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New Types of Ideals in Q-algebra

انواع جديدة من المثاليات في جبر- Q

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Abstract
In this paper , we introduce the notions of Complete Ideal, K-Ideal, Complete K-Ideal, Closed Complete K-Ideal in bounded Q-algebra, and as well some suggestions and relationships among them are debated .

المستخلص
في هذا البحث قدمنا المفاهيم لمكمل المثالي ، مثالي-K ، مكمل مثالي-K و مكمل مثالي-K المغلق في جبر-Qالمقيد ، و كذلك بعض الخواص والعلاقات التي تناقش فيما بينها .

1. Introduction

Neggers. J, Ahn. S. S , Kim H. S at (2001) introduced the class of Q-algebras, [1]. Abdullah. H. K, Hasan. Z. A at (2016) introduced of complete BCK-ideal in BCK-algebra, [2]. Abdullah. H. K, Radhi. K. T at (2016) introduced the connotation of T- filter in BCK-algebra, [3]. Abdullah. H. K, Atshan. A. A at (2017) introduced the notation of B-algebra, complete ideal and n-ideal , [4]. The aim of this paper is to introduce new types of ideals and also some of theorems which explain relationships with ideal in bounded Q-algebra.

2. Basic Concepts and Notations

In this part, we introduce definitions of Q-algebra , bounded, ideal, Q-subalgebra, Q-ideal, and some of their properties.

Definition (2.1) [1]:

Let $(U,*,0)$ be a set with a binary operation $*$ and a constant 0. Then $(U,*,0)$ is called a Q-algebra if it satisfies the following axioms :

1. $u*u=0$
2. $u*0=u$
3. $(u*v)*w=(u*w)*v \ ,\forall\, u,v,w \in U$.

We can define a binary relation $\leq$ on U by putting $u \leq v$ if and only if $u*v=0$, for all $u,v \in U$.

Example (2.2) [1]:

Let Z and R be the set of all integers and real numbers, respectively. Then$(Z,-,0)$
and $(R,\div,1)$ are Q-algebras where $"-"$ is the usual subtraction and $"\div"$ is the usual division.

Proposition (2.3) [1]:
In any Q-algebra $(U,*,0)$, the following hold :
1. $\big(u*(u*v)\big)*v=0$
2. $\Big((u*w)*\big((u*w)*v\big)\Big)*v=0$, for any $u,v,w\in U$.

Remark (2.4) :
Let $(U,*,0)$ be a Q-algebra , then
1. If $u\leq 0$, $\forall u\in U$, then U contains only 0.
2. If $u\leq v$ implies $u*(u*(u*v))=0$, for any $u,v\in U$
3. If $u\leq v$ and $u*w\leq v$, then $0\leq w$, for any $u,v,w\in U$

Proof:
1. Since $u\leq 0$, then $u*0=0$, $\forall u\in U$. Thus $u=0$, $\forall u\in U$, i.e , U contains only 0.
2. Since $u\leq v$, then $u*v=0$, $\forall u,v\in U$.
Thus $u*\big(u*(u*v)\big)=u*(u*0)=u*u=0.$
3. Since $u\leq v$ and $u*w\leq v$, then $u*v=0$ and $(u*w)*v=0$, $\forall u,v,w\in U$. Thus $0=(u*w)*v=(u*v)*w=0*w$. So $0\leq w$.

Definition (2.5):
If $(U,*,0)$ is a Q-algebra, we call U is bounded if there is an element $e\in U$ satisfying $u\leq e$ for all $u\in U$, then e is called an unit of U .

In bounded Q-algebra U, we denoted $e*u$ by u^* for every $u\in U$.

Example (2.6):
Let $U=\{0,a,b,c\}$ be a set with the following table

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	0	0	0
c	c	c	c	0

Therefore $(U,*,0)$ is Q-algebra, [1] . Notice that U is bounded with unit c.

Remark (2.7):
The unit in bounded Q-algebra not to be an unique and the following example shows this .

Example (2.8) :
Consider the following Q-algebra U , [1] with the following table

$*$	0	a	b
0	0	0	0
a	a	0	0
b	b	0	0

Notice that U is bounded with two units a,b .

Remark (2.9):
In Q-algebra, we will study the bounded with one unit only.

Proposition (2.10):
In a bounded Q-algebra U , for any $u, v \in U$, the following are hold :
1. $e^* = 0$, $0^* = e$
2. $u^* * v = v^* * u$
3. $0 * v = 0$
4. $e^* * u = 0$
5. $u^{**} \leq u$

Proof :
1. $e^* = e * e = 0$, also $0^* = e * 0 = e$
2. $u^* * v = (e * u) * v = (e * v) * u = v^* * u$
3. let $v \in U$, then
$0 = (0 * v) * e$ (since U is bounded)
$= (0 * e) * v = 0 * v$
4. $e^* * u = 0 * u$ (by 1)
$= 0$ (by 3)
5. $u^{**} * u = (e * u^*) * u = \big(e * (e * u)\big) * u = 0$

Definition (2.11) [1]:
Let $(U, *, 0)$ be a Q-algebra and L be a nonempty subset of U. Then L is called an ideal of U if for any $u, v \in U$,
1. $0 \in L$
2. $u * v \in L$ and $v \in L$ imply $u \in L$.

Obviously, $\{0\}$ and U are ideals of U. We call $\{0\}$ and U are the zero ideal and the trivial ideal of U , respectively.

Definition (2.12) [1]:
An ideal L of U is said to be proper if L is not trivial ideal of U.

Proposition (2.13):
Consider L be an ideal from Q-algebra U such that $u \leq v$, for all $u, v \in U$. If $v \in L$, therefore $u \in L$.

Proof :
Since $u * v = 0 \in L$ and $v \in L$, then $u \in L$ (since L is ideal)

Definition (2.14) [1]:
Let $(U, *, 0)$ be a Q-algebra and L be a nonempty subset of U. Then L is called a Q-subalgebra of U if $u * v \in L$, for any $u, v \in L$.

Proposition (2.15):
If L be Q-subalgebra in a Q-algebra U then $0 \in L$.

Proof :
Since $\emptyset \neq L \subseteq U$, thereafter $\exists\, u \in L$, hence $0 = u * u \in L$ (since L is Q-subalgebra) .

Corollary (2.16):

In bounded Q-algebra, every ideal is Q-subalgebra.

Proof :

Assume L is an ideal of a bounded Q-algebra U and $u, v \in L$, since L is ideal , we have : $(u * v) * u = (u * u) * v = 0 * v = 0 \in L$, but $u \in L$ implies $u * v \in L$. Thus L is a Q-subalgebra of U .

Note that the converse of this corollary needs not be true in general as in the following example .

Example (2.17):

Consider $U = \{0, a, b, c\}$ be a set as shown in the following table

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	0	c	0

Then $(U,*, 0)$ is a bounded Q-algebra with unit c .

If $L = \{0, a\}$, then L is a Q-subalgebra from U, but it's not ideal from U , because $c * a = 0 \in L$, $a \in L$ however $c \notin L$.

Definition (2.18) [5]:

A nonempty subset L of a Q-algebra U is called a Q-ideal of U if satisfies :

1. $0 \in L$
2. $(u * v) * w \in L$, $v \in L$ imply $u * w \in L$, for all $u, v, w \in U$.

Proposition (2.19):

Let L be a subset of Q-algebra U, then the following are equivalent :

1. L is an ideal
2. L is a Q-ideal

Proof :

$1 \Longrightarrow 2$: suppose L is an ideal in U and $(u * v) * w \in L$, $v \in L$, then

$$\Big((u * w) * \big((u * v) * w\big)\Big) * v = \Big((u * v) * \big((u * v) * w\big)\Big) * w = 0 \in L$$

Since L is ideal and $v \in L$, so $(u * w) * ((u * v) * w) \in L$

But $(u * v) * w \in L$, thus $u * w \in L$. Hence L is a Q-ideal of U.

$2 \Longrightarrow 1$: Let L be a Q-ideal in U and $u * v \in L$, $v \in L$

Since $(u * v) * 0 = u * v \in L$ and $v \in L$, then $u = u * 0 \in L$ (since L is Q-ideal). Thus L is an ideal of U .

3. The Main Results

In this part , we provide definitions of complete ideal , K-ideal , complete K-ideal and study its relationships with ideal in bounded Q-algebra .

Definition (3.1):

A nonempty subset L of a bounded Q-algebra U is called complete ideal (briefly, c-ideal) , if

1. $0 \in L$
2. $u * v \in L$, $\forall\, v \in L$ such that $v \neq 0$ implies $u \in L$.

Example (3.2):

Let $U = \{0, a, b, c\}$ and a binary operation $*$ is defined by

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	0	0	0

Therefore $(U, *, 0)$ is a bounded Q-algebra with unit c. The subset $L = \{0, a, c\}$ is a c-ideal , while $M = \{0, a, b\}$ is not c-ideal , because $c * v = 0 \in M$, $\forall\, v \in M$ such that $v \neq 0$ but $c \notin M$.

Remark (3.3):

In general , $\{0\}$ and U are trivial c-ideals .

Proposition (3.4):

Every ideal of bounded Q-algebra is c-ideal .

Proof :

Suppose that L be an ideal from a bounded Q-algebra U and $u * v \in L$, $\forall\, v \in L$ such that $v \neq 0$

1. If $L = \{0\}$, then L is c-ideal .
2. If $L \neq \{0\}$, therefore $\exists\, v \in L$ such that $v \neq 0$. Thus $u \in L$ (Since L is ideal). Hence L is c-ideal.

Notice that, in general, the converse of this proposition is not correct and the following example shows that .

Example (3.5):

In example (3.2), the subset $L = \{0, a, c\}$ is a c-ideal , while L is not ideal , because $b * c = 0 \in L$, $c \in L$ but $b \notin L$.

Remark (3.6):

Note that the intersection and the union of two c-ideals are not necessary to be c-ideal as shown in the following example .

Example (3.7):

Let $U = \{0, a, b, c, d, h\}$ and a binary operation $*$ is defined by

$*$	0	a	b	c	d	h
0	0	0	0	0	0	0
a	a	0	a	0	0	0
b	b	b	0	0	0	0
c	c	0	c	0	0	c
d	d	d	d	d	0	d
h	h	b	a	0	0	0

Then $(U, *, 0)$ is bounded Q-algebra with unit d. Now, let $L = \{0, a, c, d\}$ and $M = \{0, b, c, d\}$ then L, M are c-ideals, notice that $L \cap M = \{0, c, d\}$ is not c-ideal, because $a * v = 0 \in L \cap M$, $\forall\, v \in L \cap M$ such that $v \neq 0$ but $a \notin L \cap M$. Also $L \cup M = \{0, a, b, c, d\}$ is not c-ideal , since $h * w \in L \cup M$, $\forall\, w \in L \cup M$ such that $w \neq 0$ but $h \notin L \cup M$.

Definition (3.8):
Let $(U,*,0)$ be a bounded Q-algebra and L be a nonempty subset of U. Then L is called K-ideal of U if it satisfies:
1. $0 \in L$
2. $v^* * u \in L$ and $v \in L$ implies $u^* \in L$, for all $u \in U$.

In bounded Q-algebra, there are a trivial K-ideals, U and $\{0\}$.

Example (3.9):
In example (2.17), if $L_1 = \{0, c\}$ and $L_2 = \{0, b\}$, then L_1 , L_2 are K-ideals, while $L_3 = \{0, a, b\}$ is not K-ideal , because $a^* * b = 0 \in L_3$, $a \in L_3$ but $b^* = c \notin L_3$.

Proposition (3.10):
Every ideal of bounded Q-algebra is K-ideal .
Proof :
Assume L be an ideal in a bounded Q-algebra U such that $v^* * u \in L$ and $v \in L$. Since $v^* * u = u^* * v$. Thus $u^* * v \in L$ and $v \in L$. Since L is ideal , then $u^* \in L$. Thus L is a K-ideal of U.

Notice that the converse of this proposition needs not be true and we can show that in the following example.

Example (3.11):
In example (3.9) , a subset $L_1 = \{0, c\}$ is K-ideal , but it's not ideal , since $a * c = 0 \in L_1$, $c \in L_1$ but $a \notin L_1$.

Definition (3.12):
Let U be bounded Q-algebra . An element $u \in U$ satisfies $u^{**} = u$, then u is called an involution . If every element $u \in U$ is an involution, we call U is an involutory Q-algebra .

Example (3.13):
Let $U = \{0, a, b, c, d\}$ and a binary operation $*$ is defined by

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	b	a	0	d
d	d	0	0	0	0

Then $(U,*,0)$ is a bounded Q-algebra with unit c . Notice that U is an involutory .

Proposition (3.14):
If L is K-ideal of an involutory Q-algebra U. Then L is an ideal .
Proof :
let $u * v \in L$ and $v \in L$, then $v^* * u^* = u^{**} * v = u * v \in L$ (because U is an involutory) . Since L is K-ideal & $v \in L$, then $u^{**} \in L$ i.e $u \in L$. Thus L is an ideal of U .

Proposition (3.15):
Consider L is K-ideal in bounded Q-algebra U . If $u \in L$, then $u^{**} \in L$
Proof :
Let L be K-ideal in U, $u \in L$, implies $u^* * u^* = 0 \in L$. Thus $u^{**} \in L$.

Notice that the converse of this proposition may not be true and the following example explained that .

Example (3.16):
Let $U = \{0, a, b, c, d\}$ and operation $*$ is defined on U as follows

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	0	0
c	c	c	0	0	0
d	d	c	a	a	0

Then $(U,*,0)$ is bounded Q-algebra with unit d.
Let $L = \{0, a, c\}$, then $u^{**} \in L$ is holds , for all $u \in L$. But L is not K-ideal , because $a^* * 0 = c \in L$, $a \in L$ but $0^* = d \notin L$.

Proposition (3.17):
Let L be K-ideal from a bounded Q-algebra U . If $u^* \leq v$ and $v \in L$ implies $u^* \in L$.
Proof :
Let L be K-ideal of U and $u^* \leq v$. Since $v^* * u = u^* * v = 0 \in L$, then $u^* \in L$.

Proposition (3.18):
Let L be K-ideal from a bounded Q-algebra U ,
1. If $e \in L$, then $u^* \in L$, for all $u \in U$.
2. If $u \in L$ and $u^* = 0$, then $e \in L$.

Proof :
1. let L be K-ideal of U and $e \in L$, then for all $u \in U$
$e^* * u = 0 * u = 0 \in L$. Thus $u^* \in L$.
2. let L be K-ideal of U and $u \in L$ such that $u^* = 0$, then $u^* * 0 = u = 0 \in L$. Thus $e = 0^* \in L$.

Proposition (3.19):
The intersection of a family of K-ideals is a K-ideal .
Proof :
Let L_i , $i \in \Delta$ be a family of K-ideals in bounded Q-algebra U, so $0 \in L_i$, $\forall\, i \in \Delta$, then $0 \in \cap_{i\in\Delta} L_i$
Now , let $v^* * u \in \cap_{i\in\Delta} L_i$, $v \in \cap_{i\in\Delta} L_i$, then $v^* * u \in L_i$, $v \in L_i$, $\forall\ i \in \Delta$. Since L_i is K-ideal , then $u^* \in L_i$, $\forall\, i \in \Delta$. Thus $u^* \in \cap_{i\in\Delta} L_i$. Hence $\cap_{i\in\Delta} L_i$ is K-ideal .

Remark (3.20):
Note that the union of two K-ideals is not necessary to be K-ideal as shown in the following example .

Example (3.21):
In example (3.16) , if $L = \{0, a\}$ and $M = \{0, b, c\}$, then L , M are K-ideals, while $L \cup M = \{0, a, b, c\}$ is not K-ideal , because $a^* * 0 = c \in L \cup M$, $a \in L \cup M$ however $0^* = d \notin L \cup M$.

Definition (3.22):
Let $(U,*,0)$ be a bounded Q-algebra and $u \in U$. Define $u * U = \{ u * v : v \in U \}$. Then U is said to be edge if $u * U = \{0, u\}$, $\forall\, u \in U$.

Example (3.23):
In example (2.17) , then it is clear that $(U,*,0)$ is an edge Q-algebra .

Proposition (3.24):

Let L and M be K-ideals in edge Q-algebra U. Then $L \cup M$ is K-ideal .

Proof :

Let $v^* * u \in L \cup M$, $v \in L \cup M$, then $v^* * u \in L$ (or $v^* * u \in M$), $v \in L$ (or $v \in M$)
Since L and M are K-ideals , then $u^* \in L$ (or $u^* \in M$) . Thus $u^* \in L \cup M$.
Now , if $v^* * u \in L$, $v \in M$ and $v \notin L$, since $v^* * u = u^* * v$, then $u^* * v \in L$, $v \in M$ and $v \notin L$, but U is an edge Q-algebra , then either $u^* * v = 0$ or $u^* * v = u^*$
If $u^* * v = 0$ then $u^* * v \in M$ (since $0 \in M$) . Thus $v^* * u \in M$, $v \in M$. Since M is K-ideal , then $u^* \in M$. Thus $u^* \in L \cup M$.
If $u^* * v = u^*$, since $v^* * u = u^* * v = u^*$ & $v^* * u \in L \cup M$. Then $u^* \in L \cup M$.
Similarly , if $v^* * u \in M$, $v \in L$ and $v \notin M$, we can prove that $u^* \in L \cup M$. Thus $L \cup M$ is a K-ideal of U .

Definition (3.25):

A K-ideal L of a bounded Q-algebra U is called proper if $L \neq U$.

Proposition (3.26):

A K-ideal L of an involutory Q-algebra U is proper if and only if $e \notin L$

Proof :

If $e \notin L$, then it is clear that L is proper .
Conversely , let L be a proper K-ideal
If $e \in L$, then for any $u \in U$, $e^* * u^* = 0 * u^* = 0 \in L$ (since L is K-ideal) .Thus $u^{**} \in L$, i.e , $u \in L$ (since u is involution). It follows that $L = U$, a contradiction. Therefore, $e \notin L$.

Definition (3.27):

A nonempty subset L of a bounded Q-algebra U is called complete K-ideal (briefly, c-K-ideal) if,

1. $0 \in L$
2. $v^* * u \in L$, $\forall\, v \in L$ such that $v \neq 0$ implies $u^* \in L$.

Notice that in bounded Q-algebra, there are trivial c-K-ideals, $\{0\}$ and U .

Example (3.28):

In example (3.16) . A subset $L = \{0, a, b, d\}$ is a c-K-ideal of U .
While $M = \{0, a, c\}$ is not c-K-ideal of U , because $v^* * 0 \in M$, $\forall\, v \in M$ such that $v \neq 0$ but $0^* = d \notin M$.

Proposition (3.29):

Every K-ideal of bounded Q-algebra is c-K-ideal .

Proof :

Let L be K-ideal in bounded Q-algebra U and $v^* * u \in L$, $\forall\, v \in L$ and $v \neq 0$

1. If $L = \{0\}$, thus L is a c-K-ideal .
2. If $L \neq \{0\}$, therefore $\exists\, v \in L$ & $v \neq 0$. Since L is K-ideal & $v^* * u \in L$, then $u^* \in L$. Thus L is a c-K-ideal .

Notice that the converse of this proposition needs not be true in general as shown in the following example.

Example (3.30):

In example (3.16) , $L = \{0, a, b, d\}$ is a c-K-ideal , but it's not K-ideal , because $b^* * a = 0 \in L$, $b \in L$ however $a^* = c \notin L$.

Definition (3.31):

Let L be c-K-ideal of a bounded Q-algebra U. Then U is called zero star with respect to L if it satisfies $u^* * v = 0$, $\forall\, v \in L$ such that $v \neq 0$, for any $u \in U$.

Example (3.32):

Let $U = \{0, a, b, c, d\}$ and a binary operation $*$ is defined by

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	a
b	b	0	0	b	b
c	c	0	c	0	c
d	d	0	d	d	0

It is clear that $(U, *, 0)$ is a bounded Q-algebra with unit a .
If $L = \{0, a, b, c\}$, then L is c-K-ideal from U. Notice that U is a zero star with respect to L.

Proposition (3.33):

Let L be c-K-ideal in a bounded Q-algebra U . If U is a zero star with respect to L , then L is K-ideal .

Proof :

Let $v^* * u \in L$ and $v \in L$ such that $v \neq 0$, but $v^* * u = u^* * v = 0 \in L$ (since U is zero star with respect to L). Thus $v^* * u \in L$, $\forall\, v \in L$ such that $v \neq 0$. Since L is c-K-ideal , therefore $u^* \in L$. Thus L is K-ideal in U .

Corollary (3.34):

Every ideal of bounded Q-algebra is a c-K-ideal .

Proposition (3.35):

Any c-ideal from bounded Q-algebra is a c-K-ideal .

Proof :

Let L be c-ideal from a bounded Q-algebra U and $v^* * u \in L$, $\forall\, v \in L$ such that $v \neq 0$. Since $u^* * v = v^* * u$, implies $u^* * v \in L$, $\forall\, v \in L$ such that $v \neq 0$. Since L is c-ideal from U, then $u^* \in L$. Thus L is a c-K-ideal .

In general, the converse of this proposition needs not be true as shown in the following example.

Example (3.36):

In example (2.17) , if $L = \{0, c\}$, then L is a c-K-ideal from U , while L is not c-ideal , because $a * c = 0 \in L$, $c \in L$ and $c \neq 0$ but $a \notin L$.

Proposition (3.37):

If L is c-K-ideal in an involutory Q-algebra U, therefore L is c-ideal .

Proof :

Suppose L is c-K-ideal of U and $u * v \in L$, $\forall\, v \in L$ and $v \neq 0$. Since U is involutory and $v^* * u^* = u^{**} * v$, then $v^* * u^* = u * v \in L$, $\forall\, v \in L$ such that $v \neq 0$. Since L is c-K-ideal , then $u^{**} \in L$. But $u^{**} = u$, $\forall\, u \in U$, then $u \in L$. Thus L is a c-ideal of U .

Proposition (3.38):

The union from a family from c-K-ideals is a c-K-ideal .

Proof :

Let L_i , $i \in \Delta$ be a family of a c-K-ideals of a bounded Q-algebra U , so $0 \in L_i$, $\forall\, i \in \Delta$, then $0 \in \cup_{i\in\Delta} L_i$.

Assume $v^* * u \in \cup_{i\in\Delta} L_i$, $\forall\, v \in \cup_{i\in\Delta} L_i$ and $v \neq 0$. Then $\exists\ j \in \Delta$ such that $v^* * u \in L_j$, $\forall\, v \in L_j$ and $v \neq 0$. Therefore $u^* \in L_j$. Thus $u^* \in \cup_{i\in\Delta} L_i$. Hence $\cup_{i\in\Delta} L_i$ is a c-K-ideal.

Remark (3.39):

The intersection of two c-K-ideals may not be a c-K-ideal as shown in the following example.

Example (3.40):

Let $U = \{0, a, b, c, d, h\}$. Define the operation $*$ on U by

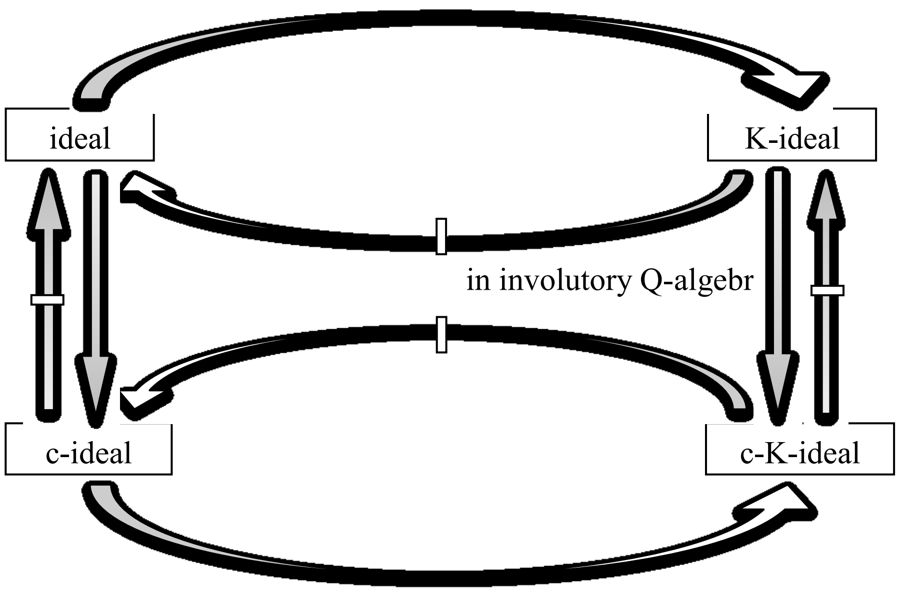
$*$	0	a	b	c	d	h
0	0	0	0	0	0	0
a	a	0	a	0	0	0
b	b	b	0	b	0	0
c	c	c	0	0	0	0
d	d	d	c	0	0	0
h	h	h	d	0	c	0

Then $(U,*,0)$ is a bounded Q-algebra with unit h .

If $L = \{0, a, c\}$ and $M = \{0, c, d, h\}$, then L , M are c-K-ideals of U. While $L \cap M = \{0, c\}$ is not c-K-ideal , because $c^* * 0 = 0 \in L \cap M$, $c \in L \cap M$, $c \neq 0$ however $0^* = h \notin L \cap M$.

Remark (3.41):

The following diagram shows the relation among ideal , K-ideal , c-ideal , and c-K-ideal in bounded Q-algebra :



Definition (3.42):
Let L be a c-K-ideal in a bounded Q-algebra U, we say that L is closed if it is Q-subalgebra .

Example (3.43):
Let $U = \{0, a, b, c\}$ and a binary operation $*$ is defined by

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	0	a	0

Therefore $(U,*, 0)$ is bounded Q-algebra with unit c .
A c-K-ideal $L = \{0, a, c\}$ is closed (because L is Q-subalgebra) .

Remark (3.44):
In general , there exists a c-K-ideal which is not Q-subalgebra. As it is shown in the following example .

Example (3.45):
In example (3.43) , a c-K-ideal $M = \{0, b, c\}$ is not Q-subalgebra , since $b , c \in M$ but $c * b = a \notin M$.

Definition (3.46):
Let L and M be two subsets from a bounded Q-algebra U such that $L \subseteq M$, then L is said to be closed with respect to M , if $v^* * u \in M$, $\forall\, v \in L$ such that $v \neq 0$ implies $v^* * u \in L$, for all $u \in U$.

Example (3.47):
In example (2.6) , if $L_1 = \{0, c\}$ and $M = \{0, b, c\}$ such that $L_1 \subseteq M$, then L_1 is closed with respect to M . If $L_2 = \{0, b\}$ such that $L_2 \subseteq M$, then L_2 is not closed with respect to M , since $b^* * 0 = c \in M$, $b \in L_2$ and $b \neq 0$ but $b^* * 0 = c \notin L_2$.

Proposition (3.48):
The union of family of closed with respect to M is a closed with respect to M .
Proof :
Let $\{ L_i : i \in \Delta \}$ be a family of closed with respect to M and $v^* * u \in M$, $\forall\, v \in \cup_{i\in\Delta} L_i$, $v \neq 0$, implies $v^* * u \in M$, $\forall\, v \in L_i$, $v \neq 0$. Since L_i is closed with respect to M , then $v^* * u \in L_i$. Thus $v^* * u \in \cup_{i\in\Delta} L_i$. Hence $\cup_{i\in\Delta} L_i$ is closed with respect to M.

Remark (3.49):
The intersection of two closed with respect to M is not closed with respect to M as it is shown in the following example .

Example (3.50):

Let $U = \{0, a, b, c, d, h\}$ with the table as follows

$*$	0	a	b	c	d	h
0	0	0	0	0	0	0
a	a	0	0	a	0	0
b	b	b	0	0	0	0
c	c	c	b	0	b	0
d	d	b	0	0	0	0
h	h	c	d	a	b	0

Notice that $(U,*,0)$ is a bounded Q-algebra with unit h .
If $M = \{0, a, c, d\}$, $L_1 = \{0, a, c\}$ and $L_2 = \{0, a, d\}$, then L_1 and L_2 are closed with respect to M, but $L_1 \cap L_2 = \{0, a\}$ is not closed with respect to M, since $a^* * 0 = c \in M$, $a \in L_1 \cap L_2$ and $a \neq 0$ but $a^* * 0 = c \notin L_1 \cap L_2$.

Remark (3.51):

If $L \subseteq M$,and L is c-K-ideal, then M is not in general c-K-ideal, as in the following example .

Example (3.52):

In example (3.43) , if $L = \{0, b\}$, $M = \{0, a, b\}$ and $L \subseteq M$. Notice that L is c-K-ideal , while M is not c-K-ideal , because $v^* * 0 \in M$, $\forall\, v \in M$ and $v \neq 0$ but $0^* = c \notin M$.

Proposition (3.53):

Let L be c-K-ideal in bounded Q-algebra U and $\emptyset \neq L \subseteq M$. If L is closed with respect to M , then M is c-K-ideal .

Proof :

Let $v^* * u \in M$, $\forall\, v \in M$ and $v \neq 0$. Since $L \subseteq M$ implies $v^* * u \in M$, $\forall\, v \in L$ and $v \neq 0$, since L is closed with respect to M , then $v^* * u \in L$. Since L is c-K-ideal, $\forall\, v \in L$ and $v \neq 0$, then $u^* \in L$, consequently $u^* \in M$. Thus M is a c-K-ideal .

The following example shows that the converse of this proposition is false, in general.

Example (3.54):

Let $U = \{0, a, b, c, d\}$ and a binary operation $*$ is defined by

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	d
b	b	b	0	0	b
c	c	b	0	0	b
d	d	0	d	0	0

Therefore $(U,*,0)$ is a bounded Q-algebra with unit c.
If $L = \{0, a, b\}$ and $M = \{0, a, b, c\}$ such that $L \subseteq M$, then L is closed with respect to M , and M is c-K-ideal , but L is not c-K-ideal , because $v^* * 0 \in L$, $\forall\, v \in L$, $v \neq 0$ however $0^* = c \notin L$.

Proposition (3.55):

Let L be closed with respect to M in an involutory Q-algebra U. If M is Q-subalgebra , then L is Q-subalgebra .

Proof :

Let $u, v \in L$,

1. If $v = 0$, then $u * v = u * 0 = u \in L$.
2. If $v \neq 0$, but $L \subseteq M$, then $u, v \in M$. Since M is Q-subalgebra implies $u * v \in M$. But $v^* * u^* = u^{**} * v = u * v$ (since U is an involutory) . Thus $v^* * u^* \in M$, $\forall\, v \in L$ and $v \neq 0$. Since L is closed with respect to M , then $v^* * u^* \in L$. Thus $u * v \in L$. Hence L is Q-subalgebra .

Notice that the converse of this proposition needs not to be true in general as shown in the following example .

Example (3.56):

In example (3.13) , if $L = \{0, a, d\}$, $M = \{0, a, c, d\}$ and $L \subseteq M$, then L is closed with respect to M , and L is Q-subalgebra , while M is not Q-subalgebra , since $a, c \in M$ but $c * a = b \notin M$.

Corollary (3.57):

Let L be closed with respect to M in an involutory Q-algebra U, and L be c-K-ideal . If M is closed c-K-ideal , then L is closed c-K-ideal .

Proof:

It is directly from proposition (3.55) and definition (3.42).

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