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Improve the approximation order of Bernstein type operators

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ABSTRACT

In this study, we present a generalization of the well-known Bernstein operators based on an odd positive integer r denoted by K_(n,r) (f;x), first, we begin by studying the simultaneous approximation where we prove that the operator K_(n,r)^((s)) (f;x) convergence to the function f^((s)) (x) then we introduce and prove the Voronovskaja-type asymptotic formula when (r=3) giving us the order of approximation O(n^(-2)) which is better than the order of the classical Bernstein operators O(n^(-1)) followed by the error theorem and at the end, we give a numerical example to show the error of a test function and its first derivative taking different values of r.

1. Introduction

The classical Bernstein polynomials are defined as [1].

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad (1)$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $f \in C[0,1]$ $and\ x \in [0,1]$.

Several generalizations and modifications was presented by many researchers [2,2,3, and 5] some researchers studied other operators [6,7,8,9, and 10]. In 2005 Pallini [11] defined a new modified Bernstein operators involving the parameter $> -\frac{1}{2}$, $f \in C[0,1], x \in [0,1]$.

$$B_{n,s}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(x + \frac{1}{n^s}\left(\frac{k}{n} - x\right)\right), \quad (2)$$

Recently, Mohammad and Hassan [12] gave a new sequence of integral types based on two parameters, for $f \in C_\alpha[0,\infty), n, r, s \in N$ and $y \in [0, y)$.

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$$M_{n,r,s}(f;y)=\frac{1}{A_r(ny)}\int_0^y A'_r(n\tau)f(y+(\tau-y)^s)d\tau, \tag{3}$$

where $A_{r,i}(y)=\sum_{l=0}^{\infty}\frac{y^{rl+i}}{(rl+i)!}, i\in N^0$. In our study, we present the following generalization for $f\in C_\gamma[0,1], x\in[0,1]$ and $r=\{1,3,\dots\}$, we have

$$K_{n,r}(f;x)=\sum_{k=0}^{n}p_{n.k}(x)f\left(x+\left(\frac{k}{n}-x\right)^r\right). \tag{4}$$

Where $C_\gamma[0,1]=\{f\in c[0,1], |f|\le M|t|^\gamma\ for\ some\ \gamma>0, M\ constant\}$ after introducing some preliminaries, we investigate three main theorem starting with the simultaneous approximation moving to Voronovskaja-type asymptotic formula proving that the approximation order is improved, followed by the error estimation and support this by providing numerical example showing the behavior of the error curve for $r=1, r=3$.

2. Preliminary Results

The following preliminaries are used in the main results of the next sections.

Lemma 2.1. [13] For $x\in[0,1]$ and $m\in\{0,1,2,\dots\}$ the moment function For $B_n(f;x)$ is defined as.

$$T_{n.m}(x)=B_n((t-x)^m;x)=\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{k}{n}-x\right)^m.$$

then,

(i) $\quad T_{n,0}(x)=1, T_{n,3}(x)=\dfrac{x-3x^2+2x^3}{n^2}.$

(ii) $\quad T_{n,m+1}(x)=\dfrac{x(1-x)}{n}\left(T'_{n,m}(x)+mT_{n,m-1}(x)\right)$, for $m\ge 1$

(iii) $\quad T_{n,m}(x)=O\left(n^{-\left[\frac{m+1}{2}\right]}\right).$

Lemma 2.2. For $x\in[0,1]$ and $\in\{0,1,2,\dots\}$, $r=\{1,3,5,7,\dots\}$,

$$Y_{n,m,r}(x)=K_{n,r}((t-x)^m;x)=\sum_{k=0}^{n}p_{n,k}(x)\left(x+\left(\frac{k}{n}-x\right)^r-x\right)^m.$$

Then,

(i) $\quad Y_{n,0,r}(x)=T_{n,0}(x), Y_{n,3,r}(x)=T_{n,3r}(x).$

(ii) $\quad Y_{n,m,r}(x)=O\left(n^{-\left[\frac{mr+1}{2}\right]}\right).$

Proof.

$$Y_{n,m,r}(x)=\sum_{k=0}^{n}p_{n,k}(x)\left(x+\left(\frac{k}{n}-x\right)^r-x\right)^m$$
$$=\sum_{k=0}^{n}p_{n,k}(x)\left(\left(\frac{k}{n}-x\right)^r\right)^m,$$

$Y_{n,m,r}(x)=T_{n,mr}(x),$

As a consequences of the above formulas $(i),(ii)$ hold.

Lemma 2.3. [13] For $c\in N$ and $x\in[0,1]$, then there is a polynomial $R_{a,e,c}(x)$ that is independent of n and k,then.

$$\left(p_{n,k}(x)\right)^{(c)}=\frac{1}{x^c(1-x)^c}\sum_{\substack{2a+e\le c\\ a,e\ge 0}}n^a(k-nx)^e\,R_{a,e,c}(x)\,p_{n,k}(x)$$

Lemma 2.4. For $q \in N^0$, we have

$$K_{n,r}(t^q;x) = x^q T_{n,0}(x) + q x^{q-1} T_{n,r}(x) + O(n^{-r}).$$

And $\lim_{n\to\infty} K_{n,r}(t^q;x) = x^q$

Proof.

$$K_{n,r}(t^q;x) = \sum_{k=0}^{n} p_{n,k}(x)\left(x + \left(\frac{k}{n} - x\right)^r\right)^q,$$

$$K_{n,r}(t^q;x) = \sum_{k=0}^{n} p_{n,k}(x) \sum_{j=0}^{q} \binom{q}{j} x^{q-j} \left(\frac{k}{n} - x\right)^{jr},$$

By simplification of the above formula, we get

$$K_{n,r}(t^q;x) = x^q T_{n,0}(x) + q x^{q-1} T_{n,r}(x) + O(n^{-r}).$$

3. Main Text

This section includes the three main theorems for the operators $K_{n,r}^{(s)}(f;x)$.

Theorem 3.1

Let $f \in C_\gamma[0,1]$, $s \in N^0$ and $f^{(s)}(x)$ exists for $x \in (0,1)$ the following relation holds.

$$\lim_{n\to\infty} K_{n,r}^{(s)}(f(t);x) = f^{(s)}(x), \quad (5)$$

Proof. We have The Taylor expansion of the function $f(t)$ is

$$f(t) = \sum_{p=0}^{s} \frac{f^{(p)}(x)}{p!}(t-x)^p + (t-x)^s \varepsilon(t,x),$$

as $t \to x$ the terms $\varepsilon(t,x)$ goes to zero

$$K_{n,r}^{(s)}(f(t);x) = \sum_{p=0}^{s} \frac{f^{(p)}(x)}{p!} K_{n,r}^{(s)}((t-x)^p;x) + K_{n,r}^{(s)}((t-x)^s \varepsilon(t,x);x) := M_1 + M_2$$

$$M_1 = \sum_{p=0}^{s} \frac{f^{(p)}(x)}{p!} K_{n,r}^{(s)}((t-x)^p;x)$$

$$= \sum_{p=0}^{s} \frac{f^{(p)}(x)}{p!} \sum_{j=0}^{p} \binom{p}{j} (-x)^{p-j} K_{n,r}^{(s)}(t^j;x),$$

By using lemma 2.4

$$= \sum_{p=0}^{s} \frac{f^{(p)}(x)}{p!} \sum_{j=0}^{p} \binom{p}{j} (-x)^{p-j} \frac{d^s}{dx^s}\left(x^j T_{n,0}(x) + j x^{j-1} T_{n,r}(x) + O(n^{-r})\right)$$

$$= \sum_{p=0}^{s} \frac{f^{(p)}(x)}{p!} \sum_{j=0}^{p} \binom{p}{j} (-x)^{p-j} \frac{d^s}{dx^s}\left(x^j\right) + \sum_{p=0}^{s} \frac{f^{(p)}(x)}{p!} \sum_{j=0}^{p} \binom{p}{j} (-x)^{p-j} \frac{d^s}{dx^s}\left(j x^{j-1} T_{n,r}(x) + O(n^{-r})\right)$$

$:= W_1 + W_2$,

As for W_1 if $j < s$ then $\frac{d^s}{dx^s}\left(x^j\right) = 0$, then

$W_1 \to f^{(s)}(x) \; as \; n \to \infty$

And the terms $W_2 \to 0$ as $n \to \infty$

Now we take,

$$M_2 = K_{n,r}^{(s)}((t-x)^s\varepsilon(t,x);x) = \sum_{k=0}^{n}\left(p_{n,k}(x)\right)^{(s)}\left(\frac{k}{n}-x\right)^{rs}\varepsilon\left(\frac{k}{n},x\right)$$

By lemma 2.3,

$$M_2 = \sum_{k=0}^{n}\frac{1}{x^s(1-x)^s}\sum_{\substack{2a+e\le s\\ a,e\ge 0}} n^a(k-nx)^e R_{a,e,s}(x)\,p_{n,k}(x)\left(\frac{k}{n}-x\right)^{rs}\varepsilon\left(\frac{k}{n},x\right)$$

$$|M_2| \le \sup_{\substack{2a+e\le s\\ a,e\ge 0}}\frac{\left|R_{a,e,s}(x)\right|}{x^s(1-x)^s}\sum_{\substack{2a+e\le s\\ a,e\ge 0}} n^a\left(\sum_{\left|\frac{k}{n}-x\right|<\delta} p_{n,k}(x)|k-nx|^e\left|\left(\frac{k}{n}-x\right)^{rs}\varepsilon\left(\frac{k}{n},x\right)\right|\right)$$
$$+\left(\sum_{\left|\frac{k}{n}-x\right|\ge\delta} p_{n,k}(x)|k-nx|^e\left|\left(\frac{k}{n}-x\right)^{rs}\varepsilon\left(\frac{k}{n},x\right)\right|\right)$$

$$|M_2| \le \rho\sum_{\substack{2a+e\le s\\ a,e\ge 0}} n^{a+e}\left(\sum_{\left|\frac{k}{n}-x\right|<\delta} p_{n,k}(x)\left|\frac{k}{n}-x\right|^{rs+e}\left|\varepsilon\left(\frac{k}{n},x\right)\right|\right)$$
$$+\left(\sum_{\left|\frac{k}{n}-x\right|\ge\delta} p_{n,k}(x)\left|\frac{k}{n}-x\right|^{e}\left|\left|\frac{k}{n}-x\right|^{rs}\varepsilon\left(\frac{k}{n},x\right)\right|\right) := Z_1 + Z_2$$

where $\rho = \sup_{\substack{2a+e\le s\\ a,e\ge 0}}\frac{\left|R_{a,e,s}(x)\right|}{x^s(1-x)^s}$ and then write Z_1 as

$$Z_1 = \rho\sum_{\substack{2a+e\le s\\ a,e\ge 0}} n^{a+e}\left(\sum_{\left|\frac{k}{n}-x\right|<\delta}\left(p_{n,k}(x)\right)^{\frac{1}{2}}\left(p_{n,k}(x)\right)^{\frac{1}{2}}\left|\frac{k}{n}-x\right|^{rs+e}\left|\varepsilon\left(\frac{k}{n},x\right)\right|\right)$$

For $\epsilon > 0, \exists\delta > 0$ such that $\left|\frac{k}{n}-x\right| < \delta \to \left|\varepsilon\left(\frac{k}{n},x\right)\right| < \epsilon$ and by applying Cauchy-Schwarz inequality,

$$Z_1 = \rho\epsilon\sum_{\substack{2a+e\le s\\ a,e\ge 0}} n^{a+e}\left(\sum_{\left|\frac{k}{n}-x\right|<\delta} p_{n,k}(x)\right)^{\frac{1}{2}}\left(\sum_{\left|\frac{k}{n}-x\right|<\delta} p_{n,k}(x)\left(\frac{k}{n}-x\right)^{2(rs+e)}\right)^{\frac{1}{2}}$$

$$Z_1 \le \rho\epsilon\sum_{\substack{2a+e\le s\\ a,e\ge 0}} n^{a+e}\left(O\left(n^{-(rs+e)}\right)\right)^{\frac{1}{2}}$$

$$Z_1 \le \rho\epsilon\sum_{\substack{2a+e\le s\\ a,e\ge 0}} O\left(n^{\frac{-rs+e+2a}{2}}\right)$$

As $n\to\infty$ then $Z_1 \to 0$

Finally, we have

$$Z_2 = \rho\sum_{\substack{2a+e\le s\\ a,e\ge 0}} n^{a+e}\left(\sum_{\left|\frac{k}{n}-x\right|\ge\delta} p_{n,k}(x)|k-nx|^e\left|\left(\frac{k}{n}-x\right)^{rs}\varepsilon\left(\frac{k}{n},x\right)\right|\right)$$

For $\left|\frac{k}{n}-x\right| \ge \delta$, $\exists\, L > 0$ such that $\left|\left(\frac{k}{n}-x\right)^{rs}\varepsilon\left(\frac{k}{n},x\right)\right| \le L\left|\frac{k}{n}-x\right|^{\lambda}$

$$Z_2 \le \rho L\sum_{\substack{2a+e\le s\\ a,e\ge 0}} n^{a+e}\left(\sum_{\left|\frac{k}{n}-x\right|\ge\delta} p_{n,k}(x)\left|\frac{k}{n}-x\right|^{e}\left|\frac{k}{n}-x\right|^{\lambda}\right).$$

Applying the same simplifications in Z_1,
$Z_2 \to 0 \;\; as \;\; n \to \infty$
Hence $K_{n,r}^{(s)}((t-x)^s \varepsilon(t,x);x) \to 0 \;\; as \;\; n \to \infty$,then (5) is obtained.
Theorem 3.2
Let $f \in C_\gamma[0,1]$. and $f^{(s+2)}(x)$ exists and continuous, where $s \in N^0$, then

$$\lim_{n\to\infty} n^2 \left(K_{n,r}^{(s)}(f;x) - f^{(s)}(x)\right) = 2s(s-2)(s-1)f^{(s-2)} + (3(s-1)s - (2s^3x - 2s^2x))f^{(s-1)}(x) + (s + 6sx + 4s^3x^2 + 2sx^2)f^{(s)}(x) + \left(x - 3\left(\frac{2x^2 + sx^2(3-2s)}{2}\right) + 2x^3\right) f^{(s+1)}(x) \quad (6)$$

proof. As a result of the Taylor expansion of the function$f(t)$, we get

$$f(t) = \sum_{v=0}^{s+2} \frac{f^{(v)}(x)}{v!}(t-x)^v + (t-x)^{s+2}\varepsilon(t,x),$$

where $\varepsilon(t,x) \to 0 \; as \; t \to x. Then,$

$$K_{n,r}^{(s)}(f(t);x) = \sum_{v=0}^{s+2} \frac{f^{(v)}(x)}{v!} K_{n,r}^{(s)}((t-x)^v;x) + K_{n,r}^{(s)}((t-x)^{s+2}\varepsilon(t,x);x) := E_1 + E_2$$

$$E_1 = \sum_{v=0}^{s+2} \frac{f^{(v)}(x)}{v!} \sum_{j=0}^{v} \binom{v}{j} (-x)^{v-j} \frac{d^s}{dx^s}\left(K_{n,r}(t^j;x)\right).$$

By lemma 2.3

$$E_1 = \sum_{v=0}^{s+2} \frac{f^{(v)}(x)}{v!} \sum_{j=0}^{v} \binom{v}{j} (-x)^{v-j} \frac{d^s}{dx^s}(x^j) + \sum_{v=0}^{s+2} \frac{f^{(v)}(x)}{v!} \sum_{j=0}^{v} \binom{h}{j} (-x)^{v-j} \frac{d^s}{dx^s}\left(jx^{j-1}T_{n,r}(x) + O(n^{-r})\right).$$

Now by taking $r = 3$ and using lemma 2.1.

$$E_1 = f^{(s)}(x) + \sum_{v=0}^{s+2} \frac{f^{(v)}(x)}{v!} \sum_{j=0}^{v} \binom{v}{j} (-x)^{v-j} \frac{d^s}{dx^s}\left(j\frac{x^j - 3x^{j+1} + 2x^{j+4}}{n^2} + O(n^{-3})\right).$$

Doing some simplification, we get

$$A_1 = f^{(s)}(x) + 2s(s-2)(s-1)f^{(s-2)} + (3(s-1)s - (2s^3x - 2s^2x))f^{(s-1)}(x) + (s + 3s^2x + 4s^3x^2 + 2sx^2)f^{(s)}(x) + \left(x - 3\left(\frac{2x^2+sx^2(3-2s)}{2}\right) + 2x^3\right) f^{(s+1)}(x) + O(n^{-3}).$$

As for A_2using the same technique used in theorem 1.3, we have $A_2 \to 0$ as $t \to \infty$
Hence we obtained equation (6)
Theorem 3.3
Let $f \in C_\gamma[0,1]$ with $\gamma > 0$ and f continuous and differentiable η-times on the interval $(a-\alpha, b+\alpha) \subset (0,1)$, for some $\alpha > 0$ where $0 \le \eta \le s+2$ the following inequality holds,

$$\left\|K_{n,r}^{(s)}(f;x) - f^{(s)}(x)\right\|_{c[a,b]} \le w_1 n^{-\left[\frac{r+1}{2}\right]} \sum_{p=0}^{\eta} \left\|f^{(p)}\right\|_{c[a,b]} + w_2 n^{\frac{-r\eta}{2}} \omega_{f(t)} + O(n^{-r}), \quad (7)$$

where w_1, w_2 are two independent constants of f and n and $\omega_f(\delta)$ defined as the modulus of continuity of f.
Proof. Using the finite Tylor's expansion of (t) , we get $f(t) = \sum_{p=0}^{\eta} \frac{f^{(p)}(x)}{p!}(t-x)^p + \frac{f^{(\eta)}(\xi) - f^{(\eta)}(x)}{\eta!}(t-x)^\eta H(t) + q(t,x)\left(1 - H(t)\right)$,

where $\xi \in (t,x)$and $H(t)$ is The characteristic function of $(a-\sigma,b+\sigma)$, for $x\in(0,1)$ and $t\in(a-\sigma,b+\sigma)$, we have

$$f(t)=\sum_{p=0}^{\eta}\frac{f^{(p)}(x)}{p!}(t-x)^p+\frac{f^{(\eta)}(\xi)-f^{(\eta)}(x)}{\eta!}(t-x)^{\eta},$$

where

$$q(t,x)=f(t)-\sum_{p=0}^{\eta}\frac{f^{(p)}(x)}{p!}(t-x)^p.$$

Then,

$$\begin{aligned}K_{n,r}^{(s)}(f(t);x)-f^{(s)}(x)\\ &=\sum_{p=0}^{\eta}\frac{f^{(p)}(x)}{p!}K_{n,r}^{(c)}((t-x)^p;x)-f^{(s)}(x)+K_{n,r}^{(s)}\left(\frac{f^{(\eta)}(\xi)-f^{(\eta)}(x)}{\eta!}(t-x)^{\eta}H(t);x\right)\\ &+K_{n,r}^{(s)}\big(q(t,x)\big(1-H(t)\big);x\big),\end{aligned}$$

$:=\beta_1+\beta_2+\beta_3,$

Now,

$$\beta_1=\sum_{p=0}^{\eta}\frac{f^{(p)}(x)}{p!}\sum_{j=0}^{p}\binom{p}{j}(-x)^{p-j}K_{n,r}^{(s)}\big(t^j;x\big)-f^{(s)}(x)$$

Using lemma 2.2 and lemma 2.4, we get

$$\|\beta_1\|_{C[a,b]}\le w_1 n^{-\left[\frac{r+1}{2}\right]}\sum_{p=0}^{\eta}\left\|f^{(p)}\right\|_{C[a,b]}+O(n^{-r})$$

Next we estimate the term

$$|\beta_2|\le\left|K_{n,r}^{(s)}\left(\frac{f^{(\eta)}(\xi)-f^{(\eta)}(x)}{\eta!}(t-x)^{\eta}H(t);x\right)\right|$$

$$|\beta_2|\le\frac{\omega_{f^{(\eta)}}(\delta)}{\eta!}K_{n,r}^{(s)}\left(\left(1+\frac{|t-x|}{\delta}\right)|t-x|^{\eta};x\right)$$

$$|\beta_2|\le\frac{\omega_{f^{(\eta)}}(\delta;)}{\eta!}\left[\frac{d^s}{dx^s}\left(\sum_{k=0}^{n}p_{n,k}(x)\left|\frac{k}{n}-x\right|^{r\eta}+\delta^{-1}\sum_{k=0}^{n}p_{n,k}(x)\left|\frac{k}{n}-x\right|^{r\eta+1}\right)\right]$$

Choosing $\delta=n^{-\frac{1}{2}}$ and using Schwarz inequality We obtain,

$$\begin{aligned}|\beta_2|\le\frac{\omega_{f^{(\eta)}}\left(n^{-\frac{1}{2}}\right)}{\eta!}\frac{d^s}{dx^s}&\left[\sum_{k=0}^{n}\left(p_{n,k}(x)\right)^{\frac{1}{2}}\left(p_{n,k}(x)\left(\frac{k}{n}-x\right)^{2r\eta}\right)^{\frac{1}{2}}\right.\\ &\left.+n^{-\frac{1}{2}}\sum_{k=0}^{n}\left(p_{n,k}(x)\right)^{\frac{1}{2}}\left(p_{n,k}(x)\left(\frac{k}{n}-x\right)^{2r\eta+2}\right)^{\frac{1}{2}}\right]\end{aligned}$$

$$|\beta_2|\le\frac{\omega_{f^{(\eta)}}\left(n^{-\frac{1}{2}}\right)}{\eta!}\left[O\left(n^{\frac{-r\eta}{2}}\right)+n^{-\frac{1}{2}}O\left(n^{\frac{-(r\eta+1)}{2}}\right)\right]$$

$$|\beta_2|\le\frac{\omega_{f^{(\eta)}}\left(n^{-\frac{1}{2}}\right)}{\eta!}O\left(n^{\frac{-r\eta}{2}}\right)$$

$$|\beta_2|\le w_2 n^{\frac{-r\eta}{2}}\omega_{f^{(\eta)}}\left(n^{-\frac{1}{2}}\right),$$

Finally, we have

$$|\beta_3| \leq \frac{d^s}{dx^s}\left(\sum_{k=0}^{n} p_{n,k}(x)\left|q\left(\frac{k}{n},x\right)\right|\right),$$

Since $t \in [0,1] \setminus (a-\sigma, b+\sigma)$ we choose $\delta > 0$ in a way to have $|t-x| \geq \delta$ therefore there is $M > 0$ such that $|q(t,x)| \leq A|t-x|^{\tau}$, and for all $x \in [a,b]$, then

$$|\beta_3| \leq \frac{d^s}{dx^s}\left(\sum_{|t-x|\geq\delta} p_{n,k}(x)\left|q\left(\frac{k}{n},x\right)\right|\right)$$

$$|\beta_3| \leq \frac{d^s}{dx^s}\left(\sum_{k=0}^{n} p_{n,k}(x)\, A\left|\frac{k}{n}-x\right|^{\tau}\right),$$

Using Cauchy-Schwartz inequality again and lemma 2.3 , we conclude

$|\beta_3| = O(n^e), \quad e > 0$

By combining $\beta_1, \beta_2, \beta_3$ we get (7).

4. numerical data

In this section we gave some numerical example comparing the error carve of approximation two test functions by $K_{n,1}(f;x)$ and $K_{n,3}$.

Example 4.1

Suppose that $f(x) = \cos(11\pi x)\, , x \in [0,1]$, is the test function.

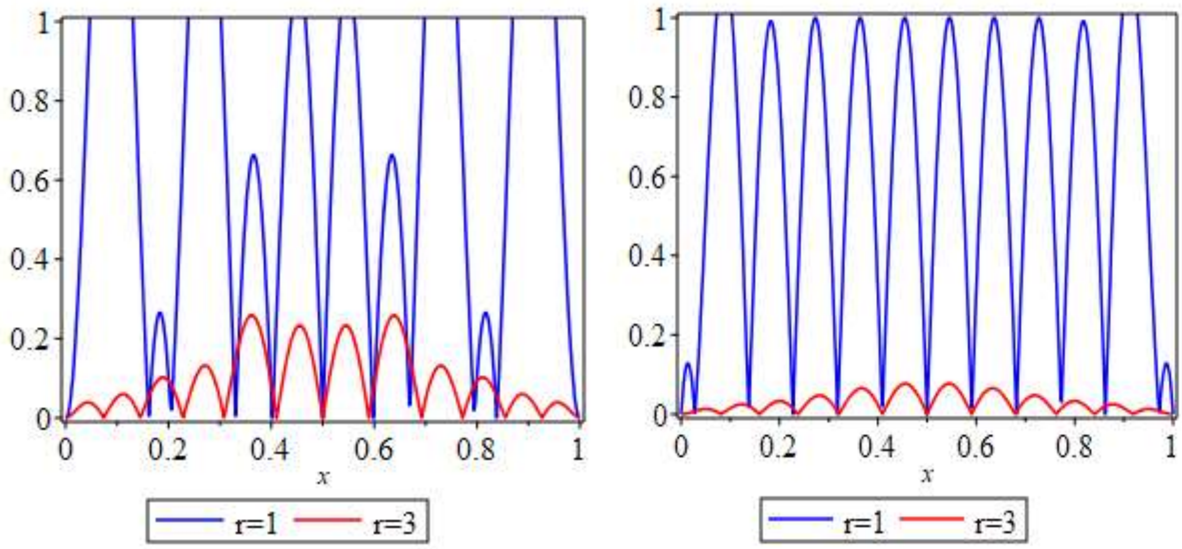
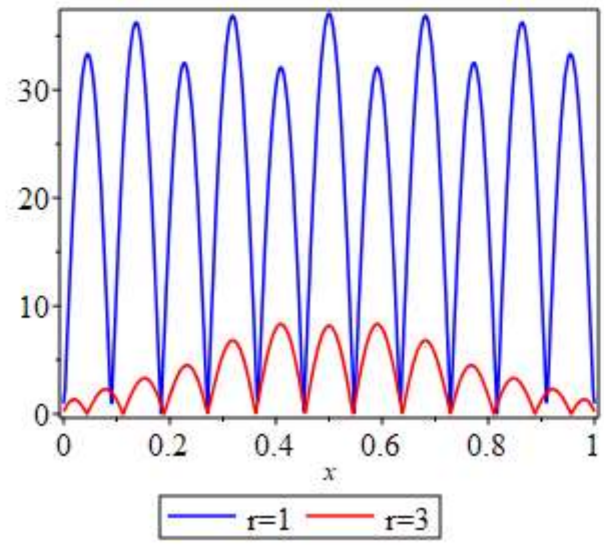


Fig 4.1 The error curve $|K_{n,1}(x) - f(x)|$ and $|K_{n,3}(x) - f(x)|$when $n = 5$ in the left and $n = 10$ on the right.



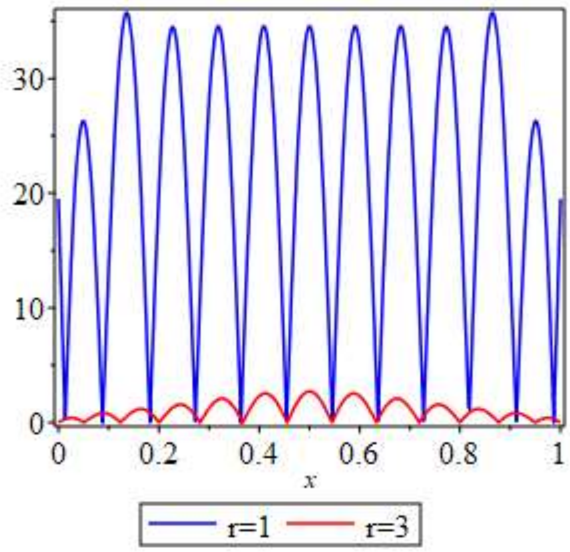


Fig 4.2 The error curve $\left|K'_{n,1}(x) - f'(x)\right|$ and $\left|K'_{n,3}(x) - f'(x)\right|$when $n = 5$ on the left and $n = 10$ on the right.

5. Conclusion

This paper gives a generalization of the well-known Bernstein operator with the goal of improving the order of approximation. also we provide two numerical example that shows the error function $\left|K_{n,r}(x) - f(x)\right|$ and $\left|K'_{n,r}(x) - f'(x)\right|$ when $n = 5, n = 10$ demonstrating that approximation become more accurate when $r = 3$ the $r = 1$.

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تحسين رتبة التقارب لمؤثر من نوع Bernstein

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سلطانقسم الرياضيات، كلية العلوم، جامعة البصرة، البصرة، العراق.

معلومات البحث

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القبول 17 آب 2022
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الكلمات المفتاحية

موثر برنستين، التقريب المتزامن، صيغة فرونوفسكي، مقياس الاستمرارية.

الملخص

في هذه الدراسة سوف نقدم تعميم لمؤثر Bernstein المعروف بالاعتماد على عدد صحيح موجب فردي r . اولاً نبدى بدراسة التقريب المتزامن ثم نقدم ونثبت صيغة Voronovskaja عند (r=3) لتعطينا رتبة تقارب (O(n^(-2) وهي افضل من رتبة تقارب موثر برنسين الاعتيادي(O(n^(-1) تتبعها مبرهنة الخطأ وفي النهاية نعطي مثالا عدديا لبيان دالة الخطأ لدالة اختبار مع مشتقتها الأولى لقيم مختلفة من r.

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