FREDHOLEM COMPACT OPERATOR ON SEQUENCE SPACES OF POWER ONE

مؤثر فريدهولم المدمج المعرف على فضاءات المتتابعة المرفوعة للاس واحد

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Abstract.

Sequence spaces ℓ_p^1 and ℓ_p^{-1} , $p \in [1, \infty)$ were introduced and were proved as Banach spaces [3] . In this paper, these spaces are studied as Hilbert spaces and quasi- Hilbert spaces. Not all these spaces are quasi- Hilbert spaces or Hilbert spaces. Bounded linear operators which were defined on these spaces are used to define others types of operators such as compact operators and Fredholem operators.

Keywords:, G$\hat{a}$teaux derivative, quasi-inner product space , quasi-Hilbert space, compact operator, Fredholem operator.

الخلاصة

برهنت فضاءات المتتابعة المرفوعة للاس 1 و -1 والتي رمزت ب $p \in [1, \infty)$ ، ℓ_p^{-1} و ℓ_p^1 : كفضاءات بناخ . اما في بحثنا هذا فسوف ندرسها كفضاءات هلبرت او شبه هلبرت وسنجد ان بعض هذه الفضاءات لاينطبق عليها هذين المفهومين .على هذه الفضاءات نعرف بعض انواع المؤثرات مثل المؤثرات المدمجة ومؤثرت فريدهولم

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1. Introduction

A sequence space ℓ_p ، where $1 \leq p < \infty$ is a Banach spaces , but only ℓ_2 is Hilbert space [1 , 2] . In [3] ,we were introduced a set of all sequence spaces of power one ℓ_p^1 and a set of all sequence spaces of power minus one ℓ_p^{-1} , and. were proved that these spaces as Banach spaces . Also, some of types of operators were studied such as bounded operators which are defined on these spaces.

In this paper , we study these spaces with concepts of Hilbert spaces and quasi- Hilbert spaces and define others types of operators such as compact operators and Fredholem operators.

This paper contains two sections. section one includes concept of quasi- Hilbert space for sequence spaces ℓ_p^1 and ℓ_p^{-1} which are Hilbert spaces only when p =2 . Section two presents compact linear operators and Fredholem operators on these spaces with some results and examples.

2. Quasi-Hilbert spaces

We begin with the notions of the quasi-Hilbert space , the Banach space and the Hilbert space for the sequence spaces ℓ_p^1 and ℓ_p^{-1} .

Definition 2.1. [4]

Let V be a vector space over the field $\mathbb{R}$ equipped with $\|.\|$. A Gâteaux derivative of $\| v \|$ is a functional $\delta(v, w)$ at $v \in V$ in the direction $w \in V$ which is defined as:

$\delta(v, w) = \frac{\| v \|}{2} (\delta_1(v, w) + \delta_2(v, w)$ such that: $\| w \|$

$\delta_1(v, w) = \lim_{h\to +0} h^{-1} (\| v + hw \| - \| v \|)$, and $\delta_2(v, w) = \lim_{h\to -0} h^{-1} (\| v + hw \| - \| v \|)$, where $h \in \mathbb{R}$. In similar way, $\delta(w, v)$. is defined as:

$$\delta(w, v) = \frac{\| w \|}{2} \left(\lim_{h\to +0} h^{-1} (\| w + hv \| - \| w \|) + \lim_{h\to -0} h^{-1} (\| w + hv \| - \| w \|) \right)$$

A space V is said to be a quasi-inner product space if the next equality is satisfied:

$$\| v + w \|^4 - \| v - w \|^4 = 8 (\| v \|^2 \delta(v, w) + \| w \|^2 \delta(w, v)), \forall v, w \in V \ldots\ldots \quad (1)$$

A Banach space is called a quasi-Hilbert space if it is a quasi-inner product space.

Remark 2.2.

(1) According to $\| v \| = (< v, v >)^{1/2}$, $\forall$ $v, w \in V$, it is easy to show, every Hilbert space V is a quasi-Hilbert space, but the converse is true only if $\delta(v, w)$ is an inner product function.

(2) A Hilbert space is a Banach space V if and only if. the equation :

$\| v + w \|^2 + \| v - w \|^2 = 2\| v \|^2 + 2\| w \|^2, \forall$ $v, w \in V$.. . ……(2) , is satisfied [1]

Definition 2.3. [3]:

Let $\{\lambda_k\} \subset \mathbb{R}_+$ is monotonically increasing sequence such that $\lim_{K\to\infty} \lambda_k = +\infty$, the spaces ℓ_p^1 and ℓ_p^{-1} are sequence spaces, $p \in (0, \infty)$, which are defined as :

$$\ell_P^1 = \left\{ v = \{v_k\} : \sum_{k=1}^{\infty} \lambda_k^{\frac{p}{2}} |v_k|^p < +\infty \right. .$$

$$\ell_P^{-1} = \left\{ v = \{v_k\} : \sum_{k=1}^{\infty} \lambda_k^{\frac{-p}{2}} |v_k|^p < +\infty \right. .$$

Theorem 2.4. [3].

For every $p \in [1, \infty)$,The sequence spaces ℓ_p^1 and ℓ_p^{-1} are Banach spaces with the functions :

$$\| v \| = \left(\sum_{k=1}^{\infty} \lambda_k^{\frac{p}{2}} |v_k|^p \right)^{1/p} . \| v \| = \left(\sum_{k=1}^{\infty} \lambda_k^{\frac{-p}{2}} |v_k|^p \right)^{1/p}, \forall v \in \ell_p^1 \text{ or } \forall v \in \ell_p^{-1} \text{ respectively.}$$

Lemma 2.5.
For every positive integer p , the functional $\delta\,(v\,,w)$ in a space ℓ_p^1 exists and defines as : $\delta\,(v\,,w) = \|\,v\,\|^{2-p} \sum_k \lambda_k \;|v_k|^{p-1}(\text{sng}\, v_k\;)w_k,\;\; \forall\, v \in \ell_p^1 \,/\, \{0\}$ where,

$$\text{sng}\;\, v_k \;=\begin{cases} 1, & v_k > 0 \\ 0, & v_k = 0 \\ -1, & v_k < 0 \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (3)$$

And,
$\delta\,(w\,,v) = \|\,w\,\|^{2-p} \sum_k \lambda_k \;|w_k|^{p-1}(\text{sng}\, w_k\;)v_k,\;\; \forall\, w \in \ell_p^1 \,/\, \{0\}$ where,

$$\text{sng}\;\, w_k \;=\begin{cases} 1, & w_k > 0 \\ 0, & w_k = 0 \\ -1, & w_k < 0 \end{cases}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (4)$$

Proof:

In definition 2.1, we use properties of limits of functions and applying a norm function of ℓ_p^1 in theorem 2.4 with help the binomial theorem, we get Eq. (3).

Remark 2.6.

In similar to lemma 2.5, we get the functionals $\delta\,(v\,,w)$ and $\delta\,(w\,,v)$ in a space ℓ_p^{-1} , where $\delta\,(v\,,w)$ is defined as

$\delta(v\,,w) = \|\,v\,\|^{2-p} \sum_k \lambda_k{}^{-1}|v_k|^{p-1}(\text{sng}\, v_k\;)w_k,\; \forall\, v \in \ell_p^{-1} \,/\, \{0\}$, and similarly, $\delta\,(w\,,v)$ is

defined.

Remark 2.7.

Only if p = 2 , spaces ℓ_p^1 and ℓ_p^{-1}, where p $\in [1,\infty)$, are quasi-Hilbert spaces and Hilbert spaces . Otherwise, there are spaces which are not quasi-Hilbert spaces, it as shown in the following results:

Theorem 2.8.
The sequence spaces ℓ_2^1 and ℓ_2^{-1} are quasi-Hilbert spaces .

Proof:
According to lemma 2.5، we get $\delta(v,w) = \sum_k \lambda_k \;\; |v_k|(sng\; v_k)\;\; w_k,$ and $\delta(w,v) = \sum_k \lambda_k \;\; |w_k| \;\; (sng\; w_k)v_k$. It is clear that the functional $\delta(v,w)$ is positive and equal 0 if $v = w = 0$, $\delta(v,w) = \delta(w,v)$, and also, it is linear .Hence, $\delta(v,w)$ is an inner product function, Since ℓ_2^1 is a Banach space then it is Hilbert space. By remark 2.2, it is a quasi- Hilbert space, since equation (1) is satisfied, where $\|\,v\,\|^2\,\delta(v,w) = \sum_k \lambda_k{}^2|v_k|^3(\text{sng}\, v_k)\;\, w_k$ and $\|\,w\,\|^2\,\delta(w,v) = \sum_k \lambda_k{}^2|w_k|^3(sng\; w_k)v_k$.

Similarly, ℓ_2^{-1} is quasi-Hilbert space , where $\delta\,(v\,,w)$ in ℓ_2^{-1} is defined as $\delta\,(v\,,w) = \sum_k \lambda_k{}^{-1}|v_k|\ (\text{sng}\, v_k\ )w_k$, and $\delta\,(w\,,v) = \sum_k\ \lambda_k{}^{-1}|w_k|\ (\text{sng}\, w_k\ )v_k$.

Example 2.9:
Suppose $v\,,w \in \ell_4^1$, where v = $\{v_k\} = \{1,0,0,0,\ldots\}$, $w = \{w_k\} = \{1,1,0,0,\ldots\}$ and take $\{\lambda_k\} = \{k\}, k \in \mathbb{N}$. Then, the left hand of Eq. (2) equals 6.472135954999579 , while the right hand equals 8.472135954999579 , then ℓ_4^1 is not Hilbert space, but it is quasi- Hilbert space, where the left and right hand of Eq. (1) are equal to 16 with $\delta(v,w) = \delta(w,v) = 1$.

Now, if replace space ℓ_4^1 by ℓ_3^1 , then we have , the left hand of equation (1) equals 16 , while the right equals 19.9260368210839 , then equation (2) is not satisfied , so this space is not quasi-inner product space, and also it is not Hilbert space, since equation (2) is not satisfied.

3. Some types of operators on sequence spaces

Let $\{\lambda_k\} \subset \mathbb{R}_+$ is monotonically increasing sequence such that $\lim_{K\to\infty} \lambda_k = +\infty$, An operator $T\colon \ell_p^1 \longrightarrow \ell_p^{-1}$, $1 \le p < \infty$, which is defined as $Tv = \lambda_k\, v_k\ \ \forall\ \ v = \{v_k\} \in \ell_p^1$ is a bijective continues linear operator where, (kernel of T) ker $T=\{0\}$ and (image of T) img $T = \ell_p^{-1}$, and has continues inverse $T^{-1}w = \lambda_k^{-1}\, w_k\ \forall\ w = \{w_k\} \in \ell_p^{-1}$ [3]

Definition 3.1. [5]
A bounded operator $T : U \longrightarrow V$, where U and V are Banach spaces, is compact if for every bounded sequence $\{v_k\}$ in U, $\{Tv_k\}$ has a convergent subsequence in V.

Lemma 3.2. [2]
(a)-Any subspace of a Banach space is closed if and only if it is a Banach space .

(b)- Any operator from a Banach space into another is bounded if and only if it is continues

Theorem 3.3.
A bounded operator $T : \ell_p^1 \to \ell_p^{-1}$, $1 \le p < \infty$, such that $Tv = \lambda_k v_k$, $k \in \mathbb{N}$ is a compact operator.

Proof
Let B be a closed subset in ℓ_p^1 and $\{u_k\}$ be any bounded sequence in B, then $\{u_k\}$ has $\{u_{ks}\}$ as a convergent subsequence . Since B is a Banach space by theorem 2.4 and lemma 3.2 , then $\{u_k\}$ converges to an element $u^* = \{u_k{}^*\}$ in B. Thus, $\{u_{ks}\}$ converges to u^* in B, that is, $u_{ks} \to u^*$.

Now, since T is continuous, then $\lim_{k\to\infty} \| Tu_{ks} - Tu^* \| =$

$\lim_{k\to\infty} \| \lambda_k u_{ks} - \lambda_k u_k^* \| = 0$ as $ks \to \infty$, that is, $Tu_{ks} \to Tu_k^*$ .Thus, $\{Tu_k\}$ contains a subsequence converges to Tu_k^* . Hence, a linear operator T is compact.

Definition 3.4 [6]

Let U and V be Banach spaces. A bounded linear operator $T : U \longrightarrow V$, is called Fredholm operator if dim ker L < ∞ and dim coker T < ∞: where coker T = Y/ img T. That is, The index of T (ind T) is finite, where ind T = dim ker - dim coker T .

Remark 3.5:

It is known, bijective property of operator gives finity to dim ker T and dim coker T, but it is not necessary in order to be a operator as a Fredholm operator . The following example explains this remark:

Example 3.6

Let $T : \ell_2^1 \longrightarrow \ell_2^1$, be a operator defined by $T(v_1, v_2, v_3, \ldots) = (0, v_1, 2v_2, 3v_3, \ldots)$,where $v = \{v_K\} \in \ell_2^1$ and $\{\lambda_k\} = \{k\}, k \in \mathbb{N}$.

Clearly, T (0, 0, 0, . . .) = (0, 0, 0, 0, . .), then ker T = {0}, so T is injective and dim ker T = 0. Also, it is clear that img T is not equal ℓ_p^1 then T is not surjective, hence, T is not bijective. Also, coker T = span{x}, where v = {1,0,0,0,…} and so dim coker T = 1 < ∞. Thus, ind T = dim ker T − dim coker T = 0 − 1 = −1. Hence, T is a Fredholm operator.

Theorem 3.7:

A bounded linear operator $T : \ell_p^1 \to \ell_p^{-1}$, $1 \le p < \infty$ such that $Tv = \lambda_k v_k , k \in \mathbb{N}$ is a Fredholem operator.

Proof:

Since img of a operator T= ℓ_p^{-1} where T is serjective, then coker $T = \ell_p^{-1}$ / img T = {0} , so dim coker T = 0. Also, T is injective, this implies that ker T = {0} and dim ker T= 0 . Thus , ind T = 0 , so T is Fredholem operator.

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