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$\mathcal{ML}(\theta\mathcal{C})$-Space in Topological Spaces

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ABSTRACT
The purpose of this paper is to introduced a new concept of spaces which is called minimal $L(\theta\mathcal{C})$-space, namely $\mathcal{M}\mathrm{in}L(\theta\mathcal{C})$-space or $\mathcal{M}L(\theta\mathcal{C})$-space, also given some properties, examples, theorems and the topological property of $\mathcal{M}\mathrm{in}L(\theta\mathcal{C})$-space are discussed.

KEYWORDS: θ-closed set, Lindelof, , $L(\theta\mathcal{C})$-space, $\mathcal{M}\mathrm{in}\theta\mathcal{T}_2$-space, $\mathcal{M}\mathrm{in}\mathcal{K}(\theta\mathcal{C})$-space

الخلاصة
الهدف من هذا البحث هو تقديم مفهوم جديد وهو فضاء minimal $L(\theta\mathcal{C})$-space ونرمز له $\mathcal{M}\ \mathrm{in}L(\theta\mathcal{C}) -$ space او $\mathcal{M}L(\theta\mathcal{C})$-space, وكذلك اعطينا بعض الامثلة, النظريات و الخاصية التوبولوجية نوقشت.

INTRODUCTION

The concept of Lindelof space was introduced in 1929 by Alexandrof and Urysohn [2], this space is an important in a topological space. Later, in 1979 Mukherji and Sarkar [10], provide the concept of LC-space. (A topological space $\mathcal{X}$ is called LC-space, if every Lindelof subset of a space $\mathcal{X}$ is a closed set). LC-space studied by many researchers such as [3].

Notices that LC-space is also know under the name L-closed such as [6, 9, 13].

The concept of θ-closed and θ-open set were first introduced by Velicko [16] in 1969. **(**Let $(\mathcal{X},\mathcal{T})$ be a topological space, $\mathcal{F}$ be a subset of $\mathcal{X}$ and $x\in\mathcal{X}$. A point x is called θ-interior point of $\mathcal{F}$, if there is $\mathcal{C}\in\mathcal{T}$, such that $x\in\mathcal{C}$ and $x\in\bar{\mathcal{C}}\subseteq\mathcal{F}$. θ-interior set which denoted by $Int_\theta(\mathcal{F})$ is the set of all θ-interior points. A subset $\mathcal{F}$ of $\mathcal{X}$ is called θ-open set iff $Int_\theta(\mathcal{F})=\mathcal{F}$. And **(**Let $(\mathcal{X},\mathcal{T})$ be topological space, $H\subseteq\mathcal{X}$, a point $b\in X$ is said to be θ-adherent point for a subset H of $\mathcal{X}$, if $H\cap\bar{G}\neq\emptyset$ for any open set G of $\mathcal{X}$ and $b\in G$. The set of θ-adherent points is said to be θ-closure of H which denoted by $Cl_\theta(H)$. A subset H of $\mathcal{X}$ is called θ-closed set iff $H=Cl_\theta(H)$). These concepts have been studied by many authors such as [8, 12]. In 2011, Al-Taai and Haider [4], study the new term called $L(\theta\mathcal{C})$-space. (A topological space $\mathcal{X}$ is called $L(\theta\mathcal{C})$-space, if every Lindelof subset of a space $\mathcal{X}$ is θ-closed set), which is a strong than LC-space. And since the union of θ-closed set may be not θ-closed set. Encourage the author to define $\mathcal{F}_\sigma$-θ-closed set which is a countable union many θ-closed sets.

In 2005, H. J. Ali [3], introduce Minimal LC-space, any LC-space $(\mathcal{X},\mathcal{T})$ is $\mathcal{M}inLC$-space, if $\mathcal{T}^*\subset\mathcal{T}$ on $\mathcal{X}$, then $(\mathcal{X},\mathcal{T}^*)$ is not LC-space studied by [14, 15].

The aim of this paper is to introduce a minimal $L(\theta\mathcal{C})$-space (denoted by $\mathcal{M}\mathrm{in}L(\theta\mathcal{C})$-space), that is a space $\mathcal{X}$ which is $L(\theta\mathcal{C})$-space is called $\mathcal{M}\mathrm{in}L(\theta\mathcal{C})$-space, if $\mathcal{T}^*\subset\mathcal{T}$ on $\mathcal{X}$, then $(\mathcal{X},\mathcal{T}^*)$ is not $L(\theta\mathcal{C})$-space. Note that every $\mathcal{M}inL(\theta\mathcal{C})$-space is $L(\theta\mathcal{C})$-space, and study some properties of this space, also study the relation between this concept with $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space and $\mathcal{M}in\theta\mathcal{T}_2$-space. Also study some important property such as, a topological property of $\mathcal{M}\mathrm{in}L(\theta\mathcal{C})$-space.

PRELIMINARIES

Definition (2.1) [5]: A space $\mathcal{X}$ is called $\mathcal{R}_1$-space, if e and d have a disjoint neighborhoods, whenever $Cl(e)\neq Cl(d)$.

Remark (2.2) [5]: A space $\mathcal{X}$ is $\mathcal{T}_2$-space iff $\mathcal{X}$ is $\mathcal{R}_1$ and $\mathcal{T}_1$-space.

Definition (2.3) [16]: Let $(\mathcal{X},\mathcal{T})$ be a topological space, $\mathcal{F}$ be a subset of $\mathcal{X}$ and $x \in \mathcal{X}$. A point x is called θ-interior point of $\mathcal{F}$, if there is $\mathcal{C} \in \mathcal{T}$, such that $x \in \mathcal{C}$ and $x \in \bar{\mathcal{C}} \subseteq \mathcal{F}$. θ-interior set which denoted by $Int_\theta(\mathcal{F})$ is the set of all θ-interior points. A subset $\mathcal{F}$ of $\mathcal{X}$ is called θ-open set iff $Int_\theta(\mathcal{F}) = \mathcal{F}$.

Definition (2.4) [16]: Let $(\mathcal{X}, \mathcal{T})$ be topological space, $H \subseteq \mathcal{X}$, a point $b \in X$ is said to be θ-adherent point for a subset H of $\mathcal{X}$, if $H \cap \bar{G} \neq \emptyset$ for any open set G of $\mathcal{X}$ and $b \in G$. The set of θ-adherent points is said to be θ-closure of H which denoted by $Cl_\theta(H)$. A subset H of $\mathcal{X}$ is called θ-closed set iff $H = Cl_\theta(H)$.

Example (2.5): Any subset of a discrete space $(\mathcal{R},D)$ on a real numbers $\mathcal{R}$ is θ-closed set and θ-open set.

Remark (2.6) [16]: Every θ-closed (resp. θ-open) set is a closed (resp. open) set.

Lemma (2.7) [3]: Let $\mathcal{Y}$ be a subspace of a space $\mathcal{X}$. If $\mathcal{P}$ is θ-closed in $\mathcal{X}$ then $\mathcal{P}$ is θ-closed in $\mathcal{Y}$, whenever $\mathcal{P} \subseteq \mathcal{Y}$.

Definition (2.8) [1, 4]: A subset $\mathcal{F}$of a space $\mathcal{X}$ is said to be $\mathcal{F}_\sigma$-θ-closed, if it is a countable union of θ-closed sets. The complement of $\mathcal{F}_\sigma$-θ-closed is said to be G_δ-θ-open set.

Remark (2.9) [1]: Every θ-closed set is $\mathcal{F}_\sigma$-θ-closed set. But the converse need not be true.

Example (2.10): Let $(\mathcal{R}, \mathcal{T}_\mathcal{U})$ be a usual topology on a real line $\mathcal{R}$, and $G_n = [1/n\ ,1]$, where $(n = 2, 3, 4, \dots)$, be a θ-closed sets, then $\bigcup_{n=2}^{\infty} G_n = (0,1]$ is $\mathcal{F}_\sigma$-θ-closed, but neither closed nor θ-closed.

Definition (2.11) [1, 3, 4]: A space $\mathcal{X}$ is said to be:

1. θP-space, if every $\mathcal{F}_\sigma$-θ-closed is θ-closed.
2. $\mathcal{K}(\theta\mathcal{C})$-space, if every compact subset of $\mathcal{X}$ is θ-closed set.
3. $L(\theta\mathcal{C})$-space, if every Lindelof subset of $\mathcal{X}$ is θ-closed set.

Example (2.12): Let $(\mathcal{Z}, \mathcal{T}_D)$ be a topological space where $\mathcal{T}_D$ be a discrete topology on an integer numbers$\mathcal{Z}$, $(\mathcal{Z}, \mathcal{T}_D)$ is $L(\theta\mathcal{C})$-space.

Definition (2.13) [4]: A subset $\mathcal{A}$ of a space $\mathcal{X}$ is said to be θ-dense, if $Cl_\theta(\mathcal{A}) = \mathcal{X}$.

Proposition (2.14) [3]: The property of $L(\theta\mathcal{C})$-space is a topological property.

Proposition (2.15) [3]: The property of $L(\theta\mathcal{C})$-space is a hereditary property.

Theorem (2.16) [4]:

1. If a space $\mathcal{X}$ is θL_1-space and θL_3-space, then $\mathcal{X}$ is $L(\theta\mathcal{C})$-space.
2. Every θP-space is θL_1-space.

Definition (2.17) [7]: A space $\mathcal{X}$ is called $\theta\mathcal{T}_1$ (resp. $\mathcal{T}_1$)-space, if every two distinct points a, b belong to $\mathcal{X}$, there is two θ-open (resp. open) sets each one contain one point but not contain the other.

Theorem (2.18) [7]: A space $\mathcal{X}$ is called $\theta\mathcal{T}_1$-space if and only if every singleton set is θ-closed set.

Definition (2.19) [7]: A space $\mathcal{X}$ is called $\theta\mathcal{T}_2$ (resp. $\mathcal{T}_2$)-space, if every two points a, b belong to $\mathcal{X}$, $a \neq b$ there is two disjoint θ-open (resp. open) sets M and N containing a and b respectively.

Remarks (2.20):

1. Every $L(\theta\mathcal{C})$-space is $\theta\mathcal{T}_1$-space.
2. Every $\theta\mathcal{T}_1$ -space is $\mathcal{T}_1$-space.
3. Every $L(\theta\mathcal{C})$-space is $\mathcal{T}_1$-space.

Proof:

1. Let $\{x\}$ be a Lindelof subset of a space $\mathcal{X}$, for each $x \in \mathcal{X}$, which is $L(\theta\mathcal{C})$-space, so $\{x\}$ is θ-closed set, then from Theorem (2.37), a space $\mathcal{X}$ is $\theta\mathcal{T}_1$-space.

2. Let a, b be two distinct point in a space $\mathcal{X}$ which is $\theta\mathcal{T}_1$ -space, so there exist two θ-open sets G and H containing a, b respectively with $a \notin H$ and $b \notin G$, from Remark 2.21, G and H are open set in $\mathcal{X}$, containing a, b respectively with $a \notin H$ and $b \notin G$, that means $\mathcal{X}$ is $\mathcal{T}_1$-space.

3. Let a space $\mathcal{X}$ be $L(\theta\mathcal{C})$-space, from part (1) of this Remark, $\mathcal{X}$ is $\theta\mathcal{T}_1$-space and from part (2), $\mathcal{X}$ is $\mathcal{T}_1$-space.

Definition (2.21) [11]: A space $\mathcal{X}$ is called $\theta\mathcal{R}_1$-space, if e and d have a disjoint θ-neighbourhood, whenever $Cl_\theta(e) \neq Cl_\theta(d)$.

Remark (2.22) [11]: A space $\mathcal{X}$ is $\theta\mathcal{T}_2$-space iff $\mathcal{X}$ is $\theta\mathcal{R}_1$ and $\theta\mathcal{T}_1$-space.

Definition (2.23) [8]: Let $(\mathcal{X}, \mathcal{T})$ and $(\mathcal{Y}, \mathcal{T}')$ be two topological space and $f: (\mathcal{X}, \mathcal{T}) \longrightarrow (\mathcal{Y}, \mathcal{T}')$ be a function. Then f is called:

1. θ-closed function [1], if $f(F)$ is θ-closed set in $\mathcal{Y}$ for each closed subset F of $\mathcal{X}$.
2. Closed function [10], if $f(F)$ is closed set in $\mathcal{Y}$ for each closed subset F of $\mathcal{X}$.

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Remark (2.24) [1]: Every θ-closed function is closed function.

Definition (2.25) [11]: Let $(\mathcal{X},\mathcal{T})$ be $\mathcal{K}(\theta\mathcal{C})$-space, a space $\mathcal{X}$ is said to be $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space, if $\mathcal{T}^*\subset\mathcal{T}$ on $\mathcal{X}$, then $(\mathcal{X},\mathcal{T}^*)$ is not $\mathcal{K}(\theta\mathcal{C})$-space.

Example (2.26): Let $(\mathcal{R},\mathcal{T}_{\mathcal{U}})$ be a usual topology defined on the real numbers,$(\mathcal{R},\mathcal{T}_{\mathcal{U}})$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.

Theorem (2.27) [11]: If a space $\mathcal{X}$ is compact $\mathcal{K}(\theta\mathcal{C})$-space, then it is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.

Proposition (2.28) [11]: If a space $\mathcal{X}$ is Locally compact, $\mathcal{K}(\theta\mathcal{C})$-space then $\mathcal{X}$ is $\theta\mathcal{T}_2$-space.

Definition (2.29) [11]: A space $\mathcal{X}$ is $\theta\mathcal{T}_2$-space, we say that $\mathcal{X}$ is $\mathcal{M}in\theta\mathcal{T}_2$-space, if there is $\mathcal{T}^*\subset\mathcal{T}$ on $\mathcal{X}$, then $(\mathcal{X},\mathcal{T}^*)$ is not $\theta\mathcal{T}_2$-space.

Theorem (2.30) [11]: If a space $\mathcal{X}$ is $\theta\mathcal{T}_2$ and $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space then $\mathcal{X}$ is $\mathcal{M}in\theta\mathcal{T}_2$-space.

$\mathcal{M}inL(\theta\mathcal{C})$-Spaces

Definition (3.1): Let $(\mathcal{X},\mathcal{T})$ be $L(\theta\mathcal{C})$-space, a space $\mathcal{X}$ is said to be $\mathcal{M}inL(\theta\mathcal{C})$-space, if $\mathcal{T}^*\subset\mathcal{T}$ on $\mathcal{X}$, then $(\mathcal{X},\mathcal{T}^*)$ is not $L(\theta\mathcal{C})$-space.

Example (3.2): Let $(\mathcal{X},\mathcal{T}_D)$ be a discrete topology defined on countable set $\mathcal{X}$, $(\mathcal{X},\mathcal{T}_D)$ is $\mathcal{M}inL(\theta\mathcal{C})$-space, since, if we take any subset $\mathcal{H}$of a space $\mathcal{X}$, which is countable then $\mathcal{H}$ is countable, so $\mathcal{H}$ is Lindelof, let $x\not\in\mathcal{H}$, also $\{x\}$ is open set containing x, also $\overline{\{x\}}\cap\mathcal{H}=\emptyset$, so $\mathcal{H}$ is θ-closed set and then $\mathcal{X}$ is $L(\theta\mathcal{C})$-space, also since $\mathcal{T}_{ind}\subset\mathcal{T}_D$, but $\mathcal{T}_{ind}$ is not $L(\theta\mathcal{C})$-space. Therefore $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Theorem (3.3): If a space $\mathcal{X}$ is Lindelof $L(\theta\mathcal{C})$-space, then it is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Proof: Let $(\mathcal{X},\mathcal{T})$ be $L(\theta\mathcal{C})$-space and suppose $\mathcal{X}$ is not $\mathcal{M}inL(\theta\mathcal{C})$-spac, that is there is a topology $\mathcal{T}^*\subset\mathcal{T}$ on $\mathcal{X}$and $(\mathcal{X},\mathcal{T}^*)$ is $L(\theta\mathcal{C})$-space. Let $I_x:(\mathcal{X},\mathcal{T})\longrightarrow(\mathcal{X},\mathcal{T}^*)$ be the identity function on $\mathcal{X}$. Now I_x is continuous, bijective and θ-closed function since (if $\mathcal{N}$ is a closed subset of $\mathcal{X}$, and $\mathcal{X}$ is Lindelof, so $\mathcal{N}$ is Lindelof), also I_x is continuous, then $I_x(\mathcal{N})$ is Lindelof subset of $(\mathcal{X},\mathcal{T}^*)$ which is $L(\theta\mathcal{C})$-space, hence $I_x(\mathcal{N})$ is θ-closed and then I_x is θ-closed function, by Remark 2.24, I_x is a closed function that is I_x is homeomorphism function, so $\mathcal{T}^*\cong\mathcal{T}$ and this is contradiction, so $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Example (3.4): Let $\mathcal{X}=\mathcal{R}$ be a real numbers, and $\mathcal{T}_{Exc}=\{\mathcal{U}\subseteq\mathcal{R}: x\not\in\mathcal{U}$, for some $x\in\mathcal{R}\}\cup\{\mathcal{R}\}$, be excluded point topology, $(\mathcal{R},\mathcal{T}_{Exc})$ is not $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space, since $(\mathcal{R},\mathcal{T}_{Exc})$ is compact, so $(\mathcal{R},\mathcal{T}_{Exc})$ is Lindelof, but not $L(\theta\mathcal{C})$-space because, if we take $x=5$ and $C=\{\{x\}\}_{x\neq5}\cup\mathcal{R}$ is an open cover to $\mathcal{R}$, then we can reduce to just $\mathcal{R}$ that is $(\mathcal{R},\mathcal{T}_{Exc})$ is Lindelof, also $\{1,5\}$ is finite set, then it is countable, so it is Lindelof set and $2\not\in\{1,5\}$, so there is open set $\{2\}$ in $\mathcal{R}$ and $\overline{\{2\}}=\{2,5\}\cap\{1,5\}\neq\emptyset$, then $2\in\theta$-adherent point, that is $\{1,5\}$ is not θ-closed set, hence $(\mathcal{R},\mathcal{T}_{Exc})$ is not $L(\theta\mathcal{C})$-space and from Theorem 3.3, this topological space is not $\mathcal{M}inL(\theta\mathcal{C})$-space.

Corollary (3.5): Every compact and $L(\theta\mathcal{C})$-space is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Proof: From Theorem 3.3. And every compact space is Lindelof space.

Remark (3.6): The continuous image of $\mathcal{M}inL(\theta\mathcal{C})$ is not necessarily $\mathcal{M}inL(\theta\mathcal{C})$, the following example explain this Remark:

Example (3.7): Let $f:(\mathcal{R},\mathcal{T}_D)\longrightarrow(\mathcal{R},\mathcal{T}_{ind})$ be a function from a discrete topology $\mathcal{T}_D$ into indiscrete topology $\mathcal{T}_{ind}$, defined by $f(x)=x,\forall x\in\mathcal{R}$, so f is continuous and $(\mathcal{R},\mathcal{T}_D)$ is $\mathcal{M}inL(\theta\mathcal{C})$, also $\mathcal{T}_{ind}\subset\mathcal{T}_D$, but $(\mathcal{R},\mathcal{T}_{ind})$ is not $L\mathcal{C}$, implies that, it is not $L(\theta\mathcal{C})$. Therefore $(\mathcal{R},\mathcal{T}_{ind})$ is not $\mathcal{M}inL(\theta\mathcal{C})$-space.

Proposition (3.8): The property of being $\mathcal{M}inL(\theta\mathcal{C})$-space is a topological property.

Proof: Let $(\mathcal{X},\mathcal{T})$ be $\mathcal{M}inL(\theta\mathcal{C})$-space, $f:(\mathcal{X},\mathcal{T})\longrightarrow(\mathcal{Y},\mathcal{T}')$ is a homeomorphism function, to prove $(\mathcal{Y},\mathcal{T}')$ is $\mathcal{M}inL(\theta\mathcal{C})$-space. Now from Proposition 2.14, $(\mathcal{Y},\mathcal{T}')$ is $L(\theta\mathcal{C})$-space, suppose $(\mathcal{Y},\mathcal{T}')$ is not $\mathcal{M}inL(\theta\mathcal{C})$-space, then there is a topology $\mathcal{T}^*\subset\mathcal{T}'$ on $\mathcal{Y}$, implies $(\mathcal{Y},\mathcal{T}^*)$ is $L(\theta\mathcal{C})$-space.

Define $\mathcal{T}_1=\{f^{-1}(\mathcal{U}):\mathcal{U}\in\mathcal{T}^*\}$, so $(\mathcal{X},\mathcal{T}_1)$ is a topology on $(\mathcal{X},\mathcal{T})$ and $\mathcal{T}_1\subset\mathcal{T}$ and $(\mathcal{X},\mathcal{T}_1)$ is $L(\theta\mathcal{C})$-space, (let $\mathcal{S}$ be a Lindelof subset of $\mathcal{X}$, to prove $\mathcal{S}$ is θ-closed in $\mathcal{X}$, since f is continuous and then we have $f(\mathcal{S})$ is Lindelof set in $\mathcal{Y}$ which is $L(\theta\mathcal{C})$-space, then $f(\mathcal{S})$ is θ-closed in $(\mathcal{Y},\mathcal{T}^*)$, to show $\mathcal{S}$ is θ-closed set, that is to show $\mathcal{S}=Cl_\theta(\mathcal{S})$, since $\mathcal{S}\subseteq Cl_\theta(\mathcal{S})$, let $s\in Cl_\theta(\mathcal{S})$ and $s\not\in\mathcal{S}$, since f is injective,then $f(s)\not\in f(\mathcal{S})$ and f is surjective, so $w\not\in f(\mathcal{S})$ where $w=f(s)$, but $f(\mathcal{S})$ is θ-closed in $\mathcal{Y}$, then there is open set $\mathcal{W}$ in $\mathcal{Y}$ with $w\in\mathcal{W}$ and $\overline{\mathcal{W}}\cap f(\mathcal{S})=\phi$, so $f^{-1}(\overline{\mathcal{W}}\cap f(\mathcal{S}))=f^{-1}(\phi)=\phi$, and $f^{-1}(\overline{\mathcal{W}})\cap f^{-1}(f(\mathcal{S}))=\phi$, then $f^{-1}(\overline{\mathcal{W}})\cap\mathcal{S}=\phi$, since f is homeomorphism, then $\overline{f^{-1}(\mathcal{W})} \cap \mathcal{S} = \phi$, we have s is not θ-adherent point to $\mathcal{S}$. Therefore $\mathcal{S}$ is θ-closed in $\mathcal{X}$), which is contradiction, since $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space. Hence $(\mathcal{Y}, \mathcal{T}')$ is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Lemma (3.9): In Lindelof space, any θ-closed set is Lindelof set.

Proof: Let $\mathcal{X}$ be a Lindelof space and $\mathcal{A}$ be θ-closed subset of $\mathcal{X}$. From Remark 2.6, $\mathcal{A}$ is a closed subset of $\mathcal{X}$. Then $\mathcal{A}$ is Lindelof set.

Proposition (3.10): Let $(\mathcal{Y}, \mathcal{T})$ be a subspace of a Lindelof $L(\theta\mathcal{C})$-space $(\mathcal{X}, \mathcal{T})$, $\mathcal{Y}$ is Lindelof iff $\mathcal{Y}$ is θ-closed.

Proof: Suppose $\mathcal{Y}$ is Lindelof subspace of $\mathcal{X}$, since $\mathcal{X}$ is $L(\theta\mathcal{C})$-space, then $\mathcal{Y}$ is θ-closed. Conversely, suppose $\mathcal{Y}$ is θ-closed in $\mathcal{X}$, which is Lindelof, then by Lemma 3.9, $\mathcal{Y}$ is Lindelof.

Example (3.11): The discrete topology $\mathcal{T}_D$ on an integer numbers $\mathcal{Z}$, $(\mathcal{Z}, \mathcal{T}_D)$ is Lindelof $L(\theta\mathcal{C})$-space, also subspace $(\mathcal{N}, \mathcal{T}_D)$ is Lindelof and θ-closed, where $\mathcal{N}$ is a natural number.

Proposition (3.12): If $(\mathcal{X}, \mathcal{T})$ is a Lindelof $L(\theta\mathcal{C})$-space, then every θ-closed subspace of $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Proof: Let $\mathcal{Y}$ be θ-closed in $\mathcal{X}$, but $\mathcal{X}$ is Lindelof, then by Proposition 3.10, $\mathcal{Y}$ is Lindelof. Now let $\mathcal{N}$ be a Lindelof subset of $\mathcal{Y}$, then $\mathcal{N}$ is Lindelof in $\mathcal{X}$, but $\mathcal{X}$ is $L(\theta\mathcal{C})$-space, so $\mathcal{N}$ is θ-closed in $\mathcal{X}$. Now $\mathcal{N} = \mathcal{N} \cap \mathcal{Y}$, since $\mathcal{N} \subseteq \mathcal{Y}$, by Lemma 2.7, $\mathcal{N}$ is θ-closed in $\mathcal{Y}$, hence $\mathcal{Y}$ is $L(\theta\mathcal{C})$-space and by Theorem 3.3, $\mathcal{Y}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Lemma (3.13): A subset $\mathcal{H}$ of a space $\mathcal{X}$ is G_δ-θ-open set if and only if every point in $\mathcal{H}$ is G_δ-θ-interior point to $\mathcal{H}$.

Proof: Suppose $\mathcal{H}$ is G_δ-θ-open set and $x \in \mathcal{H}$, then there exists $\mathcal{A} = \mathcal{H}$ which is G_δ-θ-open set and $x \in \mathcal{A} = \mathcal{H} \subseteq \mathcal{H}$, so x is G_δ-θ-interior point to $\mathcal{H}$, but x is an arbitrary point, so any point in $\mathcal{A}$ is G_δ-θ-interior point to $\mathcal{A}$. Conversely, suppose any point in $\mathcal{H}$ is G_δ-θ-interior point to $\mathcal{H}$, that is, for each $x_i \in \mathcal{H}$, there is $\mathcal{A}_{x_i}$ is G_δ-θ-open subset of $\mathcal{H}$, we get $\mathcal{H} = \bigcup_{x_i \in \mathcal{H}} \mathcal{A}_{x_i}$, then $\mathcal{H}$ is G_δ-θ-open set.

Proposition (3.14): Every Lindelof set in $\theta\mathcal{T}_2$-space is $\mathcal{F}_\sigma$-θ-closed set.

Proof: Let $\mathcal{A}$ be a Lindelof subset of a space $\mathcal{X}$, and $p \notin \mathcal{A}$, then for each $q \in \mathcal{A}$, $p \neq q$ and $p, q \in \mathcal{X}$, since $\mathcal{X}$ is $\theta\mathcal{T}_2$-space, then there exist two θ-open sets $\mathcal{U}$ and $\mathcal{V}$, with $q \in \mathcal{U}$, $p \in \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$. Let $\bigcup_{q \in \mathcal{A}} \mathcal{U}_q$ is θ-open cover to $\mathcal{A}$, then it is open cover to $\mathcal{A}$ which is Lindelof, so $\mathcal{A} \subseteq \bigcup_{i \in N} \mathcal{U}_{qi}$, then $\mathcal{U}^* = \bigcup_{i \in N} \mathcal{U}_{qi}$ is open and $\mathcal{V}^* = \bigcap_{i \in N} \mathcal{V}_{qi}(p)$, since $\mathcal{V}^*$ is the intersection of countable many θ-open set, then $\mathcal{V}^*$ is G_δ-θ-open set and $\mathcal{V}^* \cap \mathcal{U}^* = \emptyset$, so $p \in \mathcal{V}^* \subseteq \mathcal{A}^c$, then p is G_δ-θ-interior point to $\mathcal{A}^c$, from Lemma 3.13, $\mathcal{A}^c$ is G_δ-θ-open set. Therefore $\mathcal{A}$ is $\mathcal{F}_\sigma$-θ-closed set.

Proposition (3.15): Every $\mathcal{F}_\sigma$-θ-closed set in Lindelof space is Lindelof.

Proof: Let $\mathcal{H}$ be $\mathcal{F}_\sigma$-θ-closed subset of a space $\mathcal{X}$, that is $\mathcal{H} = \bigcup_{i \in N} \mathcal{F}_i$, where $\mathcal{F}_i$ is θ-closed set in $\mathcal{X}$, but $\mathcal{X}$ is Lindelof space, so by Lemma 3.9, $\mathcal{F}_i$, $i \in N$, is Lindelof. Now, $\bigcup_{i \in N} \mathcal{F}_i$ is Lindelof and $\mathcal{H} = \bigcup_{i \in N} \mathcal{F}_i$, so $\mathcal{H}$ is Lindelof.

Remark (3.16): Let $(\mathcal{R}, \mathcal{T}_D)$ be a discrete topology on a real numbers $\mathcal{R}$. Every singleton set is θ-closed, then it is $\mathcal{F}_\sigma$-θ-closed set and Lindelof, but $(\mathcal{R}, \mathcal{T}_D)$ is not Lindelof.

Theorem (3.17): Let a space $\mathcal{X}$ is $\theta\mathcal{T}_2$, Lindelof space, then $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space iff $\mathcal{X}$ is θP-space.

Proof: Let $\mathcal{X}$ be $\mathcal{M}inL(\theta\mathcal{C})$-space, to prove $\mathcal{X}$ is θP-space. Let $\mathcal{A}$ be $\mathcal{F}_\sigma$-θ-closed subset in $\mathcal{X}$, which is Lindelof, by Proposition 3.15, $\mathcal{A}$ is Lindelof subset of $\mathcal{X}$, which is $L(\theta\mathcal{C})$-space, then $\mathcal{A}$ is θ-closed set in $\mathcal{X}$, Therefore $\mathcal{X}$ is θP-space. Conversely, suppose $\mathcal{X}$ is θP-space, to prove $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space, let $\mathcal{H}$ be a Lindelof subset of $\mathcal{X}$, but $\mathcal{X}$ is $\theta\mathcal{T}_2$-space, then by Proposition 3.14 , $\mathcal{H}$ is $\mathcal{F}_\sigma$-θ-closed set, also $\mathcal{X}$ is θP-space, then $\mathcal{H}$ is θ-closed subset of $\mathcal{X}$, that means $\mathcal{X}$ is $L(\theta\mathcal{C})$-space and it is Lindelof, so from Theorem 3.3, $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Proposition (3.18): Every $\theta\mathcal{T}_2$-space and θP-space is $L(\theta\mathcal{C})$-space.

Proof: Let $\mathcal{M}$ be a Lindelof subset of $\mathcal{X}$, but $\mathcal{X}$ is $\theta\mathcal{T}_2$-space, so by Proposition 3.14, $\mathcal{M}$ is $\mathcal{F}_\sigma$-θ-closed set in $\mathcal{X}$, which is θP-space, hence $\mathcal{M}$ is θ-closed set in $\mathcal{X}$, therefore $\mathcal{X}$ is $L(\theta\mathcal{C})$-space.

Theorem (3.19): Every Lindelof $\theta\mathcal{T}_2$ and θP-space is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Proof: Let a space $\mathcal{X}$ be $\theta\mathcal{T}_2$ and θP-space, by Proposition 3.18, $\mathcal{X}$ is $L(\theta\mathcal{C})$-space and it is Lindelof, so by Theorem 3.3, $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Proposition (3.20): Every $L(\theta\mathcal{C})$-space is $\mathcal{K}(\theta\mathcal{C})$-space.

Proof: Let $\mathcal{B}$ be a compact subset of a space $\mathcal{X}$, then $\mathcal{B}$ is Lindelof in $\mathcal{X}$, but $\mathcal{X}$ is $L(\theta\mathcal{C})$-space, so $\mathcal{B}$ is θ-closed. Hence $\mathcal{X}$ is $\mathcal{K}(\theta\mathcal{C})$-space.

The convers of Proposition 3.20, is not true as shown by the following example.

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Example (3.21): Let $(\mathcal{R},\mathcal{T}_\mathcal{U})$ be a usual topology on a real numbers $\mathcal{R}$. The compact subset of this space is only finite sets or closed interval, also they are θ-closed. Therefore, $(\mathcal{R},\mathcal{T}_\mathcal{U})$ is $\mathcal{K}(\theta\mathcal{C})$-space. Also, the rational numbers $\mathbb{Q}$ is Lindelof but not θ-closed. Hence$(\mathcal{R},\mathcal{T}_\mathcal{U})$ is not $L(\theta\mathcal{C})$-space.

Theorem (3.22): If a space $\mathcal{X}$ is compact and θP-space, then $\mathcal{X}$ is $\mathcal{M}in\theta\mathcal{T}_2$-space iff $\mathcal{X}$ is $\theta\mathcal{T}_2$-space and $\mathcal{M}inL(\theta\mathcal{C})$-space.

Proof: Suppose a space $\mathcal{X}$ is $\mathcal{M}in\theta\mathcal{T}_2$-space, then $\mathcal{X}$ is $\theta\mathcal{T}_2$-space, by Proposition 3.18, $\mathcal{X}$ is $L(\theta\mathcal{C})$-space. Also $\mathcal{X}$ is compact, then $\mathcal{X}$ is Lindelof, hence by Theorem 3.3, $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space. Conversely, suppose $\mathcal{X}$ is $\theta\mathcal{T}_2$-space and $\mathcal{M}inL(\theta\mathcal{C})$-space, so $\mathcal{X}$ is $\theta\mathcal{T}_2$-space and $L(\theta\mathcal{C})$-space, by Proposition 3.20, $\mathcal{X}$ is $\theta\mathcal{T}_2$-space and $\mathcal{K}(\theta\mathcal{C})$-space, and since $\mathcal{X}$ is compact $\mathcal{K}(\theta\mathcal{C})$-space, so from Theorem 2.27**,** $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space, and by Theorem 2.30, $\mathcal{X}$ is $\mathcal{M}in\theta\mathcal{T}_2$-space.

Corollary (3.23): If a space $\mathcal{X}$ is compact and $\mathcal{M}inL(\theta\mathcal{C})$-space, then $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.

Proof: Suppose $\mathcal{X}$ is compact and $\mathcal{M}inL(\theta\mathcal{C})$-space, so $\mathcal{X}$ is compact and $L(\theta\mathcal{C})$-space, by Proposition 3.20, $\mathcal{X}$ is compact and $\mathcal{K}(\theta\mathcal{C})$-space, so from Theorem 2.27, we have $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.

Corollary (3.24): If a space $\mathcal{X}$ is compact and $L(\theta\mathcal{C})$-space, then $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$- space.

Proof: Suppose $\mathcal{X}$ is compact and $L(\theta\mathcal{C})$-space, by Proposition 3.20, $\mathcal{X}$ is compact and $\mathcal{K}(\theta\mathcal{C})$-space, so from Theorem 2.27, $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.

Corollary (3.25): Every countably compact, Lindelof and $L(\theta\mathcal{C})$-space is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.

Proof: Suppose $\mathcal{X}$ is countably compact and Lindelof space, then $\mathcal{X}$ is compact, from Corollary 3.24**,** $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.

Theorem (3.26): If a space $\mathcal{X}$ is compact and $L(\theta\mathcal{C})$-space, then a closed subspace of $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space and $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.

Proof: Let $\mathcal{Y}$ be a closed subspace of a compact space $\mathcal{X}$, so $\mathcal{Y}$ is compact set in $\mathcal{X}$, then $\mathcal{Y}$ is Lindelof. Also, $\mathcal{X}$ is $L(\theta\mathcal{C})$-space, so by Proposition 2.15, $\mathcal{Y}$ is $L(\theta\mathcal{C})$-space. Hence from Theorem 3.3, $\mathcal{Y}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space. Now, from Proposition 3.20, $\mathcal{X}$ is $\mathcal{K}(\theta\mathcal{C})$-space, so from Theorem 2.27, $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.

Corollary (3.27): If a space $\mathcal{X}$ is Lindelof and $L(\theta\mathcal{C})$-space, then a closed subspace of $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Proof: Let $\mathcal{Y}$ be a closed subset of a Lindelof space $\mathcal{X}$, then $\mathcal{Y}$ is Lindelof in $\mathcal{X}$, and then by Proposition 2.15, $\mathcal{Y}$ is $L(\theta\mathcal{C})$-space, from Theorem 3.3, $\mathcal{X}$ is $\mathcal{M}$inL(θC)-space.

Corollary (3.28): If a space $\mathcal{X}$ is Lindelof and $L(\theta\mathcal{C})$-space, then a θ-closed subspace of $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Corollary (3.29): If a space $\mathcal{X}$ is hereditarily Lindelof and $L(\theta\mathcal{C})$-space, then any subspace of $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Proof: Let $\mathcal{Y}$ be a subspace of a space $\mathcal{X}$, since $\mathcal{X}$ is hereditarily Lindelof, so $\mathcal{Y}$ is Lindelof, also by Proposition 2.15, $\mathcal{Y}$ is $L(\theta\mathcal{C})$-space, from Theorem 3.3, $\mathcal{X}$ is $\mathcal{M}$inL(θC)-space.

Theorem (3.30): If a space $\mathcal{X}$ is compact θP-space, then $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space if and only if $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Proof: Suppose $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space, that means $\mathcal{X}$ is $\mathcal{K}(\theta\mathcal{C})$-space and by hypothesis $\mathcal{X}$ is compact, so $\mathcal{X}$ is locally compact space and then from Proposition 2.28, $\mathcal{X}$ is $\theta\mathcal{T}_2$-space, also $\mathcal{X}$ is Lindelof. Therefore, by Theorem 3.19, $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space. Conversely, suppose $\mathcal{X}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space, so $\mathcal{X}$ is $L(\theta\mathcal{C})$-space, by Proposition 3.20, $\mathcal{X}$ is $\mathcal{K}(\theta\mathcal{C})$-space and it is compact, hence from Theorem 2.27, $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.

Definition (3.31): A space $\mathcal{X}$ is said to be θQ-set space, if any subset of $\mathcal{X}$ is $\mathcal{F}_\sigma$-θ-closed set in $\mathcal{X}$.

Proposition (3.32):

1. Every θQ-set space is θL_3-space.
2. Every θQ-set space and θL_1-space is $L(\theta\mathcal{C})$-space.
3. Every θQ-set space and θP-space is $L(\theta\mathcal{C})$-space.
4. Every Lindelof θL_1-space is θP-space.
5. Every θP-space and θL_3-space is $L(\theta\mathcal{C})$-space.

Proof:

1. Let $\mathcal{H}$ be a Lindelof subset of θQ-set space $\mathcal{X}$, then $\mathcal{H}$ is $\mathcal{F}_\sigma$-θ-closed set in $\mathcal{X}$. Therefore, $\mathcal{X}$ is θL_3-space.
2. Let $\mathcal{X}$ be a θQ-set space $\mathcal{X}$, by part(1), $\mathcal{X}$ is θL_3-space, and from Theorem 2.16 part (1), $\mathcal{X}$ is $L(\theta\mathcal{C})$-space.

3. Let L be a Lindelof subset of θQ-set space $\mathcal{X}$, then L is $\mathcal{F}_\sigma$-θ-closed set in $\mathcal{X}$ which is θP-space, then L is θ-closed set in $\mathcal{X}$. Therefore, $\mathcal{X}$ is $L(\theta\mathcal{C})$-space.
4. Let $\mathcal{K}$ be $\mathcal{F}_\sigma$-θ-closed set in a Lindelof space $\mathcal{X}$, then $\mathcal{K} = \bigcup_{i\in N}\mathcal{H}_i$, where $\mathcal{H}_i$ is θ-closed set in a space $\mathcal{X}$, for each $i \in N$, by Lemma 3.9, $\mathcal{H}_i$ is Lindelof, so $\mathcal{K}$ is Lindelof and $\mathcal{F}_\sigma$-θ-closed set, since $\mathcal{X}$ is θL_1-space, then $\mathcal{K}$ is θ-closed set. Therefore, $\mathcal{X}$ is θP-space.
5. Suppose $\mathcal{X}$ is θP-space, by Theorem 2.16, part(2), $\mathcal{X}$ is θL_1-space and it is θL_3-space, so by Theorem 2.16, part(1), $\mathcal{X}$ is $L(\theta\mathcal{C})$-space.

Proposition (3.33):
1. Every Lindelof θL_1-space and θL_3-space is $\mathcal{M}inL(\theta\mathcal{C})$-space.
2. Every Lindelof θL_1-space and $\theta\mathcal{T}_2$-space is $\mathcal{M}inL(\theta\mathcal{C})$-space.
3. Every Lindelof θQ-set and θL_1-space is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Proof:
1. Let a space $\mathcal{X}$ is θL_1-space and θL_3-space, the by Theorem 2.16, part(1), $\mathcal{X}$ is $L(\theta\mathcal{C})$-space and it is Lindelof, so from Theorem 3.3, $\mathcal{X}$ is $\mathcal{M}$inL(θ$\mathcal{C}$)-space.
2. Let a space $\mathcal{X}$ is Lindelof θL_1-space, the by Proposition 3.32 part(4), $\mathcal{X}$ is θP-space, and from Proposition 3.18, $\mathcal{X}$ is $L(\theta\mathcal{C})$-space, also from Theorem 3.3, $\mathcal{X}$ is $\mathcal{M}$inL(θ$\mathcal{C}$)-space.
3. Let a space $\mathcal{X}$ is Lindelof θL_1-space, the by Proposition 3.32, part (4), $\mathcal{X}$ is θP-space, and from Proposition 3.32, part (3), $\mathcal{X}$ is $L(\theta\mathcal{C})$-space, also from Theorem 3.3, $\mathcal{X}$ is $\mathcal{M}$inL(θ$\mathcal{C}$)-space.

Theorem (3.34): Every $L(\theta\mathcal{C})$-space having θ-dense Lindelof subset is $\mathcal{M}$inL(θ$\mathcal{C}$)-space.
Proof: Let $\mathcal{A}$ be a θ-dense Lindelof subset of a space $\mathcal{X}$, but $\mathcal{X}$ is $L(\theta\mathcal{C})$-space, then $\mathcal{A}$ is θ-closed, then $\mathcal{A} = Cl_\theta(\mathcal{A}) = \mathcal{X}$, hence $\mathcal{X}$ is Lindelof and it is $L(\theta\mathcal{C})$-space, so from Theorem 3.3, $\mathcal{X}$ is $\mathcal{M}$inL(θ$\mathcal{C}$)-space.
Proposition (3.35): Every Lindelof θQ-set space and θP-space is $\mathcal{M}inL(\theta\mathcal{C})$-space.
Proof: From Proposition 3.32, part(3) and Theorem 3.3.
Proposition (3.36): Every compact θQ-set space and θP-space is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.
Proof: Let a space $\mathcal{X}$ is θQ-set space and θP-space, then from Proposition 3.32, part(3), $\mathcal{X}$ is $L(\theta\mathcal{C})$-space and by proposition 3.20, $\mathcal{X}$ is $\mathcal{K}(\theta\mathcal{C})$-space, since $\mathcal{X}$ is compact and $\mathcal{K}(\theta\mathcal{C})$-space so by Theorem 2.27, $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.
Theorem (3.37): Every compact θQ-set space and θL_1-space is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.
Proof: Let a space $\mathcal{X}$ be θQ-set space and θL_1-space, so by Proposition 3.32 part (3), $\mathcal{X}$ is $L(\theta\mathcal{C})$-space, and from Proposition 3.20, $\mathcal{X}$ is $\mathcal{K}(\theta\mathcal{C})$-space, so we have a space $\mathcal{X}$ is compact $\mathcal{K}(\theta\mathcal{C})$-space, hence by Theorem 2.27, $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.
Corollary (3.38): Every compact θL_1-space and θL_3-space is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.
Proof: Let $\mathcal{X}$ be θL_1 and θL_3-space, from Proposition 2.16, part(1), $\mathcal{X}$ is $L(\theta\mathcal{C})$-space, also by Proposition 3.20, $\mathcal{X}$ is $\mathcal{K}(\theta\mathcal{C})$-space and from Theorem 2.27, $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.
Corollary (3.39): Every compact θP-space and θL_3-space is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.
Proof: Let $\mathcal{X}$ be θP-space, from Theorem 2.16, part(2), $\mathcal{X}$ is θL_1-space and from Corollary 3.38, $\mathcal{X}$ is $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.
Theorem (3.40): If $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{T}_2$-spaces, $L(\theta\mathcal{C})$-spaces, then $\mathcal{X}\times\mathcal{Y}$ is $L(\theta\mathcal{C})$-space.
Proof: Let L be a Lindelof subset of $\mathcal{X}\times\mathcal{Y}$, and let $(x_o, y_o) \notin L$, for each $(x, y) \in L$, then there exists open neighbourhoods $\mathcal{U}_x$ and $\mathcal{V}_y$ of x and y respectively, such that $(x_o, y_o) \notin \overline{\mathcal{U}_x}\times\overline{\mathcal{V}_y}$, since $L \subseteq \cup\{\mathcal{U}_x\times\mathcal{V}_y : (x, y) \in L\}$, we have $L \subseteq \cup \{\mathcal{U}_{x_n}\times\mathcal{V}_{y_n} : n \in \mathcal{Z}^+$, for some $(x_n, y_n) \in L$, $n \in \mathcal{Z}^+\}$. Now, let $E_1 = \{n \in \mathcal{Z}^+ : x_o \notin \overline{\mathcal{U}_{x_n}}\}$ and $E_2 = \{n \in \mathcal{Z}^+ : y_o \notin \overline{\mathcal{U}_{y_n}}\}$, then $E_1 \cup E_2 = \mathcal{Z}^+$. And, if $L_1 = \cup\{L \cap (\overline{\mathcal{U}_{x_n}} \times \overline{\mathcal{U}_{y_n}} : n \in E_1\}$ and $L_2 = \cup\{L \cap (\overline{\mathcal{U}_{x_n}} \times\overline{\mathcal{V}_{y_n}} : n \in E_2\}$, then L_1 and L_2 are Lindelof subset of $\mathcal{X}\times\mathcal{Y}$, such that $L_1 \cup L_2 = L$. Clearly $x_o \notin \pi_1(L_1)$ and since L_1 is Lindelof and π_1 is continuous, then $\pi_1(L_1)$ is Lindelof in $\mathcal{X}$, and since $\mathcal{X}$ is $L(\theta\mathcal{C})$-space, then $\pi_1(L_1)$ is θ-closed, by Remark 2.6, $\pi_1(L_1)$ is closed in $\mathcal{X}$, so there is an open neighbourhood $G \subseteq \mathcal{X}$ of x_o, with $G \cap \pi_1(L_1) = \emptyset$. In the same way, since $y_o \notin \pi_2(L_2)$ and L_2 is Lindelof in $\mathcal{Y}$, with π_1 is continuous, so $\pi_2(L_2)$ is Lindelof in $\mathcal{Y}$, and since $\mathcal{Y}$ is $L(\theta\mathcal{C})$-space, then $\pi_2(L_2)$ is θ-closed, so $\pi_2(L_2)$ is closed in $\mathcal{Y}$, so there is an

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open neighbourhood $H \subseteq \mathcal{Y}$ of x_o, with $H \cap \pi_2(L_2) = \emptyset$, we now claim $(G \times H) \cap L = \emptyset$, since $(x,y) \in L$, suppose $(x,y) \in (G \times H)$, then $x \in G$, but $G \cap \pi_1(L_1) = \emptyset$, then $x \notin \pi_1(L_1)$, so $(x,y) \notin L_1$, also $y \notin \pi_2(L_2)$, hence $(x,y) \notin L_2$, since $L_1 \cup L_2 = L$ That is $(x,y) \notin L$ and this is contradiction, so $\mathcal{X} \times \mathcal{Y}$ is $L(\theta\mathcal{C})$-space.

Corollary (3.41): If $\mathcal{X}$ and $\mathcal{Y}$ are compact $\mathcal{T}_2$-spaces and $L(\theta\mathcal{C})$-spaces, then $\mathcal{X} \times \mathcal{Y}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space and $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.

Proof: Let $\mathcal{X}$ and $\mathcal{Y}$ are compact $\mathcal{T}_2$-spaces and $L(\theta\mathcal{C})$-space, then by Theorem 3.40, $\mathcal{X} \times \mathcal{Y}$ is $L(\theta\mathcal{C})$-space and then $\mathcal{X} \times \mathcal{Y}$ is compact, also $\mathcal{X} \times \mathcal{Y}$ is Lindelof and $\mathcal{X} \times \mathcal{Y}$ is $L(\theta\mathcal{C})$-space, by Theorem 3.3, $\mathcal{X} \times \mathcal{Y}$ is $\mathcal{M}$inL(θ$\mathcal{C}$)-space. Now, by Proposition 3.20, $\mathcal{X} \times \mathcal{Y}$ is $\mathcal{K}(\theta\mathcal{C})$-space and it is compact, then by Theorem 2.27, $\mathcal{X} \times \mathcal{Y}$ is $\mathcal{M}$in$\mathcal{K}$(θ$\mathcal{C}$)-space.

Corollary (3.42): If $\mathcal{X}$ and $\mathcal{Y}$ are Lindelof $\mathcal{T}_2$-spaces and $L(\theta\mathcal{C})$-space, then $\mathcal{X} \times \mathcal{Y}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space.

Proof: Let $\mathcal{X}$ and $\mathcal{Y}$ are compact $\mathcal{T}_2$-spaces and $L(\theta\mathcal{C})$-spaces, then by Theorem 3.40, $\mathcal{X} \times \mathcal{Y}$ is $L(\theta\mathcal{C})$-space and from hypothesis $\mathcal{X} \times \mathcal{Y}$ is Lindelof, and the by Theorem 3.3, $\mathcal{X} \times \mathcal{Y}$ is $\mathcal{M}$inL(θ$\mathcal{C}$)-space.

Proposition (3.43): If $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{R}_1$, $L(\theta\mathcal{C})$-spaces, then $\mathcal{X} \times \mathcal{Y}$ is $L(\theta\mathcal{C})$-space.

Proof: Let $\mathcal{X}$ and $\mathcal{Y}$ are $L(\theta\mathcal{C})$-spaces, by Remarks 2.20, part (3), $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{T}_1$-spaces, but $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{R}_1$-spaces, then $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{T}_2$-space and by Theorem 3.40, $\mathcal{X} \times \mathcal{Y}$ is $L(\theta\mathcal{C})$-space.

Theorem (3.44) If $\mathcal{X}$ and $\mathcal{Y}$ are compact $\mathcal{R}_1$ and $L(\theta\mathcal{C})$-space, then $\mathcal{X} \times \mathcal{Y}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space and $\mathcal{M}in\mathcal{K}(\theta\mathcal{C})$-space.

Proof: Let $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{R}_1$, $L(\theta\mathcal{C})$-space, then by Proposition 3.43, $\mathcal{X} \times \mathcal{Y}$ is $L(\theta\mathcal{C})$-space. Also, $\mathcal{X}$ and $\mathcal{Y}$ are compact spaces, so $\mathcal{X} \times \mathcal{Y}$ is compact and then $\mathcal{X} \times \mathcal{Y}$ is Lindelof. Therefor from Theorem 3.3, $\mathcal{X} \times \mathcal{Y}$ is $\mathcal{M}$inL(θ$\mathcal{C}$)-space. Now, from Proposition 3.20, $\mathcal{X} \times \mathcal{Y}$ is $\mathcal{K}(\theta\mathcal{C})$-space and it is compact, then by Theorem 2.27, $\mathcal{X} \times \mathcal{Y}$ is $\mathcal{M}$in$\mathcal{K}$(θ$\mathcal{C}$)-space.

Theorem (3.45) If $\mathcal{X}$ and $\mathcal{Y}$ are Lindelof $\mathcal{R}_1$ and $L(\theta\mathcal{C})$-spaces, then $\mathcal{X} \times \mathcal{Y}$ is $\mathcal{M}inL(\theta\mathcal{C})$-space

Proof: Let $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{R}_1$, $L(\theta\mathcal{C})$-spaces, then by Proposition 3.43, $\mathcal{X} \times \mathcal{Y}$ is $L(\theta\mathcal{C})$-space, also $\mathcal{X}$ and $\mathcal{Y}$ are Lindelof spaces, so $\mathcal{X} \times \mathcal{Y}$ is Lindelof. Hence from Theorem 3.3, $\mathcal{X} \times \mathcal{Y}$ is $\mathcal{M}$inL(θ$\mathcal{C}$)-space.

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