On The Product Of Positively Defined Operators

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Abstract

The main result is to prove that the product of two positively defined operators is positively defined if and only if it is normal. In general, the normality is required can not be dropped.

Keywords: Positively defined operators, Normal Operators, self-adjoint operators, Hilbert space.

حول حاصل ضرب المؤثرات المعرفة تعريفا ايجابيا

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الخلاصة

النتيجة الرئيسية هي برهنة أن حاصل ضرب مؤثرين معرفين تعريفا ايجابيا يكون مؤثرا معرفا تعريفا ايجابيا إذا وفقط إذا كان ناتج حاصل ضربهما مؤثرا طبيعيا. على العموم ناتج حاصل ضرب المؤثرين مؤثر طبيعي أمر ضرري لا يمكن إسقاطه.

الكلمات المفتاحية: المؤثرات المعرفة تعريفا ايجابيا، المؤثرات الطبيعية، مؤثر ذاتية المرافق، فضاء هلبرت.

Introduction

In 2002 Hichem M. Mortad communicated with Joseph A. Ball [2] proved the fact that if we have two self-adjoint operators (bounded or not) and if their product is normal, then it is self-adjoint provided a certain condition $\sigma(A) \cap \sigma(-A) \subseteq \{0\}$ satisfied, where A is one of two self-adjoint operators. However, nothing was said about the product of two positively defined operators, is positively defined or not? In this paper we answer this question positively, and we show that the normality is necessary for this case, we give a counter example.

We also note that one can prove this result without the condition $\sigma(A) \cap \sigma(-A) \subseteq \{0\}$ we use the positivity of operators instead of this condition. We also use the famous Fuglede-Putnam Theorem.

Results

We recall Hichem M. Mortad Theorem [2]

Theorem 1. *Let H and K be two bounded self-adjoint operators. Let K satisfies the condition* $\sigma(A) \cap \sigma(-A) \subseteq \{0\}$ *If HK is normal then it is self-adjoint.*

We also recall the Fugelede-Putnam Theorem [3]

Theorem 2. *For normal operators N_1 and N_2 and arbitrary operator A, if* $AN_1 = N_2 A$ *then* $AN_1^* = N_2^* A$.

Definition 1. *A bounded operator A is said to be positively defined if A is self-adjoint and* $<A_x , x> \geq 0$ *for all* $x \in H$*, where H is a Hilbert space.*

We also recall the following Theorems [1]:

The following theorem with proof can be found in [1, Theorem 4.6.14]

Theorem 3. *Every positively defined operator A has a unique positively defined square root B. Moreover, B commutes with every operator commuting with A.*

The following theorem with proof can be found in [1, Theorem 4.6.9]

Theorem 4. *Product of two commuting positively defined operators is positively defined.*

We present here the proof of known proposition.

Proposition 1. *Let A and B be two self-adjoint operators, then AB is normal if and only if BA is normal.*

Proof.

Let AB be normal that is

$$AB(AB)^* = (AB)^*AB$$

$BA(BA)^* = BAA^* B^*$

$= (AB)^* AB$

$= AB(AB)^*$

$= ABB^* A^*$

$= (BA)^* BA.$

Conversely, similarly we get

$AB(AB)^* = (AB)^* AB.$

Proposition 2. *Let H be a Hilbert space, if A is a positively defined in H then A^2 is also a positively defined in H.*

The main result:

Theorem 5. *Let H be a Hilbert space, let A and B be two positively defined operators on H, AB is positively defined operator if and only if AB is normal.*

Proof.

The proof of *AB* is normal when *AB* is positively defined is obvious.

Conversely, let *AB* be normal, by **Proposition 1** also *BA* is normal, corresponding to **Theorem 2** let

$$N_1 = BA \qquad \text{and} \qquad N_2 = AB$$

and *A* is an arbitrary bounded operator so we have

$$ABA = ABA$$

then

$$A(BA)^* = (AB)^* A$$

$$AA^* B^* = B^* A^* A$$

$$AAB = BAA$$

$$A^2 B = BA^2 .$$

Since A^2 commutes with *B* and *A* is positively defined **Proposition 2** implies A^2 is also positively defined, and so by **Theorem 3** we have

$AB = BA.$

So by **Theorem 4** we get *AB* is positively defined.

3- Counter Example

The normality of *AB* in **Theorem 5** is necessary for positivity of *AB*. If *AB* is not normal then *AB* is not always positively defined, we give the following counter example:

Example 1. *Let*

$$A=\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

it easy to check that both A and B are positively defined, and AB is not normal. Also it easy to check that AB is not self-adjoint and so it is not positively defined.

References

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