring if R is a right and left dual ring.

Example. Let R be the set of all 2 × 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with a, b, c, $d \in Z_2$ (The ring of integers modulo 2).

A straightforward calculation, shows that R is a dual ring.

Following [1], a ring R is said to be a right Ikeda-Nakayama ring (right IN- ring) if $l(A \cap B) = l(A) + l(B)$ for all right ideals A and B of R.

In [3], Hajarnavis and Norton Proved that:

Lemma 2.1. Every dual ring is IN- ring.

We begin this section with the following lemma.

Lemma 2.2. Let R be a right dual ring, and let M_1 and M_2 be right ideals of R. Then $M_1 \subseteq M_2$ if and only if $l(M_2) \subseteq l(M_1)$.

Proof.

If $M_1 \subseteq M_2$, then obviously $l(M_2) \subseteq l(M_1)$.

Conversely, assume that $l(M_2) \subseteq l(M_1)$. Then $rl(M_1) \subseteq rl(M_2)$. By duality of R, we have $M_1 \subseteq M_2$.

The next proposition is a direct consequence of Lemma 2.2

Proposition 2.3. Let R be a dual ring. Then

1- M is a maximal right ideal of R if and only if l(M) is minimal left ideal.

2- M is a minimal right ideal of R if and only if l(M) is maximal left ideal.

Proof

(1). Let M be a maximal right ideal of R, and let L be a left ideal of R such that $(0) \subseteq L \subseteq l(M)$. Then by Lemma 2.2 , $R=r(0) \supseteq r(L) \supseteq rl(M) = M$. By maximally of M, we have $r(L) = R$, and this implies $L = l(R)$. Therefore $L = (0)$.

Conversely, assume that l(M) is a minimal left ideal of R, and let $M \subset I \subseteq R$ for some right ideal I of R.

Then $l(M) \supset l(I) \supseteq l(R) = (0)$. Hence $l(I) =(0)$, so $I = R$

(2). Let M be a minimal right ideal of R, and let L be a left ideal of R such that $l(M) \subset L \subseteq R$.

Then $M = rl(M) \supset r(L) \supseteq r(R) = (0)$. So $r(L) =(0)$ and hence $L = R$.

Conversely, let l(M) be a maximal left ideal, and let L be a right ideal such that $(0) \subseteq L \subseteq M$, then $R = l(0) \supseteq l(L) \supseteq l(M)$. So $l(L) = R$. Hence $L = (0)$.

Recall the following result of Nicholson and Yousif [5, Lemma 1.1].

Lemma 2.4. The following conditions are equivalent
1- R_R is P-injective.
2- $lr(a)= Ra$ for all a in R.
3- If $r(b) \subseteq r(a)$, for $a, b \in R$, then $Ra \subseteq Rb$.
4- $l[bR \cap r(a)] = l(b) + Ra$, for all $a, b \in R$.

Theorem 2.5 Let R be a right Noetherian P-injective ring, and let $r(L_1 \cap L_2) = r(L_1)+r(L_2)$ for all principal left ideals L_1 and L_2 of R. Then R is a right dual ring.

Proof. Let $0 \neq a \in R$. First we claim that $aR = rl(aR)$.
Clearly $aR \subseteq rl(aR)$. Let $b \in rl(aR)$. Then $xb = 0$ for all $x \in l(aR)$. Since $l(aR) \subseteq l(bR)$,then define $f: Ra \rightarrow Rb$, by $f(xa) = xb$. Clearly f is a well defined left R – homomorphism.
Since R is P-injective, there exists $c \in R$ such that $xb = f(xa) = xac$ for all $x \in R$, whence $b = ac \in aR$, yielding $aR = rl(aR)$. Since R is right Noetherian, then by [4, Theorem 2.3.13], every right ideal I of R can be written in the form
$I = a_1R+a_2R+\ldots.+a_nR$, and this implies

$$\begin{aligned} rl(I) &= r(l(a_1R) \cap l(a_2R) \cap \ldots. \cap l(a_nR)) \\ &= rl(a_1R) + rl(a_2R) + \ldots.+ rl(a_nR) \\ &= a_1R + a_2R + \ldots.+ a_nR = I. \end{aligned}$$

3.THE CONNECTION BETWEEN DUAL RINGS AND REGULAR RINGS .

The Purpose of this section is to show the connection between dual rings, regular rings and strongly regular rings.

Recall that a ring R is strongly regular if for every $a \in R$, $a \in a^2R$. Clearly a strongly regular ring is a reduced regular ring.
We begin this section with the following result.

Theorem 3.1. Let R be a reduced left or right dual ring. Then R is strongly regular.

Proof.
Let a be a non – zero element in R. Then $r(a) = r(a^2)$ (R is reduced). Since R is a left dual ring, by [6, Theorem 11], R_R is P-injective, and hence $Ra = lr(a)$ (Lemma 2.4). Whence $Ra = lr(a) = lr(a^2) = Ra^2$. This implies that $a = ra^2$, for some $r \in R$. Therefore R is strongly regular.

Next, we give other sufficient condition for dual ring to be strongly regular.

Theorem 3.2. Let R be a semi-prime left dual ring and right duo-ring. Then R is strongly regular.

Proof.
Let $0 \neq a \in R$, and let $I = r(a) \cap aR$, first we claim that $I^2 = (0)$.Suppose that $I^2 \neq (0)$. For any $d \in I$, $d \in r(a)$ and $d \in aR = Ra$ (R is a right duo-ring), so $d = ba$ for some $b \in R$, and $aba=0$. Thus $d^2 =0$ and hence $I^2 = (0)$. Since R is semi-prime, then $I=(0)$. Next, we claim that $r(a) = r(a^2)$, clearly $r(a) \subseteq r(a^2)$. Let $x \in r(a^2)$. Then $a^2x = 0$, so $a (ax) = 0$ and hence $ax \in r(a)$, but $ax \in aR$, then $ax \in aR \cap r(a) = (0)$. Therefore $x \in r(a)$. On the other hand since R is a left dual ring then $Ra = lr(a) = lr(a^2) = Ra^2$. Therefore R is strongly regular.

The next result provides a link between dual rings and regular rings.

Theorem 3.3. Let R be a right non–singular dual ring. Then R is regular ring.

Proof.
Let $0 \neq a \in R$, then by [6. Theorem 11] and (Lemma 2.4), $Ra = lr(a)$. Since R is a right non – singular ring, then $Y(R) = 0$. Whence r(a) is not essential right ideal of R. Then there exists a

non-zero right ideal L of R such that $r(a) \oplus L$ is essential right ideal of R. Now by Lemma 2.1 R is a right IN–ring. Then we have $lr(a) + l(L) = l(r(a) \cap L) = R$. Whence it follows that $Ra + l(L) = R$, while $lr(a) \cap l(L) \subseteq l(r(a) + L) = (0)$. So $Ra \cap l(L) = (0)$.
Thus $Ra = lr(a)$ is a direct summand. Therefore R is regular [2, Theorem 1.1].
Before closing this section we present the following result.

Proposition 3.4. Let R be a regular ring.
Then $r(L_1 \cap L_2) = r(L_1) + r(L_2)$ for all principal left ideals L_1 and L_2 of R.

Proof.
Obviously $r(L_1) + r(L_2) \subseteq r(L_1 \cap L_2)$ always holds.
Let $b \in r(L_1 \cap L_2)$, define $f_i \in \mathrm{Hom}_R(L_i, {}_RR)$, i = 1.2 as follows: $f_1(a_1)=a_1$ for all $a_1 \in L_1$ and $f_2(a_2) = a_2(1-b)$ for all $a_2 \in L_2$. The mapping $f(a_1+a_2)=f_1(a_1)+f_2(a_2)$ is a well defined left R-homomorphism,indeed if, $a_1 + a_2 = a'_1 + a'_2$ then $a_1 - a'_1 = -a_2 + a'_2 \in L_1 \cap L_2$ .But $b \in r(L_1 \cap L_2)$ therefore $a_2 b = a'_2 b$. Showing that $f(a_1 + a_2) = f(a'_1 + a'_2)$.Since R is regular, then ${}_RR$ is P-injective, so there exists $c \in R$ such that $f(a_1 + a_2) = (a_1 + a_2)c$.
This implies $a_1 + a_2(1-b) = f(a_1 + a_2) = (a_1 + a_2)c$, and therefore $a_1(1-c) + a_2(1-b-c) = 0$ for all $a_1 \in L_1$ and $a_2 \in L_2$. It follows that $1-c \in r(L_1)$ and $1-b-c \in r(L_2)$.
Therefore $b=(1-c) - (1-b-c) \in r(L_1) + r(L_2)$.
This shows $r(L_1 \cap L_2) = r(L_1)+r(L_2)$ for all principal left ideals L_1 and L_2 of R.

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