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Jordan-Lie Inner Ideals of the Orthogonal Simple Lie Algebras

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ABSTRACT: Let A be an associative algebra over a field F of any characteristic with involution $*$ and let $K = skew(A) = \{a \in A | a^* = -a\}$ be its corresponding sub-algebra under the Lie product $[a, b] = ab - ba$ for all $a, b \in A$. If $A = EndV$ for some finite dimensional vector space over F and $*$ is an adjoint involution with a symmetric non-alternating bilinear form on V, then $*$ is said to be orthogonal. In this paper, Jordan-Lie inner ideals of the orthogonal Lie algebras were defined, considered, studied, and classified. Some examples and results were provided. It is proved that every Jordan-Lie inner ideals of the orthogonal Lie algebras is either $B = eKe^*$ or B is a type one point space.

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1. INTRODUCTION

Let A be a finite dimensional associative algebra over a field F. Recall that A becomes a Lie algebra $A^{(-)}$ under the Lie bracket defined by $(x, y] = xy - yx$ for all $x, y \in A$. Suppose that A has an involution $*$. Recall that an involution is a linear transformation $*$ of an algebra A satisfying $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. We denote by $K = skew(A) = (a^* = -a | a \in A\}$ to be the vector space of the skew symmetric elements of A. Recall that K is a Lie algebra with the Lie bracket defined by $(x, y] = xy - yx$ for all $x, y \in K$. If the characteristic of F is non-equal 2, then K can be represented in the form:

$$K = skew(A, *) = \{a - a^* | a \in A\}. \tag{1.1}$$

Benkart was the first to introduce an inner ideal of a Lie algebra. She defined it as a subspace B of a Lie L such that the space $[B, [B, L]]$ is a subset of B [1]. She highlighted the relationship between inner ideals and an ad-nilpotent elements [2]. Recall that an adjoint map $ad : L \to gl(L)$ is a representation from a Lie L into its general linear algebra defined by $ad(l) = ad_l$, where $ad_l : L \to L$ is a linear transformation defined by $ad_l(x) = [l, x]$ for all $x \in L$. By restricting ad-nilpotent elements, one can classify non-classical from classical simple Lie algebras over algebraically closed fields of characteristic $\neq 2, 3$. Therefore, inner ideals play a role in classifying these algebras. Commutative inner ideals have proved to be a useful tool for classifying both finite and infinite-dimensional simple Lie algebras. It is proved in [3] that inner ideals play a role similar to one-sided ideal in associative algebras and can be used to construct Artinian structure theory for Lie algebras. Inner ideals is an essential tool in the classification of Lie algebras. (see [4] and [3]). Inner ideals of classical type Lie sub-algebras of associative(simple) rings were studied by Benkart and Fernandez Lopez (see [5]) . Baranov and Shlaka [6] in 2019 classified Jordan-Lie inner ideals of the Lie sub-algebras of finite dimensional associative algebras. An inner ideal B of $A^{(k)}$ or $K^{(k)}$ is said to be Jordan-Lie if $B^2 = 0$. In recent paper, Shlaka and Mousa [7], studied Jordan-Lie inner ideals $A^{(k)}$ in the case when A is simple over an algebraically closed fields of positive characteristic. Jordan-Lie inner ideals of the Lie algebras $K^{(k)}$ in the case when A is simple with the symplectic involution over an algebraically closed fields of positive characteristic were also been studied by Kareem and Shlaka in [8].

In this paper, we study inner ideals of the orthogonal Lie algebras. We start with some preliminaries in section 2. Section 3 is devoted to proof some results about Jordan-Lie inner ideals of the orthogonal Lie algebras and point space.

2. PRELIMINARIES

Throughout this paper, F is a field (algebraically closed), $p \geq 0$ is the characteristic of F, V is a vector space (finite dimensional over F), $End(V)$ is the endomorphism algebra, $so(V)$ is the orthogonal Lie algebra, A is an associative algebra (finite dimensional over F) with an involution $*$, $K = skew(A, *)$ is the Lie subalgebra of A defined as (1.1), L is a Lie algebra (finite dimensional over F), $M_n(F)$ is the matrix algebra consisting of all $n \times n$-matrices and $so_n(F)$ is the orthogonal Lie algebra of matrix .

Recall that an involution $*$ of A is a linear transformation of A such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for any $a, b \in A$ [9]. Note that $*$ does not required to be F-linear. On the other hand, it is obvious that $*$ maps the center Z into it self. Since the restriction of $*$ over F is an automorphism of order less than or equal to 2, it maps every sub-field of Z into itself. Therefore $F^* = F$. Here we have two possibilities which are either $*$ is F-linear or not. Thus, we have the following definition.

Definition 2.1 [13, 7.2] An involution is said to be of the first kind in case that $*$ is F-linear, that is the restriction of $*$ relative to F is the identity. Otherwise, it is called of the second kind.

Remark 2.2 In this paper, we consider involution of the first kind only.

Definition 2.3 Let B be a subspace of L. Then B is said to be

1. [1] An inner ideal if $(B, (B, L]] \subseteq B$.
2. [1] A commutative inner ideal if B is an inner ideal such that $[B, B] = 0$.
3. [6] A Jordan-Lie inner ideal (or simply, J-Lie) if $L = skew(A)$ and B is an inner ideal such that $B^2 = 0$.

Example 2.4 Consider the associative algebra $A = M_n(F)$. Then $\{e_{ij} | 1 \leq i, j \leq n\}$ form a basis of A consisting of matrix units, where e_{ij} is the $n \times n$-matrix with the entry 1 in the ijth position and zero elsewhere. Thus, the Lie algebra

$K = skew(A) = so_{2n}(F)$ has the following basis $\{a_{ij}, b_{ij}, c_{ij} | 1 \leq i, j \leq n\}$, where

$$a_{ij} = (e_{ij} - e_{n+j,n+i}), \qquad b_{ij} = (e_{i,n+j} - e_{j,n+i}) \quad and \quad c_{ij} = (e_{n+i,j} - e_{n+j,i}).$$

Then $B = Fa_{12}$ is J-Lie of $skew(A, *)$. Indeed, for any $x, y \in B$, we have $x = \alpha a_{12} = \alpha(e_{12} - e_{n+2,n+1})$, $y = \beta a_{12} = \beta(e_{12} - e_{n+2,n+1})$. Since

$$x.y = \alpha(e_{12} - e_{n+2,n+1}\beta(e_{12} - e_{n+2,n+1} = 0,$$

$B^2 = 0$. It remain to show that $[x, [y, l]] \in B$ for each $l \in K$.

Let $l = \sum_{i,j=1}^{n} \zeta_{ij} a_{ij} + \sum_{i,j=1}^{n} \eta_{ij} b_{i,n+j} + \sum_{i,j=1}^{n} \gamma_{ij} c_{ij} \in K$. Then

$$xly = \alpha(e_{12} - e_{n+2,n+1})(\sum_{i,j=1}^{n} \zeta_{ij} a_{ij} + \sum_{i,j=1}^{n} \eta_{ij} b_{ij} + \sum_{i,j=1}^{n} \gamma_{ij} c_{ij})y$$

$$= \alpha \sum_{j=1}^{n} (\zeta_{2j} e_{1j} + \eta_{2j} e_{1,n+j} - \eta_{j2} e_{1,n+j} + \zeta_{j1} e_{n+2,n+j} - \gamma_{1j} e_{n+2,j} + \gamma_{j1} e_{n+2,j})y$$

$$= \alpha\beta(\zeta_{21} e_{12} - \eta_{22} e_{1,n+1} - \eta_{22} e_{1,n+1} - \zeta_{21} e_{n+2,n+1} - \gamma_{11} e_{n+2,2} + \gamma_{11} e_{n+2,2})$$

$= \alpha\beta\zeta_{21}(e_{12} - e_{n+2,n+1}) = \alpha\beta\zeta_{21} a_{12} \in Fa_{12} = B$.

and

$$ylx = \beta(e_{12} - e_{n+2,n+1})(\sum_{i,j=1}^{n} \zeta_{ij} a_{ij} + \sum_{i,j=1}^{n} \eta_{ij} b_{ij} + \sum_{i,j=1}^{n} \gamma_{ij} c_{ij})x$$

$$= \beta \sum_{j=1}^{n} (\zeta_{2j} e_{1j} + \eta_{2j} e_{1,n+j} - \eta_{j2} e_{1,n+j} + \zeta_{j1} e_{n+2,n+j} - \gamma_{1j} e_{n+2,j} + \gamma_{j1} e_{n+2,j})x$$

$$= \alpha\beta(\zeta_{21}e_{12} - \eta_{22}e_{1,n+1} - \eta_{22}e_{1,n+1} - \zeta_{21}e_{n+2,n+1} - \gamma_{11}e_{n+2,2} + \gamma_{11}e_{n+2,2})$$

$= \alpha\beta\zeta_{21}(e_{12} - e_{n+2,n+1}) = \alpha\beta\zeta_{21}a_{12} \in Fa_{12} = B$.

Therefore, $[x,[y,l]] = xyl - xly - ylx + lyx = -xly - ylx \in B$, as required.

Definition 2.5 [10] A subspace P of L is said to be point space if $(P,P] = 0$ and $ad_x^2(L) = Fx$ for every non zero element $x \in P$.

Example 2.6 Let $K = so_{2n+1}(F)$, If $n = 1$, then

$$K = so_3(F) = span\{\begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_3 & 0 \\ -\alpha_1 & 0 & -\alpha_3 \end{pmatrix} | \alpha_1, \alpha_2, \alpha_3 \in F\}$$

has a basis are

$$\{b_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\}$$

Then , we need to show that b_1, is a point space. For $x \in Fb_1$ we have

$$x = \begin{pmatrix} 0 & \zeta & 0 \\ 0 & 0 & 0 \\ -\zeta & 0 & 0 \end{pmatrix} \text{ for some } \zeta \in F. \text{ Let } l = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_3 & 0 \\ -\alpha_1 & 0 & -\alpha_3 \end{pmatrix} \in so_3(F).$$

Then, $ad_x^2(L) = (x,(x,l]]$

$$= [x, \begin{pmatrix} -\zeta\alpha_2 & \zeta\alpha_3 & 0 \\ 0 & 0 & 0 \\ 0 & -\zeta\alpha_1 & -\zeta\alpha_2 \end{pmatrix} - \begin{pmatrix} -\zeta\alpha_2 & 0 & 0 \\ 0 & -\zeta\alpha_2 & 0 \\ \zeta\alpha_3 & -\zeta\alpha_1 & 0 \end{pmatrix}]$$

$$= [\begin{pmatrix} 0 & \zeta & 0 \\ 0 & 0 & 0 \\ -\zeta & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \zeta\alpha_3 & 0 \\ 0 & \zeta\alpha_2 & 0 \\ -\zeta\alpha_3 & 0 & -\zeta\alpha_2 \end{pmatrix}]$$

$$= \begin{pmatrix} 0 & \zeta^2\alpha_2 & 0 \\ 0 & 0 & 0 \\ 0 & -\zeta^2\alpha_3 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \zeta^2\alpha_2 & -\zeta^2\alpha_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \zeta^2\alpha_2 & 0 \\ 0 & 0 & 0 \\ -\zeta^2\alpha_2 & 0 & 0 \end{pmatrix} \in Fb_1$$

Therefore, Fb_1 and also Fb_2 is a point space. while Fb_2 is not point space.

We will need the following lemma. For the proof see [11].

Lemma2.7 [11] Let B be an I-ideal of L. If $B^2 = 0$, then

(1) $b_1lb_2 + b_2lb_1 \in B$ for all $b_1, b_2 \in B$ and $l \in L$.

(2) $blb \in B$ for all $b \in B$ and $l \in L$.

Definition 2.8 [9] Let $\psi : V \times V \to F$ be a nondegenerate symmetric bilinear form. For each $x \in EndV$ define x^* by the following property $\psi(x^*(v), w) = \psi(v, x(w))$ for all $v, w \in V$. Then the map $* : EndV \to EndV$ is an involution of the algebra $EndV$, called the adjoint involution with respect to ψ.

Theorem 2.9 [10, Ch.1, introduction] The map $\psi \mapsto *$ induced one to one correspondence between equivalence classes of nondegenerate bilinear forms on V modulo multiplication by a factor in $F^\times$ and involution (of first kind) on $EndV$.

Definition 2.10 [9] Let $*$ be an involution of $EndV$. We say that $*$ is orthogonal if it is adjoint to a symmetric non-alternating bilinear form on V.

Definition 2.11 [2] Let A be an associative algebra with involution $*$ over a field $F$and let $a \in A$. Then we define the trace of a by $\tau(a) = a - a^*$.

3. JORDAN LIE INNER IDEAL OF THE ORTHOGONAL LIE ALGEBRAS

Theorem 3.1 Suppose that A is simple with involution and $p \neq 2$. Let $x \in skew(A,*)$. Then $x = xyx$ for some $y \in skew(A,*)$.

Proof. We have $x^* = -x$. Since A is V-Neumann algebra, $x = xax$ for some $a \in A$. Put $y = \frac{1}{2}(a - a^*) \in skew(A,*)$. Then

$xyx = \frac{1}{2}x(a - a^*)x = \frac{1}{2}(xax - xa^*x) = \frac{1}{2}(x - (xax)^*) = \frac{1}{2}(x - x^*) = \frac{1}{2}(2x) = x$

Lemma 3.2 Let $eKe^* \subseteq B$ be a subspace of $K = skew(A,*)$ such that $e \in BK$ and $e^* \in KB$. If e' be an idempotent in A such that $ee' = e'e = 0$, then $e'Be'^* \subseteq B$.

Proof. If $e'Be'^* = 0$. Then $e'Be'^* \subseteq B$. Suppose now that $e'Be'^* \neq 0$. Then $\exists a \in B$ such that $e'ae'^* \neq 0$.

$e'ae'^* = (1-e)a(1-e^*) = a - (ea + ae^*) + eae^*$

As $e \in BK$, $\exists b_1 \in B$ and $k_1 \in K$ such that $e = b_1k_1$. This implies that

$e^* = (b_1k_1)^* = k_1{}^*b^* = k_1b_1$

We have $a \in B$ and $eae^* \in eKe^* \subseteq B$. By Lemma 2.7,

$ea + ae^* = b_1k_1a + ak_1b_1 \in B$

Therefore, $e^{'}ae^{'*} \in B$, as required.

Recall that A is simple, so A can be identified with $End(V)$ for some vector space V. We have the following proposition.

Proposition 3.3 Let $\psi : V \times V \to F$ be a non-singular form and let $*$ be an adjoint involution of $A = End(V)$. Let $e, e^{'}$ be idempotent in A such that $ee^{'} = e^{'}e = 0$. Suppose that $eKe^* \neq 0$. Then the following hold

For each $k \in K$ such that $eke^* \neq 0$, we have

(1) $c = k + e^{'}ke^{'*} \neq 0$.

(2) $e^{'}Ke^* = 0$.

(3) $eKe^{'*} = 0$.

Proof. (1) Let $v \in V$ such that $\psi(v, eke^*(v)) \neq 0$. Such v exists because ψ is non-singular. We need to show that $\psi(e^*(v), ce^*(v) \neq 0$. Since $ee^{'} = 0$,

$\psi(e^*(v), ce^*(v)) = \psi(v, ece^*(v))$

$= \psi(v, e(k + e^{'}ke^{'*})e^*(v))$

$= \psi(v, eke^*(v)) + \psi(v, ee^{'}ke^{'*}e^*(v))$

$= \psi(v, eke^*(v)) \neq 0$.

(2) Let $w \in e^{'}Ke^*$. Then there is $k \in K$ such that $w = e^{'}ke^*$. For each $v \in V$ we have

$\psi(e^*(v), we^*(v)) = \psi(v, ewe^*(v)) = \psi(v, ee^{'}ke^*e^*(v)) = \psi(v, 0) = 0$,

so $w = e^{'}Ke^* = 0$.

(3) Let $h \in eKe^{'*}$. Then there is $k \in K$ such that $h = eke^{'*}$. For each $v \in V$ we have

$\psi(e^*(v), he^*(v)) = \psi(v, ehe^*(v)) = \psi(v, e(eke^{'*})e^*(v)) = \psi(v, 0) = 0$.

Therefore, $h = eke^{'*} = 0$.

The idea of the following lemma comes from McCrimmon's paper [2].

Lemma 3.4 Let A be an associative algebra with involution $*$ over a field F. Suppose that $L = skew\,(A, *)$. Then the trace τ that defined above by $\tau\,(a) = a - a^*$ has the following properties:

(1) τ is linear.

(2) $\tau\,(x) \in L$ for any $x \in L$.

(3) $x\tau\,(a)\,x = \tau\,(xax)$ For any $a \in A$ and $x \in L$.

(4) $a\tau\,(b) + \tau\,(b)\,a^* = \tau\,(ab) + \tau\,(ba^*)$ For any $a, b \in A$.

(5) $\tau\,(a)\,x\tau\,(a) = \tau\,(axa) - axa^* - a^*xa$ For any $a \in A$ and $x \in L$.

Proof. (1) Suppose that $a, b \in A$ and $\alpha \in F$. Then

$\tau\,(\alpha a) = \alpha a - (\alpha a)^* = \alpha\,(a - a^*) = \alpha\tau\,(a)\,;$

$$\tau\,(a + b) = (a + b) - (a + b)^* = (a - a^*) + (b - b^*) = \tau\,(a) + \tau\,(b)\,.$$

Thus, τ is linear.

(2) Let $a \in A$ Then

$(\tau\,(a))^* = (a - a*)^* = a^* - a = -\,(a - a^*) = -\tau\,(a)$.

Therefore, $\tau\,(a) \in L$.

(3) Let $a \in A$ and $x \in L$ Then we have

$x\tau\,(a)\,x = x\,(a - a^*)\,x = xax - xa^*x = xax - (x^*ax^*)^* = \tau\,(xax)\,.$

(4) Let $a, b \in A$ Then

$$a\tau\,(b) + \tau\,(b)\,a^* = a\,(b - b^*) + (b - b^*)\,a^* = ab - ab^* + ba^* - b^*a^*$$

$= (ab - b^*a^*) + (ba^* - ab^*)$

$= (ab - (ab)^*) + (ba^* - (ba^*)^*) = \tau\,(ab) + \tau\,(ba^*)$

(5) For any $a \in A$ and $x \in L$ we have

$\tau\,(a)\,x\tau\,(a) = (a - a^*)\,x\,(a - a^*) = axa + a^*xa^* - axa^* - a^*xa$

$= (axa - (axa)^*) - axa^* - a^*xa = \tau\,(axa) - axa^* - a^*xa.$

Lemma 3.5 Suppose that $p \neq 2, 3$ and $K = skew\,(EndV, *)$. Then the following hold:

(1) If $\psi\,(Kv, w) = 0$ for some nonzero vectors $v, w \in V$, then $w \in Fv$. Consequently $Kv = v^{\perp}$ for any nonzero vector $v \in V$.

(2) If U is a subspace such that $dimU > 1$, then $KU = V$.

(3) A transformation $x \in K$ satisfies $xKx^* = 0$ if and only if $rank\,(x) \leq 1$.

Proof. (1) Suppose that $v, w \in V$ be nonzero vectors such that $\psi\,(Kv, w) = 0$. For the contrary we assume that $w \notin Kv$. Then we could find a linear transformation $a \in A$ such that $a\,(w) = 0$ and $\psi\,(a\,(v)\,, w) \neq 0$. Note that $a - a^* \in K$. Thus,

$0 \neq \psi(a(v), w) = \psi(a(v), w) - 0 = \psi(a(v), w) - \psi(v, a(w))$
$= \psi(a(v), w) - \psi(a^*(v), w) = \psi((a - a^*)(v), w) = 0,$
a contradiction Therefore $w \in Kv$ Consequently, for any nonzero vector v we have $v^\perp = Kv$
(2) Suppose that U be a subspace of V such that $dimU > 1$ Then
$KU = \sum_{w \in U} Kw = \sum_{w \in U} w^\perp = V$
That is, any $w^\perp$has co-dimensional 1. Thus, if $w_1^\perp = w_2^\perp$. Then $w_1 \in Kw_2$. Hence any two independent vectors $w_i^\perp$ will span all V.
(3) If $x^*Kx = 0$ Then
$0 = \psi(x^*Kx(v), v) = \psi(Kx(v), x(v)) \quad for\ all \quad v \in V.$
This implies $K(x(V)) \neq V$, so by (2), we get that $dim(x(V)) \leq 1$.
Theorem 3.6 Let e, e', f be an idempotent in $A = EndV$ such that $ee' = e'e = 0$ and $e^*e = 0$. Let $e^*f = fe^* = 0$ and $e^{*'}f = fe^{*'} = f$,If $B = eKe'^*$ then B is a $J-$Lie.
Proof. Let $w = eke'^* \neq 0$, by Theorem 3.1, $w = wz'w$ for some $z' \in K$.
put $z = e'^*z'e$. Then
$wzw = w(e'^*z'e)w = eke'^*e'^*z'eeke'^* = eke'^*z'eke'^* = wz'w$
Let $f = zw = (e'^*z'e)(eke'^*) = e'^*z'eke'^*$. Then
$e^*f = e^*e'^*z'eke'^* = 0$
and
$fe^* = e'^*z'eke'^*e^* = (1 - e^*)z'ek(1 - e^*)e^* = 0$
$e'^*f = e'^*e'^*z'eke'^* = e'^*z'eke'^* = f$
Also
$fe'^* = e'^*z'eke'^*e'^* = f$
By Lemma 3.5 (3), since $rank(f) = 1$, so $rank(f^*) = 1$. Therefore,
$f^*Kf = (e'^*z'eke'^*)^*K(e'^*z'eke'^*) = e'k^*e^*z'^*e'Ke'^*z'eke'^* = 0$
and
$fKf^* = e'^*z'eke'^*Ke'k^*e^*z'^*e' = 0$
Moreover, for any $u \in Ker(w)$, $f(u) = zw(u) = 0$
Therefore, $Ker(w) \subseteq Ker(f)$, both have co-dimension. Then

$$Ker(w) = Ker(f) = Ker(w')$$

Recall $f = zw$ is idempotent of rank 1. Let $c \in Im(w')$ such that $c \notin Ker(w')$. Then $w'f(c) \neq 0$
(if $w'f(c) = 0$, then either $c \in Ker(f)$ or $c \in Ker(w')$ this is a contradiction)
If $w'f(c) \neq 0$, then $c \in Im(w'f)$. Since $c \in Im(w')$, so $c \in Im(w'f)$
Therefore, $Im(w') \subseteq Im(w'f)$. both have co-dimension, so
$Im(w') = Im(w'f)$
Since $Ker(f) = Ker(w')$, so
$Ker(w'f) = Ker(w')$
Therefore, $w'f = w'$ for any $w' \in eBe'^*$.
Next, we claim that
$B \subseteq B' = eKe^* + \tau(ekf),$
for any $d \in B$ we have
$d = ede^* + ede'^* + e'de^* + e'de'^*$

$$= ede^* + ede'^* - (ede'^*)^*$$

$$= ede^* + \tau(ede'^*)$$

$$= ede^* + \tau(w')$$

Since $w'(f) = w' \in eBe'^*$, we have that
$K = eke^* + \tau(w'f) = ede^* + \tau(ede'^*f)$
As $e'^*f = f$, so
$K = eke^* + \tau(edf) \in eKe^* + \tau(eKf)$

put $B^{'} = eKe^{*} + \tau(eKf)$. Then
$(e + f^{*})K(e + f^{*})^{*} = (e + f^{*})K(e^{*} + f)$

$$= eKe^{*} + eKf + f^{*}Ke^{*} + f^{*}Kf$$

$$= eKe^{*} + eKf - (eKf)^{*}$$

$$= eKe^{*} + \tau(eKf)$$

Let $g = e + f^{*}$, then
$g^{2} = (e + e^{'*}z^{'}eke^{'*})(e + e^{'*}z^{'}eke^{'*})$
$= e + e^{'*}z^{'}eke^{'*} = g$
and $g^{*}g = (e^{*} + f)(e + f^{*})$
$= (e^{*} + e^{'*}z^{'}eke^{'*})(e + e^{'}ke^{*}z^{'*}e^{'}) = 0$
Now, let $gk_1g^{*}, gk_2g^{*} \in gKg$ and $l \in K$. Then
$[gk_1g^{*}, [gk_2g^{*}, l]] = [gk_1g^{*}, gk_2g^{*}l - lgk_2g^{*}]$
$= gk_1g^{*}gk_2g^{*}l - gk_2g^{*}lgk_1g^{*} - gk_1g^{*}lgk_2g^{*} + lgk_2g^{*}gk_1g^{*}$
$= -2gk_1g^{*}lgk_2g^{*} = g(-2k_1g^{*}lgk_2)g^{*} \in gKg^{*}$
Therefore, $B^{'} = gKg^{*}$ is an I-ideal of $K$and $B^{'}$ is J-Lie of K. as required.

Theorem 3.7 Let $e, e^{'}, f$ be an idempotent in $EndV$ such that $ee^{'} = e^{'}e = 0$. Let B be a $J-$Lie of $K = skew(A, *)$ such that $bKb \neq Fb$ for all $b \in B$. Then the following hold
1. $E(V = eKe^{*}K(v_0^{\perp} forall v_0 \in V$
2. $EBe^{'*}(V = e(V)$

Proof. (1) Suppose that $bKb \neq Fb$. We have $B \subseteq B^{'} = (e + f^{*})K(e + f^{*})^{*}$
By Lemma 3.5 (1), we have $Kv_0 = v_0^{\perp}$. Thus $eKe^{*}K(v_0) = eKe^{*}(v_0^{\perp})$
Suppose that $dim(e^{*}(v_0^{\perp})) \geq 1$.
If $dim(e^{*}(v_0^{\perp})) > 1$, then by Lemma 3.5 (2),
$Ke^{*}(v_0^{\perp}) = V \Longrightarrow eKe^{*}(v_0^{\perp}) = e(V)$
Therefore, $e(V) = eKe^{*}(v_0^{\perp})$.
and if $dim(e^{*}(v_0^{\perp})) = 1$, then there exist a non-zero $u_0 \in V$ such that

$$e^{*}(v_0^{\perp}) = Fu_0. \tag{3.2}$$

Then
$eKe^{*}(v_0^{\perp}) = eK(u_0) = e(u_0^{\perp})$
for all $u \in u_0^{\perp}$, we have $e(u) \in (v_0^{\perp})^{\perp} = Fv_0$, because
$e(u) \in e(u_0^{\perp}) = eKe^{*}(v_0^{\perp}) \subseteq B(v_0^{\perp}) \subseteq K(v_0^{\perp}) = (v_0^{\perp})^{\perp} = Fv_0$,
so
$u_0^{\perp} = e^{'}(V) + Fv_0$
Thus,

$$eKe^{*}(u_0) = e(u_0^{\perp}) = e(e^{'}(V) + Fv_0) = Fv_0. \tag{3.3}$$

But for any non-zero $r \in u_0^{\perp}$ and $\alpha \in F$, we have $e^{*}(r) = \alpha u_0$

$$\alpha e^{*}(u_0) = e^{*}(\alpha u_0) = e^{*}(e^{*}(r)) = e^{*}(r) = \alpha u_0.$$

so $e^{*}(u_0) = u_0$. Thus, for any $y = ey^{'}e^{*} \in eKe^{*}$, we can assume that $y(u_0) = 0$
Let $y(V) \subseteq u_0^{\perp}$, by equation (3.3),

$$y(V) = ey^{'}e^{*}(V) = e(ey^{'}e^{*}(V)) = e(y(V)) \subseteq e(u_0^{\perp}) = Fv_0$$

By Lemma 3.5 (3), if y has rank 1, then $y^{*}Ky = yKy = 0$.
By Theorem 3.1, $\exists 0 \neq l \in K$ such that $y = yly \in yKy = 0$.
Then, $y \in eKe^{*} \subseteq B$. Therefore $y \in Fb$, but $eKe^{*} = bKb$

so $y \in bKb$. Thus, if $bKb \neq Fb$, then $eKe^{*}K(v_0) = e(V)$, as required.
(2) for any l,l^’\in K, we have ele^\ast\in eKe^\ast\subseteq B
Let $b^{''} = -[ele^{*}, [b^{'}, l^{'}]] \in [B, [B, K]] \subseteq B$
$b^{'} \in B$ is the same $b^{'}$ that satisfies $w = eb^{'}e^{*'} \neq 0$. Since
$b^{''} = -[ele^{*}, [b^{'}, l^{'}]] = -[ele^{*}, b^{'}l^{'} - l^{'}b^{'}]$
$= -(ele^{*}b^{'}l^{'} - b^{'}l^{'}ele^{*} - ele^{*}l^{'}b^{'} + l^{'}b^{'}ele^{*})$
$eb^{''}e^{'*} = -e(ele^{*}b^{'}l^{'} - b^{'}l^{'}ele^{*} - ele^{*}l^{'}b^{'} + l^{'}b^{'}ele^{*})e^{'*}$
$= -eele^{*}b^{'}l^{'}e^{'*} + eb^{'}l^{'}ele^{*}e^{'*} + eele^{*}l^{'}b^{'}e^{'*} - el^{'}b^{'}ele^{*}e^{'*}$
$= -ele^{*}b^{'}l^{'}e^{'*} + ele^{*}l^{'}b^{'}e^{'*}$
and $ele^{*}b^{'}l^{'}e^{'*} = bxlxbb^{'}l^{'}e^{'*} = 0$. As $(bb^{'} = 0)$
$eb^{''}e^{'*} = ele^{*}l^{'}b^{'}e^{'*} = ele^{*}l^{'}(e + e^{'})b^{'}e^{'*}$
$= ele^{*}l^{'}eb^{'}e^{'*} + ele^{*}l^{'}(e^{'}b^{'}e^{'*})$
By using equation , $(e^{'}b^{'}e^{'*} = 0)$, we have $eb^{''}e^{'*} = ele^{*}l^{'}(eb^{'}e^{'*})$.
Since $w = eb^{'}e^{'*}$, so $eb^{''}e^{'*} = ele^{*}l^{'}w$ for any $l, l^{'} \in K$.
Let $v \in V$. Then $eb^{''}e^{'*}(v) = ele^{'*}l^{'}w(v) = ele^{'*}l^{'}(v_0)$
Since $bKb \neq Fb$, so we must have
$eBe^{'*}(V) = eKe^{*}K(v_0)$
Since $eKe^{*}K(v_0) = e(V)$, we get that
$eBe^{'*}(V) = e(V)$
as required.

Theorem 3.8 Let e, f be an idempotent in $A = EndV$ and let B be a $J-$Lie of $K = skew(A, *)$. Suppose that $bKb = Fb$ for all $b \in B$.Then B is a type one point space.

Proof. Suppose that $bKb = Fb$. we are going to prove that B is a type one point space
Recall that $B \subseteq B^{'} = eKe^{*} + \tau(eKf)$, so
$B^{'} = eKe^{*} + \tau(eKf) = bxKxb + \tau(eKf)$

$$= bKb + \tau(eKf) \tag{3.4}$$

Since $bKb = Fb$, so
$B^{'} = Fb + \tau(eKf)$
for any $c \in B^{'}$, there exist $\lambda \in F$ and $l \in K$ such that
$c = \lambda b + \tau(elf)$
Then $\forall y \in K$, we have
$cyc = (\lambda b + \tau(elf))y(\lambda b + \tau(elf))$
$= \lambda^2 byb + \lambda by\tau(elf) + \lambda\tau(elf)yb + \tau(elf)y\tau(elf)$
$= \lambda^2 byb + \lambda(by)\tau(elf) + \lambda\tau(elf)(by)^{*} + \tau(elf)y\tau(elf)$
By Lemma 3.4 (3),
$cyc = \lambda^2 byb + \lambda\tau(byelf) + \lambda\tau(elf(by)^{*})$
$+\tau(elfyelf) - elfy(elf)^{*} - (elf)^{*}yelf$
$\lambda^2 byb + \lambda\tau(byelf) + \lambda\tau(elfyb) + \tau(elfyelf) - elfyf^{*}l^{*}e^{*} - f^{*}l^{*}e^{*}yelf$
Since $fKf^{*} = f^{*}Kf = 0$.

$$cyc = \lambda^2 byb + \lambda\tau(byelf) + \lambda\tau(elfyb) + \tau(elfyelf) \tag{3.5}$$

we need to calculate each term. Since $bKb = Fb$, so

$$byb = \alpha b \tag{3.6}$$

$\tau(byelf) = \tau(bybxlf) = \tau(\alpha bxlf) = \tau(\alpha(elf))$

$$= \alpha\tau(elf) \tag{3.7}$$

for the third one we have
$elfyb = bxlfyb \in bAb$
Since $\tau(a) \in L$ for any $a \in A$,$\tau(elfyb) \in K$, then $b\tau(xlfy)b \in bKb \subseteq B$.
By Lemma 3.5 (3),

$$\tau(elfyb) = \tau(bxlfyb) = b\tau(xlfy)b = \beta b \tag{3.8}$$

for some $\beta \in F$

for the four one we have

$elfyelf = elfybxlf = (elf)yb(xlf) = 0$

Since $f^*Lf = 0$, so $byf^*le^*xlf = 0$. Then

$$(elf)y(elf) = elfybxlf - byf^*le^*xlf$$

$$= (elfyb - (elfyb)^*)xlf$$

$$= \tau(elfyb)xlf$$

$$\beta b(xlf) = \beta elf$$

$$\tau(elfyelf) = \beta\tau(elf) \tag{3.9}$$

Substituting equation 3.6 , 3.7, 3.8 and 3.9 in 3.5, we get that

$$cyc = (\lambda^2\alpha)b + (\lambda\alpha)\tau(elf) + (\lambda\beta)b + \beta\tau(elf)$$

$$= (\lambda^2\alpha + \lambda\beta)b + (\lambda\alpha + \beta)\tau(elf)$$

$$= (\lambda\alpha + \beta)c$$

Therefore, $cKc = Fc$, B' is a point space

since B is a maximal point space, so $B = B'$

Therefore, B is a type one point space.

Theorem 3.9 Suppose that A is simple with the orthogonal involution $*$ defined on it. If $p \neq 2, 3$ and A is of dimensional greater than 16, Then every $J-$Lie B of $(K, K]$ is of the form eKe^* or B is a type one point space. where e is an idempotent in A such that $e^*e = 0$.

Proof. Let $b \in B$, Then by Theorem 3.1, $\exists x \in K$ such that $b = bxb$.

Let $e = bx$. Then $e^* = (bx)^* = x^*b^* = xb$, since B is $J-$Lie, $b^2 = 0$, so $e^*e = xbbx = 0$. By Lemma 3.2, $bKb \subseteq B$

Suppose that $bKb \subseteq B$ is maximal with the property. Since

$bKb = bxbKbxb \subseteq bxKxb = eKe^*$;

$eKe^* = bxKxb \subseteq bKb$,

We have

$$eKe^* = bKb \subseteq B \tag{3.10}$$

Next, we need to show that $B \subseteq eKe^*$

Let $e' = 1 - e$ and $e'^* = (1 - e)^* = 1 - e^*$, we have

$$b = 1b1 = (e + e')b(e^* + e'^*) = ebe^* + ebe'^* + e'be^* + e'be'^* \tag{3.11}$$

First, we need to show that $e'Ke'^* = 0$

It remains to show that $e'Ke'^* = 0$. Assume to the contrary that $e'Ke'^* \neq 0$. Then $\exists c' \in K$ such that $z = e'c'e'^* \neq 0$. By Lemma 3.2 , $e'Ke'^* \subseteq B$, so $z \in B$. Let $c = b + z \in B$. In the view of Lemma 3.3(1), we have $c \neq 0$

First, we claim that $bKb \subseteq cKc$. Since $c \in B$, by Lemma 2.7, $cKc \subseteq B$. Take any $y \in K$. Then

$ce^*yec = (b + z)e^*ye(b + z)$

$= be^*yeb + be^*yez + ze^*yeb + ze^*yez$
Since $ez = e(e'c'e'^*) = 0$ and $ze^* = (e'c'e'^*)e^* = 0$,
$ce^*yec = be^*yeb = bxbybxb = byb$
so $ce^*Kec = bKb$. As $ce^*Kec \subseteq cKc$, we get that

$$bKb = ce^*Kec \subseteq cKc \tag{3.12}$$

Next, we need to show that $zKz \subseteq cKc$. Take any $l \in K$, we have
$ce'^*le'c = (b+z)e'^*le'(b+z)$

$$= be'^*le'b + be'^*le'z + ze'^*le'b + ze'^*le'z \tag{3.13}$$

By computing mutually each term, we get that
$be'^*le'b = b(1-e^*)l(1-e)b = blb - bleb - be^*lb + be^*leb.$

$$= blb - blbxb - bxblb + bxblbxb = blb - blb - blb + blb = 0 \tag{3.14}$$

$be'^*le'z = b(1-e^*)l(1-e)z = blz - blez - be^*lz + be^*lez$

$$= blz - bxblz = blz - blz = 0 \tag{3.15}$$

$$ze'^*le'b = z(1-e^*)l(1-e)b = zlb - zleb - ze^*lb + ze^*leb = zlb - zlb = 0 \tag{3.16}$$

$$ze'^*le'z = z(1-e^*)l(1-e)z = zlz - zlez - ze^*lz + ze^*lez = zlz \tag{3.17}$$

By substituting equation 3.14, 3.15, 3.16 and 3.17 in 3.13, we get that $ce'^*le'c = zlz$. Since $l \in K$, by Lemma 3.2, $e'^*le \in K$, so

$$zKz = ce'^*Ke'c \subseteq cKc \tag{3.18}$$

Recall that $z = e'c'e'^* \in K$. By Theorem 3.1, $\exists k \in K$ such that $z = zkz \in zKz$. By equation 3.18, we get that $z \in zKz \subseteq cKc$. But $z \notin bKb \subseteq cKc$, a contradiction. Therefore,

$$e'Ke'^* = 0 \tag{3.19}$$

Therefore, $e'be'^* = 0$. Now we have to consider to two cases depending on eKe'^* whether it is zero or not
If $eKe'^* = 0$, then $(e'be^*)^* = eb^*e'^* = -ebe'^* \in eKe'^*$
substituting in equation (3.11), we get that
$b = ebe^* + ebe'^* - ebe'^* + e'be'^*$
$= ebe^* \in eKe^*$
Therefore, $B = eKe^*$.
Suppose now that $eKe'^* \neq 0$. Then $\exists k \in K$ such that $w = eke'^* \neq 0$. Since
$w^*Kw = (eke'^*)^*K(eke'^*)$
$= e'k^*e^*Keke'^* \subseteq e'Ke'^* = 0$,
By Lemma 3.5 (3), rank $w \leq 1$, so $rank(w) = 0$ or $rank(w) = 1$. Thus, $rank(w) = 1$ (because $w \neq 0$).
Hence, $dimw(V)$ must be one, fix any $v_0 \in V$ such that $w(V) = Fv_0$.
Let $v \in V$ such that

$$w(v) = v_0. \tag{3.20}$$

$V = Im(w) + Ker(w)$
$= Fv + Ker(w)$
Let $w' = ele'^* \in eBe'^*$ be a non-zero transformation. Then
$0 = e'(l(e^*Ke)k + k(e^*Ke)l)e'^*$
$= e'l(e^*Ke)ke'^* + e'k(e^*Ke)le'^*$

$= (e' le^*)K(eke'^*) + (e' ke^*)K(ele'^*)$
$= w'^* Kw + w^* Kw'$
If $u \in Ker(w)$, then
$0 = \psi(0(v), u) = \psi((w'^* Kw + w^* Kw')(v), u)$
$= \psi(w'^* Kw(v), u) + \psi(w^* Kw'(v), u)$
$= \psi(Kw(v), w'(u)) + \psi(Kw'(v), w(u))$
Since $u \in Ker(w)$, so $w(u) = 0$.
$= \psi(Kw(v), w'(u))$
By Lemma 3.5 (1), $w'(u) \in Fv_0$. Now either $w'(u) = 0$ or $w'(u) \neq 0$ for all $u \in Ker(w)$
If $w'(u) = 0$ for all $u \in Ker(w)$, then $Ker(w) \subseteq Ker(w')$
But $dim(w(v)) = dim(w'(v)) = 1$, so $Ker(w) = Ker(w')$
Suppose now that $w'(u) \neq 0$ for some $u \in Ker(w)$, then $Im(w') = Fv_0 \subseteq Im(w)$.
Since both have dimension 1, so $Im(w') = Im(w) = Fv_0$.
Then by Theorem 3.6, B' is a $J-$Lie.
Now, we need to show that $B = B'$, by Theorem 3.7,
$e(V) = eKe^* K(v_0^{\perp})$
and

$$eBe'^*(V) = e(V) \tag{3.21}$$

we claim that $eB'e'^*(V) \subseteq eBe'^*$
we have $B' = eKe^* + \tau(eKf)$
$eB'e'^* = e(eKe^* + \tau(eKf))e'^*$
$= eKe^* e'^* + eKfe'^* - (eKf)^* e'^*$
$= eKfe'^* - f^* Ke^* e'^* = eKfe'^*$
Since $e^* e'^* = 0$. Recall that $fe'^* = f$,
$eB'e^* = eBf$
Let $elf \in eKf$
$elf(v) = elzw(v) = elz(v_0) \in eKK(v_0) = eK(v_0^{\perp}) = e(v_0) \in e(V)$
because $(w(v) = v_0)$. By equation (3.21), $e(V) = eBe'^*$
$elf(v) \in eBe'^*(V)$. Therefore $\exists w' \in eBe'^*$ such that $elf(v) = w'(v)$.
Since $V = Fv_0 + Ker(w)$ and $Ker(f) = Ker(w) = Ker(w')$ fore equation (3.1), and $w' \in eBe'^*$,
for any $elf \in eKf = eB'e'^*$, there exist $w' \in eBe'^*$ such that $elf = w' \in eBe'^*$
Then $eB'e'^* \subseteq eBe'^*$ and $B' \subseteq B$. Therefore $B' = B$.
There exist idempotent $(e + f^*)$ such that $B = (e + f^*)K(e + f^*)^*$.
Now, when $bKb = Fb$, by Theorem 3.8, $B' = B$ is a type one point space.
Suppose that $Im(w') = Im(w) = Fv_0$ for any $w' \in eBe'^*$, we need to show that B is a type one point space
Recall that $wzw = w$,$z = e'^* z' e$
Let
$f = wz = eb'e'^* z' e$
Then
$fe' = eb'e'^* z' e(1 - e) = 0$
and
$e' f = (1 - e)eb'e'^* z' e = 0$
$f^2 = (eb'e'^* z' e)(eb'e'^* z' e) = eb'e'^* z' e = f$
$ef = eeb'e'^* z' e = eb'e'^* z' e = f$ and $fe = eb'e'^* z' ee = f$
Since $rank(f) = 1$, so $rank(f^*) = 1$
Recall that $f^* Kf = fKf^* = 0$. we have $Im(w) = Im(f)$
for any $w' \in eBe'^*$, we have $Im(w) = Im(f) = Im(w')$
we have going to prove that there exist point space $B' = eKe^* + \tau(fKe'^*)$ such that $B = B'$
First, we claim that $fw' = w'$ for any $w' \in eBe'^*$
Let $u \in Ker(w')$, then $fw'(u) = 0$
so $Ker(w') \subseteq Ker(f(w'))$, therefore $Ker(w') = Ker(f(w'))$ (co-dimension 1)
Since $Im(w') = Im(fw')$, so $fw' = w'$ for any $w' \in eBe'^*$.
Second, we claim that
$B \subseteq B' = eKe^* + \tau(fKe'^*)$

take $K \in B$, then
$K = eKe^* + eKe^{'*} + e^{'}Ke^* + e^{'}Ke^{'*}$
$K = eKe^* + \tau(eKe^{'*})$
$K = eKe^* + \tau(w^{'})$
Since $fw^{'} = w^{'}$, so
$K = eKe^* + \tau(feKe^{'*})$
because $fe = f$. For all $K \in B$, we have
$K = eKe^* + \tau(fKe^{'*})$
$B \subseteq B^{'} = eKe^* + \tau(fKe^{'*})$
Now, we claim that B is a point space, that is $bKb \neq Fb$. Then
$eKe^*K(v_0) \subseteq eKe^{'*}(v_0) \subseteq fKe^{'*}(v) = Fv_0$
but
$eKe^{'*}K(v_0) = e(V) \neq Fv_0$
because $ranke(V) > 1$.
Finally, we claim that $B^{'} = eKe^* + \tau(fKe^{'*})$ is point space
By using equation (3.4), and our assume that $bKb = Fb$, we have
$B^{'} = eKe^* + \tau(fKe^{'*})$
$= bxKxb + \tau(fKe^{'*})$
$= bKb + \tau(fKe^{'*}) = Fb + \tau(fKe^{'*})$
for any $c^{'} \in B^{'}, \exists l \in K, \lambda \in F$ such that
$c^{'} = \lambda b + \tau(fle^{'*})$
for all $y \in K$, we have
$c^{'}yc^{'} = (\lambda b + \tau(fle^{'*}))y(\lambda b + \tau(fle^{'*}))$
$= \lambda^2 byb + \lambda by\tau(fle^{'*}) + \lambda\tau(fle^{'*})yb + \tau(fle^{'*})y(fle^{'*})$
$= \lambda^2 byb + \lambda(by)\tau(fle^{'*}) + \lambda\tau(fle^{'*})(by)^* + \tau(fle^{'*})y\tau(fle^{'*})$
By Lemma 3.4 (3),
$= \lambda^2 byb + \lambda\tau(byfle^{'*}) + \lambda\tau(fle^{'*}(by)^*) + \tau(fle^{'*}yfle^{'*})$
$-fle^{'*}y(fle^{'*})^* - (fle^{'*})^*yfle^{'*}$
$= \lambda^2 byb + \lambda\tau(byfle^{'*}) + \lambda\tau(fle^{'*}yb) + \tau(fle^{'*}yfle^{'*})$
$-fle^{'*}ye^{'}l^*f^* - e^{'}l^*f^*yfle^{'*}$
Since $fKf^* = f^*Kf = 0$

$$c^{'}yc^{'} = \lambda^2 byb + \lambda\tau(byfle^{'*}) + \lambda\tau(fle^{'*}yb) + \tau(fle^{'*}yfle^{'*}) \tag{3.22}$$

we need to calculate each term
Since $bKb = Fb$, so

$$byb = \alpha b \tag{3.23}$$

$$\tau(byfle^{'*}) = \tau(byefle^{'*}) = \tau(bybxfle^{'*}) = \tau(\alpha bxfle^{'*})$$

$$= \tau(\alpha(efle^{'*}) = \alpha\tau(efle^{'*}) \tag{3.24}$$

For the third one we have
$fle^{'*}yb = efle^{'*}yb = bxfle^{'*}yb \in bAb$
Since $\tau(a) \in L$ for any $a \in A$, so $\tau(xfle^{'*}y) \in K$,then $b\tau(xfle^{'*}y)b \in bKb \subseteq B$
By Lemma 3.5 (3),

$$\tau(fle^{'*}yb) = \tau(bxfle^{'*}yb) = b\tau(xfle^{'*}y)b = \beta b \tag{3.25}$$

Lastly, we have
$fle^{'*}yfle^{'*} = fle^{'*}yefle^{'*} = fle^{'*}ybxfle^{'*}$
$= (fle^{'*})yb(xfle^{'*})$
Since $f^*Lf = 0$, so $bye^{'}lf^*xfle^{'*} = 0$. Then
$(fle^{'*})y(efle^{'*}) = fle^{'*}ybxfle^{'*} - bye^{'}lf^*xfle^{'*}$
$= (fle^{'*}yb - (fle^{'*}yb)^*)xfle^{'*}$

$= \tau(fle'^{*}yb)xfle'^{*} = \beta b(xfle'^{*})$

$$\Longrightarrow \tau(fle'^{*}yfle'^{*}) = \beta\tau(fle'^{*})$$

Substituting equation 3.23, 3.24, 3.25 and 3.26 in equation 3.22. We get that

$c'yc' = (\lambda^2\alpha)b + (\alpha\lambda)\tau(efle'^{*}) + (\lambda\beta)b + \beta\tau(fle'^{*})$

$= (\lambda^2\alpha + \lambda\beta)b + (\alpha\lambda + \beta)\tau(fle'^{*})$

$= (\lambda\alpha + \beta)(\lambda b + \tau(fle'^{*}))$

$c'yc' = (\lambda\alpha + \beta)c'$

Therefore, $c'Kc' = Fc'$

B' is point space and $B \subseteq B'$ but B is maximal. Therefore

$B = B'$

B is a type one point space

4. CONCLUSION

Every Jordan-Lie inner ideals of the orthogonal Lie algebras is either $B = eKe^*$ or B is a type one point space. one can find an idempotent $e \in A$ such that this inner ideal can be written in the form eKe^*. We study the relationship between these algebras and their corresponding Lie ones. Also study Jordan-Lie inner ideals of these Lie algebras. proved that every Jordan-Lie inner ideal of the orthogonal Lie algebra of an associative algebra (finite dimensional) is generated by an idempotent $e \in A$ with the property $e^*e = 0$.

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