Crossed modules of Bicomplexes

By

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Abstract: *This paper is devoted to introduce the notion of crossed modules of bicomplexes as a suitable generalization of the notions of crossed modules of complexes and crossed modules of bigroups, and the notion of bicomplexes as a suitable generalization of the notion of complexes. This is done by embedding the categories of crossed modules of complexes and crossed modules of bigroups in the category of crossed modules of bicomplexes, and the category of complexes in the category of bicomplexes(via isomorphism of categories). The notion of bichain complex and bichain map had also been introduced in the beginning of this paper. Finally, two adjoint pair of functors had been given between the category of crossed modules of bicomplexes and the category of bicomplexes.*

Key words: (Chain) complexes; Crossed modules; Bigroups; Adjoint functors.

الموديولات المتصالبة للهياكل الثنائية

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المستخلص:

يعنى هذا البحث بتقديم مفهوم الموديولات المتصالبة للهياكل الثنائية كتعميم مناسب لمفهومي الموديولات المتصالبة للهياكل والموديولات المتصالبة لزمر الثنائية. عمل هذا بغمر فصيلتي الموديولات المتصالبة للهياكل والموديولات المتصالبة لزمر الثنائية في فصيلة الموديولات المتصالبة للهياكل الثنائية (وفقا للتشاكل التقابلي للفصائل). مفهوم الهيكل الثنائي والتطبيق السلسلي للهياكل الثنائية كذلك قدم في اول البحث. أخيرا، قدم زوج مترافق من المقرنات التغايرية بين فصيلة الموديولات المتصالبة للهياكل الثنائية وفصيلة الهياكل الثنائية.

1. Introduction

Crossed modules of groups were introduced by Whitehead through his investigation of the algebraic structure of the second relative homotopy groups [3]. The notion of crossed modules of groups play an important role in homotopy theory, group presentations [7], algebraic K-theory [5] and homological algebra [1,4]. Recall that a crossed module of groups (C, G, ∂, θ) is a group homomorphism $\partial: C \to G$ together with an action $\theta: G \times C \to C$ of G on C (usually written by $(g, c) = g_c$) satisfying the following two conditions:

(CM1) ∂ is a precrossed module, i.e., $\partial(g_c) = g\partial(c)g^{-1}$, for all $c \in C$, $g \in G$.

(CM2) The Pieffer subgroup is trivial, i.e., $\partial(c_2)_{c_1} = c_2 c_1 {c_2}^{-1}$, for all $c_1, c_2 \in C$.

When the action is unambiguous, we may write the crossed module of groups (C, G, ∂, θ) simply as (C, G, ∂). A morphism of crossed modules of groups $(\mu, \eta): (C, G, \partial) \to (D, H, \delta)$ is a pair of group homomorphisms $\mu: C \to D$ and $\eta: G \to H$, such that $\delta\mu = \eta\partial$ and $\mu(g_c) = \eta(g)_{\mu(c)}$, for all $c \in C$, $g \in G$. Crossed modules of groups and morphisms as defined above form a category, $CModGrps$. For more details, we refer the reader to Brown [7] and Baues [2]. Vasantha Kandasamy in [10] defined the bigroup $(G, +, \cdot)$ that is a set with two binary operation

'$+$' and '$\cdot$' such that there exist two proper subsets G_1 and G_2 of G satisfied the following conditions: (i) $G = G_1 \cup G_2$. (ii) $(G_1, +)$ is a group. (iii) $(G_2, \cdot)$ is a group. He also defined a bigroup homomorphism $\partial: C \to G$ is a map from bigroup $C = C_1 \cup C_2$ to $G = G_1 \cup G_2$ such that $\partial_1: C_1 \to G_1$ and $\partial_2: C_2 \to G_2$ are group homomorphisms where ∂_1 and ∂_2 are restriction of ∂ on C_1 and C_2 respectively, i.e. $\partial_1 = \partial / C_1$ and $\partial_2 = \partial / C_2$. Kamil in [6] defined a crossed module of bigroups (C, G, ∂) is bigroup homomorphism $\partial : C \to G$ of bigroups $\mathrm{C} = \mathrm{C}_1 \cup \mathrm{C}_2$ and $= G_1 \cup G_2$ such that $(C_1, G_1, \partial_1, \theta_1)$ and $(C_2, G_2, \partial_2, \theta_2)$ are a crossed modules of groups, where $\partial_1 = \partial / \mathrm{C}_1$ and $\partial_2 = \partial / \mathrm{C}_2$. is also defined a morphism $(\mu, \eta): (C, G, \partial) \to (H, D, \delta)$ of bigroups is a pair of bigroup homomorphisms such that crossed modules of $(\mu_1, \eta_1): (C_1, G_1, \partial_1) \to (H_1, D_1, \delta_1)$ and $(\mu_2, \eta_2): (C_2, G_2, \partial_2) \to (H_2, D_2, \delta_2)$ are a morphism of crossed modules of groups where $\mu_1 = \mu / \mathrm{C}_1$, $\mu_2 = \mu / \mathrm{C}_2$, $\eta_1 = \eta / \mathrm{G}_1$ and $\eta_2 = \partial / \mathrm{G}_2$.
crossed modules of bigroups and morphisms as defined above form a category ,$CModBiGrps$**.**

2. Bicomplexes

PROPOSITION 2.1. For any complex of groups (C_*, μ_*), the sequence of the inners of the groups C_n for all $n \in Z$ and morhpisms between them;

... $\to Inn(C_{n+1}) \xrightarrow{\bar{\mu}_{n+1}} Inn(C_n) \xrightarrow{\bar{\mu}_n} Inn(C_{n-1}) \to \cdots$, where $\bar{\mu}_n(f_{c_n}) = f_{\mu_n(c_n)}$ for all $c_n \in C_n$, is a complex and denoted by $(Inn(C)_*, \bar{\mu}_*)$.

Proof: for all $f_{c_{n+1}} \in Inn(C_{n+1})$,

$$\bar{\mu}_n\, \bar{\mu}_{n+1}\left(f_{c_{n+1}}\right) = \bar{\mu}_n\left(\bar{\mu}_{n+1}\left(f_{c_{n+1}}\right)\right) = \bar{\mu}_n\left(f_{\mu_{n+1}(c_{n+1})}\right) = f_{\mu_n \mu_{n+1}(c_{n+1})} = f_0 \,. \qquad \blacksquare$$

According to proposition 2.3 in [**6**], each group C can be viewed as extended group $C - Inn(C)$ and vice-versa, thereby it could be exists a concept of complex of extended groups(or simply, extended complex of groups) which we will in next definition, for any complex and which is given an isomorphic between the category, $Comp$, of complexes and the category, $Comp - InnComp$, of extended complexes.

DEFINITION 2.2. A sequence of extended groups and extended group homomorphisms between them ... $\to C_{n+1} - Inn(C_{n+1}) \xrightarrow{\beta_{n+1}} C_n - Inn(C_n) \xrightarrow{\beta_n} C_{n-1} - Inn(C_{n-1}) \to \cdots$ is called complex of extended groups by inner of the groups C_n for all $n \in Z$, which is denoted by $(C - Inn(C)_*, \beta_*)$ if ... $\to C_{n+1} \xrightarrow{\mu_{n+1}} C_n \xrightarrow{\mu_n} C_{n-1} \to \cdots$ and ... $\to Inn(C_{n+1}) \xrightarrow{\acute{\mu}_{n+1}} Inn(C_n) \xrightarrow{\acute{\mu}_n} Inn(C_{n-1}) \to \cdots$ are complexes of groups, where $\mu_n = \beta_n / C_n$ and $\acute{\mu}_n = \beta_n / Inn(C_n)$ for all $n \in Z$. (For shortening in this paper, we will called it *extended complex*).

DEFINITION 2.3. A map $\rho = \left(\acute{\rho}, \acute{\acute{\rho}}\right) = \left(\{\acute{\rho}_n\}, \{\acute{\acute{\rho}}_n\}\right): (C - Inn(C)_*, \beta_*) \to (G - Inn(G)_*, \delta_*)$ is called an extended chain map if a pair $\acute{\rho} = \{\acute{\rho}_n\}: C_* \to G_*$ and $\acute{\acute{\rho}} = \{\acute{\acute{\rho}}_n\}: (Inn(C)_*) \to (Inn(G)_*)$ are chain maps such that $\rho_n \,/\, C_n = \acute{\rho}_n$ and $\rho_n \,/\, Inn(C_n) = \acute{\acute{\rho}}_n$ for all $n \in Z$.

PROPOSITION 2.4. The category of complexes of groups, $Comp$, and the category of extended complexes, $Comp - InnComp$, are isomorphic.

Proof:- define a functor $F: Comp \longrightarrow Comp - InnComp$ as follows:

(i) $F\left((C_*, \mu_*)\right) = (C - Inn(C)_*, \bar{\bar{\mu}}_*)$ where $\bar{\bar{\mu}}_n = \mu_n / C_n$ and $\bar{\bar{\mu}}_n = \bar{\mu}_n / Inn(C_n)$ for all $n \in Z$ and $(C_*, \mu_*) \in obComp$, and

(ii) $F(\partial = \{\partial_n\}) = (\rho, \acute{\rho}) = (\{\rho_n\}, \{\acute{\rho}_n\})$ for all $\partial \in Mor_{Comp}\left((C_*, \mu_*), (G_*, \eta_*)\right)$, where $\rho = \partial: (C_*, \mu_*) \to (G_*, \eta_*)$ and $\acute{\rho}: (Inn(C)_*, \bar{\mu}_*) \to (Inn(G)_*, \bar{\eta}_*)$ where $\acute{\rho}_n = \bar{\partial}_n$ for all $n \in Z$. Since ∂ is a chain map, then for all $f_{c_n} \in Inn(C_n)$;

$$\bar{\partial}_{n-1}\bar{\mu}_n\left(f_{c_n}\right) = \bar{\partial}_{n-1}\left(\bar{\mu}_n\left(f_{c_n}\right)\right) = \bar{\partial}_{n-1}\left(f_{\mu_n(c_n)}\right) = f_{\partial_{n-1}(\mu_n(c_n))} = f_{\partial_{n-1}\mu_n(c_n)} =$$
$$f_{\eta_n\partial_n(c_n)} = \bar{\eta}_n\left(f_{\partial_n(c_n)}\right) = \bar{\eta}_n\left(\partial_n\left(f_{c_n}\right)\right) = \bar{\eta}_n\partial_n\left(f_{c_n}\right)$$

i.e. $\bar{\partial}_{n-1}\bar{\mu}_n = \bar{\eta}_n\partial_n$ for all $n \in Z$, and hence $\acute{\rho} = \{\acute{\rho}_n\}: (Inn(C)_*, \bar{\mu}_*) \to (Inn(G)_*, \bar{\eta}_*)$ is a chain map (which is can be seen in this paper as extended chain map).

Also, define a functor $T: Comp - InnComp \longrightarrow Comp$as follows:

(i) $T(C - Inn(C)_*) = C_*$ for all $C - Inn(C)_* \in obComp - InnComp$, and

(ii) $T\left(\rho = \left(\acute{\rho}, \acute{\acute{\rho}}\right)\right) = \acute{\rho}$ for all $\rho = \left(\acute{\rho}, \acute{\acute{\rho}}\right) \in Mor_{Comp-InnComp}\,(C - Inn(C)_*, G - Inn(G)_*)$.

Therefore , it is easy to see that $TF = I_{Comp}$ and $FT = I_{Comp-InnComp}$. Thus $Comp \approx Comp - InnComp$. ■

The concept of extended complexes in the previous definition can be extend to a complexes of bigroups as we shall show that in the beginning of the next definition.

DEFINITION 2.5.

A *complex of bigroups* (C_*, μ_*) is a sequence of bigroups and bigroup homomorphisms between them $\ldots \to C_{n+1} \xrightarrow{\mu_{n+1}} C_n \xrightarrow{\mu_n} C_{n-1} \to \cdots$, where $C_n = \acute{C}_n \cup \acute{\acute{C}}_n$ for all $n \in Z$, Such that $\ldots \to \acute{C}_{n+1} \xrightarrow{\acute{\mu}_{n+1}} \acute{C}_n \xrightarrow{\acute{\mu}_n} \acute{C}_{n-1} \to \cdots$ and $\ldots \to \acute{\acute{C}}_{n+1} \xrightarrow{\acute{\acute{\mu}}_{n+1}} \acute{\acute{C}}_n \xrightarrow{\acute{\acute{\mu}}_n} \acute{\acute{C}}_{n-1} \to \cdots$ are complexes of groups, where $\acute{\mu}_n = \mu_n / \acute{C}_n$ and $\acute{\acute{\mu}}_n = \mu_n / \acute{\acute{C}}_n$ for all $n \in Z$ (in this paper can be seen as bicomplexes of groups).

DEFINITION 2.6. A map $\rho = (\acute{\rho}, \acute{\acute{\rho}}) = (\{\acute{\rho_n}\}, \{\acute{\acute{\rho_n}}\}): (C_*, \mu_*) \to (G_*, \eta_*)$ is called a bichain map if a pair $\acute{\rho} = \{\acute{\rho_n}\}: \acute{C}_* \to \acute{G}_*$ and $\acute{\acute{\rho}} = \{\acute{\acute{\rho_n}}\}: \acute{\acute{C}}_* \to \acute{\acute{G}}_*$ are chain maps such that $\acute{\rho_n} = \rho_n \,/\, \acute{C}_n$ and $\acute{\acute{\rho_n}} = \rho_n \,/\, \acute{\acute{C}}_n$ for all $n \in Z$.

Taking objects and morphisms as defined above, we obtain the category, $BiComp$, of bicomplexes of groups. Note that, $Comp - InnComp \subseteq BiComp$, and since $Comp \approx Comp - InnComp$, we deduce that $Comp \subseteq BiComp$, i.e. the category of complexes of groups is embedding (as a subcategory) in the category of bicomplexes of groups (via isomorphisms of categories).

3. Crossed Modules of Bicomplexes

According to proposition 2.3, we can there is an isomorphic between the category of crossed modules of complexes, $CModComp$, and the category of (which we shall call in this paper) crossed modules of extended complexes, $CModComp - InnComp$, which we will give in the next proposition.

PROPOSITION 3.1. The category of crossed modules of complexes, $CModComp$, and the category of crossed modules of extended complexes, $CModComp - InnComp$, are isomorphic.

Proof:- define a functor $F: CModComp \longrightarrow CModComp - InnComp$ as follows:

(i) $F\big((C_*, G_*, \partial)\big) = \big(C - Inn(C)_*, G - Inn(G)_*, \rho = \{\acute{\rho}, \acute{\acute{\rho}}\}\big)$ for all $(C_*, G_*, \partial) \in obCModComp$, where $\acute{\rho} = \partial: C_* \to G_*$ and $\acute{\acute{\rho}} = \bar{\rho}: Inn(C)_* \to C - Inn(G)_*$.

Note that, $\acute{\acute{\rho}} = \bar{\rho}: Inn(C_n) \to Inn(G_n)$ with the action of $Inn(G_n)$ on $Inn(C_n)$ via the action of G_n on C_n ($f_{g_{f_c}} = f_{g_c}$ for all $f_g \in Inn(G)$ and $f_c \in Inn(C)$) is a crossed modules of groups for all $n \in Z$, from [6]. And

For any $f_{c_n} \in Inn(C_n)$, $f_{g_n} \in Inn(G_n)$;

$$\acute{\acute{\mu}}_n\left(f_{{g_n}_{f_{c_n}}}\right)$$

$$= \acute{\acute{\mu}}_n\left(f_{{g_n}_{c_n}}\right) = f_{\mu_n\left({g_n}_{c_n}\right)}$$

$$= f_{\eta_n(g_n)_{\mu_n(c_n)}} = f_{\eta_n(g_n)_{f_{\mu_n(c_n)}}}$$

$$= \acute{\acute{\eta}}_n\big(f_{g_n}\big)_{\acute{\acute{\mu}}_n(f_{c_n})} .$$

This implies that $\acute{\acute{\rho}} = \bar{\rho}: Inn(C)_* \to C - Inn(G)_*$ is a crossed modules of complexes and hence $\big(C - Inn(C)_*, G - Inn(G)_*, \rho = \{\acute{\rho}, \acute{\acute{\rho}}\}\big)$ is a crossed modules of extended complexes i.e. $F\big((C_*, G_*, \partial)\big) = \big(C - Inn(C)_*, G - Inn(G)_*, \rho = \{\acute{\rho}, \acute{\acute{\rho}}\}\big) \in obCModComp - InnComp$.

(ii) $F(\alpha,\sigma) = \left((\acute{\alpha}, \acute{\acute{\alpha}}), (\acute{\sigma}, \acute{\acute{\sigma}})\right)$ for all $(\alpha,\sigma) \in Mor_{CModComp-InnComp}\left((C_*, G_*, \partial), (H_*, D_*, \omega)\right)$ where $\acute{\alpha} = \alpha$, $\acute{\sigma} = \sigma$, $\acute{\acute{\alpha}} = \bar{\alpha}$ and $\acute{\acute{\sigma}} = \bar{\sigma}$.

Note that; from proof above; $\acute{\acute{\alpha}}_n\left(f_{g_{n_{f_{c_n}}}}\right) = \acute{\acute{\sigma}}_n\left(f_{g_n}\right)_{\acute{\acute{\alpha}}_n(f_{c_n})}$ for any $f_{c_n} \in Inn(C_n)$, $f_{g_n} \in Inn(G_n)$ and this shows that $(\acute{\acute{\alpha}}, \acute{\acute{\sigma}}) : \left(Inn(C)_*, Inn(G)_*, \acute{\acute{\rho}}\right) \to \left(Inn(H), Inn(D), \acute{\acute{\gamma}}\right)$ is morphism of crossed modules of complexes.

Therefore $\left((\mu, \bar{\mu}), (\eta, \bar{\eta})\right)$ is morphism of crossed modules of extended complexes, i.e.

$$F(\alpha,\sigma) = \left((\acute{\alpha}, \acute{\acute{\alpha}}), (\acute{\sigma}, \acute{\acute{\sigma}})\right) \in Mor_{CModComp-InnComp}\left(\left(C - Inn(C)_*, G - Inn(G)_*, \rho = \{\acute{\rho}, \acute{\acute{\rho}}\}\right), \left(H - Inn(H)_*, D - Inn(D)_*, \vartheta = \{\acute{\vartheta}, \acute{\acute{\vartheta}}\}\right)\right).$$

Also, define a functor $T: CModComp - InnComp \longrightarrow CModComp$ as follows:

(i) $T\left(\left(C - Inn(C)_*, G - Inn(G)_*, \rho = \{\acute{\rho}, \acute{\acute{\rho}}\}\right)\right) = (C_*, G_*, \acute{\rho})$ for all $\left(C - Inn(C)_*, G - Inn(G)_*, \rho = \{\acute{\rho}, \acute{\acute{\rho}}\}\right) \in obCModComp - InnComp$, and

(ii) $T\left((\acute{\alpha}, \acute{\acute{\alpha}}), (\acute{\sigma}, \acute{\acute{\sigma}})\right) = (\acute{\alpha}, \acute{\sigma})$ for all $\left((\acute{\alpha}, \acute{\acute{\alpha}}), (\acute{\sigma}, \acute{\acute{\sigma}})\right) \in Mor_{CModComp-InnComp}\left(\left(C - Inn(C)_*, G - Inn(G)_*, \rho = \{\acute{\rho}, \acute{\acute{\rho}}\}\right), \left(H - Inn(H)_*, D - Inn(D)_*, \vartheta = \{\acute{\vartheta}, \acute{\acute{\vartheta}}\}\right)\right)$.

Therefore, it is easy to see that $TF = I_{CModComp}$ and $FT = I_{CModComp-InnComp}$. Thus $CModComp \approx CModComp - InnComp$. ■

The concept of crossed module of extended complexes in the previous preposition can be seen as extended crossed module of complexes , and which we can extend to a crossed module of bicomplexs as we shall show that in the beginning of the next definition.

DEFINITION 3.2. Let (C_*, μ_*) and (G_*, η_*) be a bicomplexes, where $C_n = \acute{C}_n \cup \acute{\acute{C}}_n$ and $G_n = \acute{G}_n \cup \acute{\acute{G}}_n$ for all $n \in Z$. A bichain map $\partial = \left(\acute{\partial} = \{\acute{\partial}_n\}, \acute{\acute{\partial}} = \{\acute{\acute{\partial}}_n\}\right) : (C_*, \mu_*) \to (G_*, \eta_*)$ is called a *crossed modules of bicomplexes* denoted by $(C_*, G_*, \partial, \theta)$ (or simply as (C_*, G_*, ∂)), if $\left(\acute{C}_*, \acute{G}_*, \acute{\partial}, \acute{\theta}\right)$ and $\left(\acute{\acute{C}}_*, \acute{\acute{G}}_*, \acute{\acute{\partial}}, \acute{\acute{\theta}}\right)$ are a crossed modules of complexes, where $\acute{\partial_n} = \partial_n \,/\, \acute{C}_n$ and $\acute{\partial_n} = \partial_n \,/\, \acute{C}_n$ for all $n \in Z$. If in the above definition, $\left(\acute{C}_*, \acute{G}_*, \acute{\partial}\right)$ and $\left(\acute{\acute{C}}_*, \acute{\acute{G}}_*, \acute{\acute{\partial}}\right)$ are a precrossed module of complexes, we call (C_*, G_*, ∂) a precrossed module of bicomplexes.

PROPOSITION 3.3. A bichain map $\partial = \left(\acute{\partial} = \{\acute{\partial}_n\}, \acute{\acute{\partial}} = \{\acute{\acute{\partial}}_n\}\right) : (C_*, \mu_*) \to (G_*, \eta_*)$ is called a *crossed modules of bicomplexes* iff

1- (C_n, G_n, ∂_n) is a crossed modules of bigroups for all $n \in Z$.

2- $\mu_n\left({}^{g_n}c_n\right) = {}^{\eta_n(g_n)}\mu_n(c_n)$) where $\mu_n \,/\, \acute{C}_n = \acute{\mu}_n$, $\mu_n \,/\, \acute{\acute{C}}_n = \acute{\acute{\mu}}_n$, $\eta_n \,/\, \acute{G}_n = \acute{\eta}_n$ and $\eta_n \,/\, \acute{\acute{G}}_n = \acute{\acute{\eta}}_n$ for all $n \in Z$.

Proof:-Clearly.

DEFINITION 3.4. A morphism $(\alpha,\sigma) = \left((\acute{\alpha},\acute{\acute{\alpha}}),(\acute{\sigma},\acute{\acute{\sigma}})\right): (C_*,G_*,\partial) \to (H_*,D_*,\delta)$ of crossed modules of bicomplexes is a pair of bichain maps such that $(\acute{\alpha},\acute{\sigma}): \left(\acute{C}_*,\acute{G}_*,\acute{\partial}\right) \to \left(\acute{H}_*,\acute{D}_*,\acute{\delta}\right)$ and $\left(\acute{\acute{\alpha}},\acute{\acute{\sigma}}\right): \left(\acute{\acute{C}}_*,\acute{\acute{G}}_*,\acute{\acute{\partial}}\right) \to \left(\acute{\acute{H}}_*,\acute{\acute{D}}_*,\acute{\acute{\delta}}\right)$ are a morphism of crossed modules of complexes where $\acute{\alpha}_n = \alpha_n \,/\, \acute{C}_n$, $\acute{\sigma}_n = \sigma_n \,/\, \acute{G}_n$, $\acute{\acute{\alpha}}_n = \acute{\alpha}_n \,/\, \acute{\acute{C}}_n$ and $\acute{\acute{\sigma}}_n = \sigma_n/\acute{\acute{G}}_n$ for all $n \in Z$.

Taking objects and morphisms as defined above, we obtain the category $CModBiComp$ of crossed modules of bicomplexes. Note that, $CModComp - Inn(Comp) \subseteq CModBiComp$, and since $CModComp \approx CModComp - Inn(Comp)$, we deduce that $CModComp \subseteq CModBiComp$, i.e., the category of crossed modules of complexes is embedding (as a subcategory) in the category of crossed modules of bicomplexes (via isomorphisms of categories).

EXAMPLES 3.5. (1) Let $\left(\acute{K}_*,\acute{\gamma}_*\right)$ be a normal subbicomplex of (C_*,μ_*) such that C_n has an action on $\acute{K}_n$ by conjugations for all $n \in Z$. This together with the inclusion bichain map $i_{\acute{K}_*} = \left\{i_{\acute{K}_n}\right\}: \left(\acute{K}_*,\acute{\gamma}_*\right) \hookrightarrow (C_*,\mu_*)$ is a crossed module of bicomplexes $\left(\acute{K}_*,C_*,i_{\acute{C}_*}\right)$. Accordingly, any bicomplex (C_*,μ_*) may be regarded as a crossed module of bicomplexes in two ways via the identity bichain map or the inclusion bichain map, i.e., $\left(C_*,C_*,I_{C_*}\right)$ and $\left(1_{C_*},C_*,i_{1_{C_*}}\right)$ respectively, where 1_{C_*} denotes the trivial bicomplex.

(2) Any bichain map of abelian bicomplexes $\partial = \{\partial_n\}: (C_*,\mu_*) \to (G_*,\eta_*)$ is a crossed module of bicomplexes with respect to the trivial action of G_n on C_n for all $n \in Z$.

The generalization of crossed module of bicomplexes for crossed modules of bigroups starts as follow. It is plain that each bigroup C can be viewed as a length zero bicomplex and vice-versa. This shows that the category $BiGrps$ of bigroups is isomorphic to the category $BiComp^{(0)}$ of length zero bicomplexes, i.e., $BiGrps \approx BiComp^{(0)}$, and since $BiComp^{(0)}$ is a subcategory of $BiComp$, therefore $BiGrps$ can be embedded in $BiComp$ as a subcategory. Accordingly, each crossed module of bigroups (C,G,∂) can also be viewed as a chain map ∂_0 of length zero bicomplexes C_0 and G_0 ,

$$\begin{tikzcd} C_0 \arrow[d, "\partial_0(*)"'] & \cdots \arrow[r] & O \arrow[r] \arrow[d, equal] & C \arrow[r] \arrow[d, "\partial"'] & O \arrow[r] \arrow[d, equal] & \cdots \\ G_0 & \cdots \arrow[r] & O \arrow[r] & G \arrow[r] & O \arrow[r] & \cdots \end{tikzcd}$$

which we shall call it in this paper a crossed module of length zero bicomplexes (C_0,G_0,∂_0) (in the sense that each vertical homomorphism of $(*)$ is a crossed module of bigroups and each pair of parallel horizontal homomorphisms of $(*)$ represents a morphism of crossed modules of bigroups, i.e., all crossed module information are encoded in the above diagram). In this case the category of crossed modules of bigroups $CModBiGrps$ and the category $CModBiComp^{(0)}$ of crossed modules of length zero bicomplexes are isomorphic, i.e.,$CModBiGrps \approx CModBiComp^{(0)}$. Note that, $CModBiComp^{(0)} \subseteq CModBiComp$, and since $CModBiGrps \approx CModBiComp^{(0)}$, we deduce that $CModBiGrps \subseteq CModBiComp$, i.e., the category of crossed modules of bigroups is embedding (as a subcategory) in the category of crossed modules of bicomplexes (via isomorphisms of categories).

4. Adjoint Pair of Functors

LEMMA 4.1. There are two covariant functors;

(1) The forgetful functor $F: CModBiComp \to BiComp$ derfined by:

(i) $F\big((C_*, G_*, \partial)\big) = C_*$, for all $(C_*, G_*, \partial) \in ObCModBiComp$.

(ii) $F\left((\alpha, \sigma) = \left((\acute{\alpha}, \acute{\acute{\alpha}}), (\acute{\sigma}, \acute{\acute{\sigma}})\right)\right) = \alpha = (\acute{\alpha}, \acute{\acute{\alpha}})$, for all $(\alpha, \sigma) = \left((\acute{\alpha}, \acute{\acute{\alpha}}), (\acute{\sigma}, \acute{\acute{\sigma}})\right) \in Mor_{CModBiComp}\big((C_*, G_*, \partial), (H_*, D_*, \delta)\big)$.

(2) $L: BiComp \to CModBiComp$ defined by:

(i) $L(C_*) = (C_*, C_*, I_{C_*})$ for all $C_* \in ObBiComp$.

(ii) $L\left(\alpha = (\acute{\alpha}, \acute{\acute{\alpha}})\right) = (\alpha, \alpha) = \left((\acute{\alpha}, \acute{\acute{\alpha}}), (\acute{\alpha}, \acute{\acute{\alpha}})\right)$ for all $\alpha = (\acute{\alpha}, \acute{\acute{\alpha}}) \in Mor_{BiComp}(C_*, G_*)$.

THEOREM 4.2. $L: BiComp \to CModBiComp$ is a left adjoint functor of $F: CModBiComp \to BiComp$.

Proof. We shall show that there is a natural isomorphism $\Phi: Mor_{BiCModComp}(L-, -) \to Mor_{BiComp}(-, F-)$, where

$Mor_{BiCModComp}(L-, -)$, $Mor_{BiComp}(-, F-): BiComp^{op} \times CModBiComp \to S$ are bifunctors, the notation $BiComp^{op}$ denotes the opposite (or dual) category of $BiComp$, and S the category of sets, defined respectively by the following compositions;

$$BiComp^{op} \times CModBiComp \xrightarrow{L^{op} \times I_{CModBiComp}} CModBiComp^{op} \times CModBiComp \xrightarrow{E_{CModBiComp}} S$$

and $BiComp^{op} \times CModBiComp \xrightarrow{I_{BiComp^{op}} \times F} BiComp^{op} \times BiComp \xrightarrow{E_{BiComp}} S$.

Define a function $\Phi: Mor_{CModBiComp}(L-, -) \to Mor_{BiComp}(-, F-)$ as follows;

for all $A = \left(C_*^{\ op}, (G_*, H_*, \partial)\right) \in Ob(BiComp^{op} \times CModBiComp)$,

$\Phi(A): Mor_{CModBiComp}\left((C_*, C_*, I_{C_*}), (G_*, H_*, \partial)\right) \to Mor_{BiComp}(C_*, G_*)$, is defined by $\Phi(A)\left((\alpha,\sigma) = \left((\acute{\alpha}, \acute{\acute{\alpha}}), (\acute{\sigma}, \acute{\acute{\sigma}})\right)\right) = \alpha = (\acute{\alpha}, \acute{\acute{\alpha}})$, for all $(\alpha,\sigma) = \left((\acute{\alpha}, \acute{\acute{\alpha}}), (\acute{\sigma}, \acute{\acute{\sigma}})\right) \in Mor_{CModBiComp}\left((C_*, C_*, I_{C_*}), (G_*, H_*, \partial)\right)$. We first show that Φ is a natural transformation. To do this, let $\left(\varrho^{op}, (r,t)\right) \in Mor_{CModBiComp}(A, B)$, where $A = \left(\left({C_*}^{op}, (G_*, H_*, \partial)\right)\right)$ and $B = \left({D_*}^{op}, (E_*, K_*, \lambda)\right)$. It is enough to show the commutativity of the following diagram

$$\begin{tikzcd}
A \arrow[d, "{(\varrho^{op}, (r,t))}"] & Mor_{CModBiComp}\left((C_*, C_* I_{C_*}), (G_*, H_*, \partial)\right) \arrow[r, "\Phi(A)"] \arrow[d, "{E_{CModBiComp}((\varrho^{op}, \varrho^{op}), (r,t))}"] & Mor_{BiComp}(C_*, G_*) \arrow[d, "{E_{BiComp}(\varrho^{op}, r)}"] \\
B & Mor_{CModBiComp}\left((D_*, D_* I_{D_*}), (E_*, K_*, \lambda)\right) \arrow[r, "\Phi(B)"] & Mor_{BiComp}(D_*, E_*)
\end{tikzcd}$$

Let $(\alpha,\sigma) \in Mor_{CModBiComp}\left((C_*, C_* I_{C_*}), (G_*, H_*, \partial)\right)$. Therefore

$$\left(E_{BiComp}(\varrho^{op}, r)\, \Phi(A)\right)(\alpha,\sigma) = E_{BiComp}(\varrho^{op}, r)(\alpha) = □□□ \ = \ □(□)(□□□, □□□)$$

$$= \ □(□)\left((□,□)(□,□)(□,□)\right) = \left(□(□)\ □_{□□□□□□□□m□}\left((□^{□□}, □^{□□}), (□,□)\right)\right)(□,□).$$

Also, define a function □: □□□$_{□□□□□□□}$(−, □ −) → □□□$_{□□□□□□□□□□□}$(□−, −) as follows; for all □ = $\left(□_*^{□□}, (□_*, □_*, □)\right)$ ∈ □□(□□□□□□□$^{□□}$ × □□□□□□□□□□□),

$$□(□): □□□_{□□□□□□□}(□_*^{□□}, □_*) \to □□□_{□□□□□□□□□□□}\left((□_*, □_*, □_{□_*}), (□_*, □_*, □)\right)$$

is defined by □(□)(□) = (□, □□) for all □ ∈ □□□$_{□□□□□□□}$($□_*^{□□}$, $□_*$). Note that □(□)(□) = (□, □□): $(□_*, □_*, □_{□_*}) \to (□_*, □_*, □)$ is indeed a morphism of crossed modules of complexes since □ and □□ are chain maps, $(□_□ □_□)□_{□_□} = □_□ □_□$ and $□_□(^{□_□}□_□) = {}^{□_□ □_□(□_□)}□_□(□_□)$ for all $□_□, □_□ \in □_□$ and all □ ∈ □. We turn now to show that □ is a natural transformation. , let $\left(□^{□□}, (□,□)\right) \in □□□_{□□□□□□□□□□□}(□,□)$, where □ = $\left(□_*^{□□}, (□_*, □_*, □)\right)$ and □ = $\left(□_*^{□□}, (□_*, □_*, □)\right)$. It is enough to show the commutativity of the following diagram.

□ □□□$_{□□□□□□□}$(□$_*$, □$_*$) □(□) □□□$_{□□□□□□□□□□□}$((□$_*$, □$_*$, □$_{□_*}$), (□$_*$, □$_*$, □))

(□$^{□□}$, (□, □)) □$_{□□□□□□□}$(□$^{□□}$, □) □$_{□□□□□□□□□□□}$((□$^{□□}$, □$^{□□}$), (□, □))

□ □□□$_{□□□□□□□}$(□$_*$, □$_*$) □(□) □□□$_{□□□□□□□□□□□}$((□$_*$, □$_*$, □$_{□_*}$), (□$_*$, □$_*$, □))

Let □ ∈ □□□$_{□□□□□□□}$(□$_*$, □$_*$). Therefore

(□$_{□□□□□□□□□}$((□$^{□□}$, □$^{□□}$), (□, □)) □(□))(□) = □$_{□□□□□□□□□}$((□$^{□□}$, □$^{□□}$), (□, □))(□, □□)

= (□, □)(□, □□)(□, □) = (□□□, □(□□)□) = (□□□, (□□́)□□) = (□□□, (□́□)□□) = □(□) (□□□) = □(□) (□$_{□□□□□}$(□$^{□□}$, □)(□)) = (□(□) □$_{□□□□□}$(□$^{□□}$, □))(□).

To proceed the proof, we need only to show that □□ = □$_{□□□□□□□□□□□□□□}$(□−,−) and

Let □ = ((□$_*$, □$_*$)$^{□□}$, ((□́$_*$, □́$_*$), (□́$_*$, □́$_*$), □́)) ∈ □□(□□□□□□$^{□□}$ × □□ = □$_{□□□_B□□□□□□}$(−,□−). □□□□□□□□□□). We need to prove the commutativity of the following diagram

□□□$_{□□□□□□□}$(□$_*$, □$_*$) □(□) □□□$_{□□□□□□□□□□□}$((□$_*$, □$_*$, □$_{□_*}$), (□$_*$, □$_*$, □))

□$_{□□□□□□□□□□}$(−,□−)(□) □(□)

□□□$_{□□□□□□□}$(□$_*$, □$_*$)

Let □ ∈ □□□$_{□□□□□□□}$(□$_*$, □$_*$). Therefore

(□(□)□(□))(□) = □(□)(□, □́□) = □ = □$_{□□□□□□□□□□(□_*,□_*),(□́_*,□́_*)}$(□)

= (□$_{□□□□□□□□□□}$(−,□−)(□))(□).

Likewise, □□ = □$_{□□□□□□□□□□□□□□}$(□−,−). Hence □ is a left adjoint functor of □. ■

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