Generalized mean function for n-variable

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ABSTRACT:

In this work we present the theory of an integral mean for generalized GN'-function .We will show under what conditions the mean function is a GN'-function and satisfies a $\Delta-$condition. Moreover, we examine how the minimizing points in the definition of the mean function affect a basic property of the ordinary integral mean.

Keywords :Generalized GN'-function for n-variable,Δ-condition, Generalized mean function.

الملخص:

في هذا العمل نقدم نظرية المتوسط للتكامل للدالة – GN' لnمن المتغيرات. وسنبين الشروط التي تجعلها دالة – GN' والتي تحقق شرط – Δ. وكذلك نبين ان اصغر نقطة في تعريف الدالة القيمة المتوسطة وتأثيرها بالخواص متوسط التكامل الاعتيادي.

1.Introduction and Basic Concept:

In what follows T will denote a space of point with σ-finite measure and E^n a n-dimensional Euclidean space.

Definition1.1 Orlicz(1932)

Orlicz space $\mathcal{L}_{\mathcal{M}} = \mathcal{L}_{\mathcal{M}}(\Omega, \mu)$ is a Banach space consisting of all

$f \in S(\Omega, \mu)$ *where* $S(\Omega, \mu)$ is a ring of all measurable functions on the space with bounded measure space (Ω, μ).

such that

$$\int_{\Omega} M(|f|)d\mu < \infty,$$

With the Luxemburg Nakano norm

$$\|f\|_M = \inf\{\lambda > 0 : \int_{\Omega} M(\frac{|f|}{\lambda})d\mu \le 1\}$$

Orlicz spaces L_m are natural generalization of L_p space, where $L_p(I)$ consists of all the measurable functions f defined on the interval I for which

$$\left(\int_I |f|^p\right)^{\frac{1}{p}} < \infty \qquad \text{Corothers(2000)}$$

They have very rich topological and geometrical structures; they may possess peculiar properties that do not occur in an ordinary L_p space.

Definition 1.2 Borwein (1997)

Let $M : I \to R$ be defined on some interval of the real line R. A function *M* is called convex if

$$M\left(\frac{u_1+u_2}{2}\right) \leq \frac{1}{2}(M(u_1)+M(u_2)) \qquad (1)$$

for all $u_1, u_2 \in I$

we can generalize the inequality (1) for any $u_1, u_2, \ldots, u_n$ by

$$M\left(\frac{u_1+u_2+\ldots.+u_n}{n}\right) \leq \frac{1}{n}(M(u_1)+\ldots.+M(u_n)) \qquad (2)$$

Definition 1.3 Hassen(2007)

Let $M(t,x,y)$ be a real valued non-negative function defined on $T \times E^n \times E^n$ such that:

(i) $M(t,x,y)=0$ if and only if *x,y* are the zero vectors $x, y \in E^n$, $\forall t \in T$

(ii) $M(t,x,y)$ is a continuous convex function of x, y for each *t* and a measurable function of *t* for each x, y

(iii) For each $t \in T$, $\lim\limits_{\substack{\|x\|=\infty \\ \|y\|=\infty}} \frac{M(t,x,y)}{\|x\|\|y\|} = \infty$, and (iv)There are constants $d \geq 0$ and $d_1 \geq 0$ such that

$$\inf_t \inf_{\substack{c \geq d \\ c' \geq d_1}} k(t,c,c') > 0 \qquad (1)$$

Where $k(t,c,c') = \dfrac{\underline{M}(t,c,c')}{\overline{M}(t,c,c')}$,

$\overline{M}(t,c,c') = \sup_{\substack{|x|=c \\ |y|=c'}} M(t,x,y), \underline{M}(t,c,c') = \inf_{\substack{|x|=c \\ |y|=c'}} M(t,x,y)$ and if $d > 0$ and $d_1 > 0$, then $\overline{M}(t,d,d_1)$ is an integrable function of *t*. We call the function satisfying the properties (i)-(iv) a generalized N*-function or a GN*-function.

Definition 1.4:

Let $M(t,x_1,x_2,\ldots,x_n)$ be a real valued non-negative function defined on $T \times \underbrace{E^n \times E^n \times \ \ldots \ \times E^n}_{n-times}$ such that:

(i) $M(t,x_1,x_2,\ldots,x_n) = 0$ if and only if $x_1,x_2,\ldots,x_n$ are the zero vectors $x_1,x_2,\ldots,x_n \in E^n$, $\forall t \in T$

(ii) $M(t,x_1,x_2,\ldots,x_n)$ is a continuous convex function of $x_1,x_2,\ldots,x_n$ for each *t* and a measurable function of *t* for each $x_1,x_2,\ldots,x_n$,

(iii) For each $t \in T$, $\lim_{\substack{\|x_1\|=\infty \\ \|x_2\|=\infty \\ \|x_n\|=\infty}} \dfrac{M(t,x_1,x_2,\ldots,x_n)}{\|x_1\|\|x_2\|\ldots\|x_n\|} = \infty$, and

(iv)There are constants $d_1 \geq 0, d_2 \geq 0, \ldots, d_n \geq 0$ such that

$$\inf_t \inf_{\substack{c_1 \geq d_1 \\ c_2 \geq d_2 \\ c_n \geq d_n}} k(t,c_1,c_2,\ldots,c_n) > 0 \qquad (1)$$

Where

$$k(t,c_1,c_2,\ldots,c_n)=\frac{\underline{M}(t,c_1,c_2,\ldots,c_n)}{\overline{M}(t,c_1,c_2,\ldots,c_n)},$$

$$\overline{M}(t,x_1,c_2,\ldots,c_n)=\sup_{\substack{|x_1|=c_1\\|x_2|=c_2\\|x_n|=c_n}} M(t,x_1,x_2,\ldots,x_n),$$

$$\underline{M}(t,c_1,c_2,\ldots,c_n)=\inf_{\substack{|x_1|=c_1\\|x_2|=c_2\\|x_n|=c_n}} M(t,x_1,x_2,\ldots,x_n)$$

and if $d_1>0$, $d_2>0$,…, $d_n>0$, then $\overline{M}(t,d_1,d_2,\ldots,d_n)$ is an integrable function of *t*. We call the function satisfying the properties (i)-(iv) a generalized N'-function or a GN'-function.

Definition 1.5 Hassen (2010)

We say that a GN*-function *M(t,x,y)*satisfies a $\Delta-$condition if there exists a constant $K\geq 2$ and non-negative measurable functions $\delta_1(t)$ and $\delta_2(t)$ such that the function $M(t,2\delta_1(t),2\delta_2(t))$ is integrable over the domain *T* and such that for almost all *t* in *T* we have

$$M(t,2x,2y)\leq KM(t,x,y) \qquad (1)$$

for all *x* and *y* satisfying $|x|\geq\delta_1(t)$ and $|y|\geq\delta_2(t)$.

We say that a GN*-function satisfies a Δ_0-condition if it satisfies a $\Delta-$ condition with $\delta_1(t)=0$ and $\delta_2(t)=0$ for almost all *t* in *T*.

In Definition 1.5 we could have used any constant $\tau>1$ in place of the scalar 2 in (1).

Definition 1.6:

We say that a GN'-function $M(t,x_1,x_2,\ldots,x_n)$ satisfies a $\Delta-$condition if there exists a constant $K\geq 2$ and non-negative measurable functions $\delta_1(t)$, $\delta_2(t)$,…,

$\delta_n(t)$ such that the function $M(t,2\delta_1(t),2\delta_2(t),\ldots,2\delta_n(t))$ is integrable over the domain *T* and such that for almost all *t* in *T* we have

$$M(t,2x_1,2x_2,\ldots,2x_n) \le KM(t,x_1,x_2,\ldots,x_n) \qquad (1)$$

for all $x_1, x_2,\ldots,x_n$ satisfying $|x_1| \ge \delta_1(t)$, $|x_2| \ge \delta_2(t),\ldots,|x_n| \ge \delta_n(t)$.

Thus, according to this definition, the statement above can be formulated as:

We say that a GN'-function satisfies a Δ_0-condition if it satisfies a $\Delta-$condition with $\delta_1(t)=0$, $\delta_2(t)=0,\ldots,\delta_n(t)=0$ for almost all *t* in *T*.

In Definition1.6we could have used any constant $\tau > 1$ in place of the scalar 2 in (1).

Definition 1.7 Hassen (2007)

For each *t* in *T* and *h>0* let

$$M_h(t,x,y) = \int_{E^n}\int_{E^n} M(t,x+z,y+w)J_h(z)J_h(w)dzdw,$$

where $J_h(z)$ and $J_h(w)$ are no negative, c^{∞} function with compact

support in a ball of a radius *h* such that $\int_{E^n}\int_{E^n} J_h(z)J_h(w)dtdt = 1$.

Moreover, let x_0 and y_0 are any tow points (depending on *h*, *t*) which satisfy the inequality

$$M_h(t,x_0,y_0) \le M_h(t,x,y)$$

for all x and y in E^n . Then the function $\hat{M}_h(t,x,y)$ defined for each *t* in *T* and $h>0$ by

$$\hat{M}_h(t,x,y) = M_h(t,x+x_0,y+y_0) - M_h(t,x_0,y_0)$$

is called a **mean function** for $M(t,x,y)$ relative to the minimizing point x_0 and y_0 .

Definition 1.8:

For each *t* in *T* and *h>0* let

$$M_h(t,x_1,x_2,\ldots,x_n)=$$

$$\int_{E^n}\int_{E^n}\ldots\int_{E^n} M(t,x_1+y_1,x_2+y_2,\ldots,x_n+y_n)J_h(y_1)J_h(y_2)..J_h(y_n)dy_1dy_2..dy_n$$

where $J_h(y_1)$, $J_h(y_2)$,...., $J_h(y_n)$ are no negative, c^{∞} function with compact support in a ball of a radius *h* such that

$\int_{E^n}\int_{E^n}\ldots\int_{E^n} J_h(y_1)J_h(y_2)..J_h(y_n)dtdt..dt=1$`. Moreover, let x_{01}, $x_{02},\ldots,x_{0n}$ are any points (depending on *h, t*) which satisfy the inequality

$$M_h(t,x_{01},x_{02},\ldots,x_{0n})\le M_h(t,x_1,x_2,\ldots,x_n)$$

for all $x_1,x_2,\ldots,x_n$ in E^n . Then the function $\hat{M}_h(t,x_1,x_2,\ldots,x_n)$ defined for each *t* in *T* and $h>0$ by

$$\hat{M}_h(t,x_1,x_2,\ldots,x_n)=M_h(t,x_1+x_{01},x_2+x_{02},x_n+x_{0n})-M_h(t,x_{01},x_{02},\ldots,x_{0n})$$

is called a **mean function** for $M(t,x_1,x_2,\ldots,x_n)$ relative to the minimizing points $x_{01},x_{02},\ldots,x_{0n}$.

Theorem 1.9 Hassen (2007)

If $M(t,x,y)$ is a GN*-function for which $\overline{M}(t,c,c')$ is integrable in *t* for each *c* and *c'*, then $\hat{M}_h(t,x,y)$ is a GN*-function.

Theorem 1.10 Hassen (2007)

If $M(t,x,y)$ is a GN*-function satisfying a $\Delta-$condition and for which $\overline{M}(t,c,c')$ is integrable in *t* for each *c* and c' , then $\hat{M}_h(t,x,y)$

satisfies a $\Delta-$condition.

Theorem 1.11 Hassen (2007)

For each $h>0$ let x_0^h and y_0^h be the minimizing point of $M_h(t,x,y)$

defining $\hat{M}_h(t,x,y)$.Then for each t in T and each x,y in E^n,there exists $K(t,x,y)$ such that

$$\lim_{h=0} \hat{M}_h(t,x,y) = M(t,x,y) + K(t,x,y) \lim_{h=0} \left|x_0^h\right| \lim_{h=0} \left|y_0^h\right|$$

Corollary 1.12 Hassen (2007)

Suppose $M(t,x,y)$ is a GN*-function such that $M(t,x,y) = M(t,-x,-y)$.

Then for each *t* in *T* and x,y in E^n,we have

$$\lim_{h=0} M_h(t,x,y) = \hat{M}(t,x,y)$$

Theorem 1.13 Hassen (2007)

The sets *B* and A_h are closed convex sets.

Theorem 1.14 Hassen (2007)

Let $B_e = \{(x,y) : M(t,x,y) < e\}$ for each *t* in *T*. Then for given any *e>0,*

there is a constant $h_0 > 0.$ such that $A_h \subset B_e$ for each $h \le h_0$.

Theorem 1.15 Hassen (2007)

Suppose *M(t,x,y)* is a GN*-function which is strictly convex in *x* and *y*

for each *t*. Then $h, A_h = \{(0,0)\}$ for each *h*.

Theorem 1.16 Hassen (2013)

A necessary and sufficient condition that (1.5.1) holds is that if

$\left|x_1\right| \le \left|y_1\right|, \left|x_2\right| \le \left|y_2\right|, \ldots, \left|x_n\right| \le \left|y_n\right|$ then there exists constants $K \ge 1, d_1 \ge 0, d_2 \ge 0$

,

$\ldots, d_n \geq 0$ such that $M(t, x_1, x_2, \ldots, x_n) \leq KM(t, y_1, y_2, \ldots, y_n)$ for each *t* in *T*, $|x_1| \geq d_1, |x_1| \geq d_1, \ldots, |x_n| \geq d_n$

Theorem 1.17 Hassen (2010)

A GN'-function $M(t, x_1, x_2, \ldots, x_n)$ satisfies a $\Delta-$condition if and only if given any $\tau > 1$ there exists a constant $K_\tau \geq 2$ and a non-negative measurable functions $\delta_1(t), \delta_2(t), \ldots, \delta_n(t)$ such that $\overline{M}(t, 2\delta_1(t), 2\delta_2(t), \ldots, 2\delta_n(t))$ is integrable over *T* and such that for almost all *t* in *T* we have

$$M(t, \tau x_1, \tau x_2, \ldots, \tau x_n) \leq K_\tau M(t, x_1, x_2, \ldots, x_n), \qquad (1)$$

whenever $|x_1| \geq \delta_1(t)$, $|x_2| \geq \delta_2(t), \ldots, |x_n| \geq \delta_n(t)$.

2. Generalized mean function:

Theorem 2.1:

If $M(t, x_1, x_2, \ldots, x_n)$ is a GN'-function for which $\overline{M}(t, c_1, c_2, \ldots, c_n)$ is integrable in *t* for each $c_1, c_2, \ldots, c_n$, then $\hat{M}_h(t, x_1, x_2, \ldots, x_n)$ is a GN'-function.

<u>Proof:</u>

We will show this result by justifying conditions (i)-(iv) of the definition 3.1.1. By hypothesis and the choice of $x_{01}, x_{02}, \ldots, x_{0n}$, we have for each *h*, $\hat{M}_h(t, x_1, x_2, \ldots, x_n) \geq 0$ and $\hat{M}_h(t, 0, 0, \ldots, 0) = 0$. On the other hand, if $x_1 \neq 0, x_2 \neq 0, \ldots, x_n \neq 0$, then $M(t, x_1, x_2, \ldots, x_n) > 0$, and hence there are constants $h_{01}, h_{02}, \ldots, h_{0n}$ such that

$$a = \inf_{\substack{|w_i| \leq h_{0i} \\ 1 \leq i \leq n}} M(t, x_1 + w_1, x_2 + w_2, \ldots, x_n + w_n) > 0$$

However, since $M(t,x_1,x_2,\dots,x_n)=0$ if and only if $x_1=0,x_2=0,\dots,x_n=0$, the minimizing points x_{01} tends to zero , x_{02} tends to zero,…, x_{0n} tends to zero as *h* tends to zero. Therefore, we can choose $g_{01}\le h_{01}, g_{02}\le h_{02},\dots, g_{0n}\le h_{0n}$ such that if $h\le g_{01}, h\le g_{02},\dots, h\le g_{0n}$ then $M(t,x_{01}+y_{01},x_{02}+y_{02},\dots,x_{0n}+y_{0n})<a$ for all $y_{01},y_{02},\dots,y_{0n}$ for which $\left|x_{01}+y_{01}\right|<h$, $\left|x_{02}+y_{02}\right|<h,\dots$, $\left|x_{0n}+y_{0n}\right|<h$ for this $g_{01},g_{02},\dots,g_{0n}$ we obtain the inequality

$$M(t,x_1+x_{01}+y_{01},x_2+x_{02}+y_{02},\dots,x_n+x_{0n}+y_{0n})\ge$$

$$\inf_{\substack{|w_1|\le g_{01}\\ 1\le i\le n}} M(t,x_1+w_1,x_2+w_2,\dots,x_n+w_n)\ge a$$

$$>M(t,x_{01}+y_{01},x_{02}+y_{02},\dots,x_{0n}+y_{0n})$$

whenever $|x_{01}+y_{01}|\le g_{01}, |x_{02}+y_{02}|\le g_{02},\dots,|x_{0n}+y_{0n}|\le g_{0n}$.This means for some $h\le g_{01}, h\le g_{02},\dots, h\le g_{0n}$ we have

$$M(t,x_1+x_{01}+y_{01},x_2+x_{02}+y_{02},\dots,x_n+x_{0n}+y_{0n})>$$

$$M(t,x_{01}+y_{01},x_{02}+y_{02},\dots,x_{0n}+y_{0n})$$

$$M_h(t,,x_1+x_{01},x_2+x_{02},\dots,x_n+x_{0n})>M_h(t,x_{01},x_{02},\dots,x_{0n})$$

or $\hat{M}_h(t,x_1,x_2,\dots,x_n)>0$ if $x_1\ne 0,x_2\ne 0,\dots,x_n\ne 0$ which proves property (i).

Properties (ii) and (iii) for $\hat{M}_h(t,x_1,x_2,\dots,x_n)$ follow easily from the same properties for $M(t,x_1,x_2,\dots,x_n)$. Let us now show (iv). By assumption, there are constants $d_1\ge 0$, $d_2\ge 0,\dots,d_n\ge 0$ such that

$$\tau(t)\overline{M}(t,c_1,c_2,\dots,c_n)\le \underline{M}(t,c_1,c_2,\dots,c_n) \qquad (1)$$

for all $c_1 \geq d_1, c_2 \geq d_2, \ldots, c_n \geq d_n$. Furthermore, it is not difficult to show that for all c and c' we have

$$\overline{M}(t, c_1, c_2, \ldots, c_n) \geq \sup_{\substack{|x_i| \leq c_i \\ 1 \leq i \leq n}} M(t, x_1, x_2, \ldots, x_n) \qquad (2)$$

and for some fixed $w_1, w_2, \ldots, w_n$

$$\inf_{\substack{|x_i| \geq c_i \\ 1 \leq i \leq n}} M(t, x_1 + w_1, x_2 + w_2, \ldots, x_n + w_n) \leq \inf_{\substack{|x_i| = c_i \\ 1 \leq i \leq n}} M(t, x_1 + w_1, x_2 + w_2, \ldots, x_n + w_n)$$

(3)

By using (2), we obtain (for each t in T)that

$$\tau(t) \sup_{\substack{|x_i| = c_i \\ 1 \leq i \leq n}} M(t, w_1, w_2, \ldots, w_n) \leq \tau(t) \sup_{\substack{|r_i| < c_i + |x_{0i} + x_{1i}| \\ 1 \leq i \leq n}} M(t, r_1, r_2, \ldots, r_n) \qquad (4)$$

$$\leq \tau(t) \sup_{\substack{|r_i| = c_i + |x_{0i} + x_{1i}| \\ 1 \leq i \leq n}} M(t, r_1, r_2, \ldots, r_n)$$

where $w_i = x_i + x_{0i} + r_i$ for i=1 to n. On the other hand, by (1) and (3), we achieve

$$\tau(t) \sup_{\substack{|w_i| = c_i + |x_{0i} + x_{i1}| \\ 1 \leq i \leq n}} M(t, w_1, w_2, \ldots, w_n) \leq \inf_{\substack{|w_i| = c_i + |x_{0i} + x_{i1}| \\ 1 \leq i \leq n}} M(t, w_1, w_2, \ldots, w_n) \qquad (5)$$

$$< \inf_{\substack{|x_i| \geq c_i \\ 1 \leq i \leq n}} M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n).$$

$$< \inf_{\substack{|x_i| = c_i \\ 1 \leq i \leq n}} M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n).$$

If we combine (4) and (5), then for all $c_i \geq d_i$ for i=1 to n and we arrive at

$$\tau(t) \sup_{\substack{|x_i| = c_i \\ 1 \leq i \leq n}} M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n) \leq$$

$$\inf_{\substack{|x_i| = c_i \\ 1 \leq i \leq n}} M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n)$$

From this inequality, we obtain

$$\inf_{\substack{|x_i| = c_i \\ 1 \leq i \leq n}} \hat{M}_h(t, x_1, x_2, \ldots, x_n) \geq \int_{E^n} \int_{E^n} \ldots \int_{E^n} \inf_{\substack{|x_i| = c_i \\ 1 \leq i \leq n}} \{M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n)$$

$$- M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n)\} J_h(r_1) J_h(r_2) \ldots J_h(r_n) dr_1 dr_2 \ldots dr_n$$

$$\geq \int_{E^n} \int_{E^n} \int_{E^n} \{\tau(t) \sup_{\substack{|x_i| = c_i \\ 1 \leq i \leq n}} M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n) \qquad (6)$$

$$- M(t, x_{01} + r_1, x_{02} + r_2, \ldots, x_{0n} + r_n) J_h(r_1) J_h(r_2) \ldots J_h(r_n)\} dr_1 dr_2 \ldots dr_n ,$$

and

$$\sup_{\substack{|x_i| = c_i \\ 1 \leq i \leq n}} \hat{M}_h(t, x_1, x_2, \ldots, x_n) \leq \int_{E^n} \ldots \int_{E^n} \int_{E^n} \sup_{\substack{|x_i| = c_i \\ 1 \leq i \leq n}} M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots,$$

$$x_n + x_{0n} + r_n) J_h(r_1) J_h(r_2) \ldots J_h(r_n) dr_1 dr_2 \ldots dr_n . \qquad (7)$$

Moreover, since $\lim_{\substack{c_i = \infty \\ 1 \leq i \leq n}} \sup_{\substack{|x_i| = c_i \\ 1 \leq i \leq n}} M(t, x_1 + x_{01} + r_1, x_2 + x_{02} + r_2, \ldots, x_n + x_{0n} + r_n) = \infty$

for fixed x_{0i}, r_i for $1 \leq i \leq n$ such that $|r_i| \leq h_i$ for $1 \leq i \leq n$ given

$$K_1(t) = 2\sup_{\substack{|r_i|\le h_i \\ 1\le i\le n}} M(t, x_{01}+r_1, x_{02}+r_2, \ldots, x_{0n}+r_n) / \inf_t \tau(t)$$

there are $d_i > 0, 1 \le i \le n$ such that if $c_i \ge d_i', 1 \le i \le n$,then

$$\sup_{\substack{|x_i|=c_i \\ 1\le i\le n}} M(t, x_1+x_{01}+r_1, x_2+x_{02}+r_2, ..., x_n+x_{0n}+r_n) \ge K_1.$$

Therefore, by using (3.3.8) and (3.3.9), we achieve the inequalities

$$\frac{\inf_{\substack{|x_i|=c_i \\ 1\le i\le n}} \hat{M}_h(t, x_1, x_2, \ldots, x_n)}{\sup_{\substack{|x_i|=c_i \\ 1\le i\le n}} \hat{M}_h(t, x_1, x_2, \ldots, x_n)} \ge \tau(t) -$$

$$\frac{\sup_{\substack{|r_i|\le h_i \\ 1\le i\le n}} M(t, x_1+x_{01}+r_1, x_2+x_{02}+r_2, \ldots, x_n+x_{0n}+r_n)}{\inf_{\substack{|r_i|\le h_i \\ 1\le i\le n}} \sup_{\substack{|x_i|=c_i \\ 1\le i\le n}} M(t, x_1+x_{01}+r_1, x_2+x_{02}+r_2, \ldots, x_n+x_{0n}+r_n)} \ge \tau(t) - \frac{1}{2} \inf_t \tau(t)$$

(8)

for all $c_i \ge d_{0i} = \max(d_i, d_i', |x_{0i}|)$. Taking the infimum of both sides of (2.1.8) over *t*, shows the first part of the property (iv). To show the latter part, assume $d_{0i} > 0, 1 \le i \le n$ and . Then $\sup_{\substack{|x_i|=d_{0i} \\ 1\le i\le n}} \hat{M}_h(t, x_1, x_2, \ldots, x_n)$ is integrable over *t* in *T* since it is bounded by the integrable function $\overline{M}(t, d_{21}, d_{22}, \ldots, d_{2n})$ where $d_{2i} = d_{0i} + |x_{0i}| + h$ .This proves property (iv) and

the theorem.■

In the next theorem we show under what condition $\hat{M}_h(t, x_1, x_2, \ldots, x_n)$ satisfies a $\Delta-$ condition.

Theorem 2.2:

If $M(t,x_1,x_2,\ldots,x_n)$ is a GN'-function satisfying a $\Delta-$condition and for which $\overline{M}(t,c_1,c_2,\ldots,c_n)$ is integrable in *t* for each $c_1,c_2,\ldots,c_n$ then $\hat{M}_h(t,x_1,x_2,\ldots,x_n)$ satisfies a $\Delta-$condition.

Proof:

It suffices to show that $M_h(t,x_1,x_2,\ldots,x_n)$ satisfies a $\Delta-$condition.

For, $\hat{M}_h(t,x_1,x_2,\ldots,x_n)$ is the sum of a constant and a translation of $M_h(t,x_1,x_2,\ldots,x_n)$ and neither of these operations affects the growth condition. Let us observe first that if $|x_i|\geq 2$ for $1\leq i\leq n, |z_i|\leq h\leq 1$ for $1\leq i\leq n$ then $|2x_i+z_i|\leq 3|x_i+z_i|$ for $1\leq i\leq n$. Hence, by Theorem (1.16), there are constants $K\geq 1$ and $d_1\geq 0$ such that

$$M_h(t,2x_1,2x_2,\ldots,2x_n)\leq k\int_{E^n}\int\ldots\int_{E^n} M(t,3(x_1+z_1),3(x_2+z_2),\ldots,3(x_n+z_n))$$
$$K_3M_h(t,x_1,x_2,\ldots,x_n)J_h(z_1)J_h(z_2)\ldots J_h(z_n)dz_1dz_2\ldots dz_n$$

for all x_i for $1\leq i\leq n$ such that $|x_i|\geq d_2$ for $1\leq i\leq n$ and $d_2=\max(d_1,2)$.On the other hand ,by theorem (1.17),

$$\int_{E^n}\int_{E^n}\ldots\int_{E^n} M(t,3(x_1+z_1),3(x_2+z_2),\ldots,3(x_n+z_n))J_h(z_1)J_h(z_2)\ldots J_h(z_n)dz_1dz_2\ldots dz_n\leq$$
$$K_3M_h(t,x_1,x_2,\ldots,x_n)$$

there is a constant $K_3\geq 2$, $\delta_i(t)\geq 0$ for $1\leq i\leq n$ such that for almost all *t* in *T*for all x_i,z_i for $1\leq i\leq n$such that $|x_i+z_i|\geq\delta_i(t)$ for $1\leq i\leq n$ where $|z_i|\leq h_i$ for $1\leq i\leq n$ .By combining the above two inequalities, we achieve

$$M_h(t,2x_1,2x_2,\ldots,2x_n)\leq KK_3M_h(t,x_1,x_2,\ldots,x_n)$$

for all $|x_i| > \max(d_{2i}, \delta_i(t)+h) = \delta_i'(t)$ Since $\overline{M}(t, 2\delta_1(t), 2\delta_2(t), \ldots, 2\delta_n(t))$ is integrable over T ,this yields the integrability of $\overline{M}_h(t, 2\delta_1'(t), 2\delta_2'(t), \ldots, 2\delta_n'(t))$ which proves the theorem.■

For each t in T and $x_1, x_2, \ldots, x_n$ in E^n it is known that

$$\lim_{h=0} M_h(t, x_1, x_2, \ldots, x_n) = M(t, x_1, x_2, \ldots, x_n).$$

However, the same property does not hold in general for $\hat{M}_h(t, x_1, x_2, \ldots, x_n)$. This is the point of the next theorem.

Theorem 2.3:

For each $h > 0$ let x_{0i}^h for $1 \le i \le n$ be the minimizing point of $M_h(t, x_1, x_2, \ldots, x_n)$ defining $\hat{M}_h(t, x_1, x_2, \ldots, x_n)$ .Then for each t in T and each x_i for $1 \le i \le n$ in E^n ,there exists $K(t, x_1, x_2, \ldots, x_n)$ such that

$$\lim_{h=0} \hat{M}_h(t, x_1, x_2, \ldots, x_n) = M(t, x_1, x_2, \ldots, x_n) + K(t, x_1, x_2, \ldots, x_n) \prod_{i=1}^{n} \lim_{h=0} |x_{0i}^h|$$

Proof:

By the definition of $\hat{M}_h(t, x_1, x_2, \ldots, x_n)$ we can write

$$\left|\hat{M}_h(t, x_1, x_2, \ldots, x_n) - M(t, x_1, x_2, \ldots, x_n)\right| \le$$

$$\int_{E^n}\int_{E^n}\cdots\int_{E^n} \Big| M(t, x_1 + x_{01}^h + z_1, x_2 + x_{02}^h + z_2, \ldots, x_n + x_{0n}^h + z_n) -$$
$$M(t, x_{01}^h + z_1, x_{02}^h + z_2, \ldots, x_{0n}^h + z_n) - M(t, x_1, x_2, \ldots, x_n) \Big| J_h(z_1) J_h(z_2) \ldots J_h(z_n) dz_1 dz_2 \ldots dz_n$$
(1)

However, we know that

$$\left| \begin{array}{l} M(t, x_1 + x_{01}^h + z_1, x_2 + x_{02}^h + z_2, \ldots, x_n + x_{0n}^h + z_n) - \\ M(t, x_{01}^h + z_1, x_{02}^h + z_2, \ldots, x_{0n}^h + z_n) - M(t, x_1, x_2, \ldots, x_n) \end{array} \right| \qquad (2)$$

$$\leq \left| M(t, x_1 + x_{01}^h + z_1, x_2 + x_{02}^h + z_2, \ldots, x_n + x_{0n}^h + z_n) - M(t, x_1, x_2, \ldots, x_n) \right|$$

$$+ \left| M(t, x_{01}^h + z_1, x_{02}^h + z_2, \ldots, x_{0n}^h + z_n) - M(t, z_1, z_2, \ldots, z_n) \right| + \left| M(t, z_1, z_2, \ldots, z_n) \right|.$$

Moreover , since $M(t, x_1, x_2, \ldots, x_n)$ is a convex function, it satisfies a Lipshitz condition on compact subsets of E^n (see[Skaff (1968),Th.5.1]).Therefore ,there exists $K_1(t, x_1, x_2, \ldots, x_n)$ and $K_2(t, x_1, x_2, \ldots, x_n)$ such that

$$\left| M(t, x_1 + x_{01}^h + z_1, x_2 + x_{02}^h + z_2, \ldots, x_n + x_{0n}^h + z_n) - M(t, x_1, x_2, \ldots, x_n) \right| \leq$$
$$K_1(t, x_1, x_2, \ldots, x_n) \left| x_{01}^h + z_1 \right| \left| x_{02}^h + z_2 \right| \ldots \left| x_{0n}^h + z_n \right|.$$

(3)

and

$$\left| M(t, x_{01}^h + z_1, x_{02}^h + z_2, \ldots, x_{0n}^h + z_n) - M(t, z_1, z_2, \ldots, z_n) \right| \leq K_2(t, x_1, x_2, \ldots, x_n) \left| x_{01}^h \right| \left| x_{02}^h \right| \ldots \left| x_{0n}^h \right|$$

. (4)

If we combine (3) and (4) with (2) and if we substitute the resulting expression into (1), we achieve the inequality

$$\left| \hat{M}_h(t, x_1, x_2, \ldots, x_n) - M(t, x_1, x_2, \ldots, x_n) \right| \leq \prod_{i=1}^{n} \left| x_{0i}^h \right| (K_1(t, x_1, x_2, \ldots, x_n) + K_2(t, x_1, x_2, \ldots, x_n)) +$$
$$\sum_{i=1}^{n} \int_{E^n} \int_{E^n} \ldots \int_{E^n} \left| x_{0i}^h \right| K_1(t, x_1, x_2, \ldots, x_n) \left| z_i \right| J_h(z_1) J_h(z_2) \ldots J_h(z_n) dz_1 dz_2 \ldots dz_n +$$
$$\int_{E^n} \int_{E^n} \ldots \int_{E^n} K(t, x_1, x_2, \ldots, x_n) \prod_{i=1}^{n} \left| z_i \right| J_h(z_1) J_h(z_2) \ldots J_h(z_n) dz_1 dz_2 \ldots dz_n +$$
$$\int_{E^n} \int_{E^n} \ldots \int_{E^n} M(t, z_1, z_2, \ldots, z_n) J_h(z_1) J_h(z_2) \ldots J_h(z_n) dz_1 dz_2 \ldots dz_n$$

Since the last four integrals on the right side tend to zero as *h* tends to zero, we prove the theorem by setting

$$K(t,x_1,x_2,\ldots,x_n)=K_1(t,x_1,x_2,\ldots,x_n)+K_2(t,x_1,x_2,\ldots,x_n)$$

Corollary 2.4:

Suppose $M(t,x_1,x_2,\ldots,x_n)$ is a GN'-function such that

$$M(t,x_1,x_2,\ldots,x_n)=M(t,-x_1,-x_2,\ldots,-x_n).$$

Then for each *t* in *T* and x_i in E^n for *i*=1to *n* ,we have

$$\lim_{h=0} M_h(t,x_1,x_2,\ldots,x_n)=\hat{M}(t,x_1,x_2,\ldots,x_n)$$

<u>Proof:</u>

This result is clear since $\lim\limits_{h=0}\left|x_{0i}^h\right|=0$ for *i*=1 to *n*

if $M((t,x_1,x_2,\ldots,x_n)=M(t,-x_1,-x_2,\ldots,-x_n)$.In fact, if $M(t,x_1,x_2,\ldots,x_n)$ is even in $(x_1,x_2,\ldots,x_n)$ then the $x_{0i}^h=0$ for *i*=1 to *n* for all *h*.■

For each *t* in *T* let A_h denote the set of minimizing points of $M_h(t,x_1,x_2,\ldots,x_n)$ and let *B* represents the null space of $M(t,x_1,x_2,\ldots,x_n)$ relative to points in $E^n\times E^n\times\ldots\times E^n$, i.e.,

$$B=\{(x_1,x_2,\ldots,x_n)\quad in\quad E^n\times E^n\times\ldots\times E^n : M(t,x_1,x_2,\ldots,x_n)=0\}.$$

If $M(t,x_1,x_2,\ldots,x_n)$ is a GN'-function, then *B*={(0,0,…,0)}. For the sake of argument, let us suppose that $M(t,x_1,x_2,\ldots,x_n)$ has all the properties of a GN'-function except that $M(t,x_1,x_2,\ldots,x_n)=0$ need not imply $x_i=0$ for $i=1$to$n$.

We will show the relationships that exist between A_h and B. This is the content of the next few theorems.

Theorem 2.5:

The sets B and A_h are closed convex sets.

Proof:

This result follows from the convexity and continuity of $M(t,x_1,x_2,\ldots,x_n)$ in x_i for i=1to n for each t in T.■

Theorem 2.6:

Let $B_e=\{(x_1,x_2,\ldots,x_n): M(t,x_1,x_2,\ldots,x_n)<e\}$ for each t in T. Then given any $e>0$,

there is a constant $h_0>0.$ such that $A_h\subset B_e$ for each $h\leq h_0$.

Proof:

Since $B\subseteq B_e$, we can choose h_0 sufficiently small so that if $(x_1,x_2,\ldots,x_n)$ is in B then $(x_1+z_1,x_2+z_2,\ldots,x_n+z_n)$ is in B_e for all $(z_1,z_2,\ldots,z_n)$ such that $|z_i|\leq h_0$ for i=1to n . Let $(z_{01},z_{02},\ldots,z_{0n})$ be arbitrary but fixed points in $A_h, h\leq h_0$. Then

$M_h(t,z_{01},z_{02},\ldots,z_{0n})\leq M_h(t,x_1,x_2,\ldots,x_n)$ for all x_i for $i=1$ to n . Therefore, if $(x_1,x_2,\ldots,x_n)$ in B, we have $M_h(t,z_{01},z_{02},\ldots,z_{0n})<e$ by our choice of h_0. Letting h tend to zero yields $M(t,z_{01},z_{02},\ldots,z_{0n})<e$, i.e.,$(z_{01},z_{02},\ldots,z_{0n})$ in B_e.

We have commented above that $A_h=\{(0,0,\ldots,0)\}$

$$M(t,x_1,x_2,\dots,x_n)=M(t,-x_1,-x_2,\dots,-x_n).$$

It is also true if $M(t,x_1,x_2,\dots,x_n)$ is strictly convex in *x* for each *t* in *T*.

Theorem 2.7:

Suppose $M(t,x_1,x_2,\dots,x_n)$ *is a GN*-function which is strictly convex in* x_i for $i=1$ to n *for each t. Then* $h, A_h=\{(0,0,\dots,0)\}$ for each *h*.

Proof:

Suppose that there exists $z_{0i}\neq x_{0i}$ for $1\le i\le n$ such that z_{0i}, x_{0i} for $1\le i\le n$ are in A_h. Let $z_i=\dfrac{(x_{0i}+z_{0i})}{2}$ for $1\le i\le n$. Then, since $M(t,x_1,x_2,\dots,x_n)$ is strictly convex, $M_h(t,x_1,x_2,\dots,x_n)$ is strictly convex in $x_1,x_2,\dots,x_n$, therefore, we have

$$M_h(t,z_1,z_2,\dots,z_n)<\frac{1}{2}M_h(t,x_{01},x_{02},\dots,x_{0n})+\frac{1}{2}M_h(t,z_{01},z_{02},\dots,z_{0n}). \qquad (1)$$

However, $(x_{01},x_{02},\dots,x_{0n}),(z_{01},z_{02},\dots,z_{0n})$ are in A_h reduces (1) to the inequality $M_h(t,z_1,z_2,\dots,z_n)<M_h(t,x_1,x_2,\dots,x_n)$ for all x_i for $i=1$ to n. This means $z_1,z_2,\dots$ and z_n are in A_h and are $(x_{01},x_{02},\dots,x_{0n}),(z_{01},z_{02},\dots,z_{0n})$ not in A_h which is a contradiction. Hence, $x_{0i}=z_{0i}$ for $i=1$ to n .Since $M(t,x_1,x_2,\dots,x_n)$ is a GN'-function, $B=\{(0,0,\dots,0)\}$. In this case $x_{0i}=z_{0i}=0$ for $i=1$ to n.

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