Inclusion Properties for Certain Subclass of Meromorphic Multivalent Functions and Applications on the Electromagnetic cloaking

الخواص الضمنية لفئة جزئيه من الدوال الميروموروفيةالمتعددة التكافؤ وتطبيقاتها على الاخفاء الكهرومغناطيسي

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Abstract.

For several subclasses of multivalent meromorphic functions in the punctured unit disc having a pole at the origin of p order, the subordination and some of inclusion properties are studied. Through combinations and iterations of operator $\mathcal{T}_{\lambda,p,t}^{n,m}(a,c,\delta)$ for normalized regular functions the subclasses under search are defined. In this paper the impact of the increase of diverse parameters on the size of the subclasses are discussed. Applications are specified for the subordination outcomes on the electromagnetic cloaking.

Keywords: Meromorphic function, Multivalent meromorphic function, Hadamard product, differential subordination, Electromagnetic cloaking, Refractive index

Mathematics subject classification 30C45.

المستخلص:

يتم دراسة التابعيه والخواص الضمنيه للدوال الميروموروفية والمتعددة التكافؤ المعرفة على قرص الوحدة المثقوب التي تمتلك قطب من الرتبة p عند نقطة الاصل .لقد تم تعريف فئات جزئية جديدة من فئة الدوال التحليلية المنتظمه قيد البحث من خلال دمج وتكرار للمؤثر $\mathcal{T}_{\lambda,p,t}^{n,m}(a,c,\delta)$. كمان تأثير زيادة البارامترات المتنوعة على حجم الفئات الجزئيه فد نوقشت في هذا البحث. واخيراطبقت نتائج التابعية على الاخفاء الكهرومغناطيسي.

1. Introduction and Definition

Let σ_P be the class of all functions f of the following form

$$f(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}a_{k-p}z^{k-p} \quad (p\in\mathbb{N}), \qquad (1.1)$$

which are regular in the following punctured open disk

$$\mathfrak{U}^{*}:=\mathfrak{U}-\{0\}=\{z\in\mathbb{C} \text{ and } 0<|z|<1\}.$$

For function $f\in\sigma_P$ given by (1.1) and $g\in\Sigma_P$ defined in the following form

$$g(z)=\frac{1}{z^p}+\sum_{k=1}^{\infty}a_{k-p}z^{k-p} \quad (z\in\mathfrak{U}^{*}),$$

the Hadamard product denoted by $f(z)*g(z)$ and defined below

$$f(z)*g(z)=\frac{z^pf(z)*z^pg(z)}{z^p}:=\frac{1}{z^p}+\sum_{2}^{\infty}a_{k-p}\,b_{k-p}z^{k-p}=g(z)*f(z)\,,\ (z\in\mathfrak{U}^{*}).$$

A regular function $f(z)$ is subordinate to regular function $G(z)$ if there exist Schwarz function $\mathcal{k}(z)$ which is regular and satisfying $\mathcal{k}(0) = 0, |\mathcal{k}(z)| < 1$ in $\mathfrak{U}$, such that

$$f(z) = G(\mathcal{k}(z)),$$

We denote this subordination as following

$$f \prec G \text{ or } f(z) \prec G(z) \quad (z \in \mathfrak{U}).$$

Furthermore, if the function G is univalent in $\mathfrak{U}$, then $f(z) \prec G(z)$ is equivalent to $f(0) = G(0)$ and $f(\mathfrak{U}) \subset G(\mathfrak{U})$.For more details on the concept of subordination, (see [1]).

Many authors have lately used Hurwitz-Lerech Zeta function and scrutinized several operators [2, 3]. In [2] El-Ashwah and Bulboaca defined the operator $\mathcal{L}_{p,d}^{s}$ by using Hurwitz-Lerech Zeta function as follows

$$\mathcal{L}_{p,d}^{s}: \Sigma_P \to \Sigma_P,$$

such that

$$\mathcal{L}_{p,d}^{s} f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{d}{k+p+d}\right)^s a_k z^k \qquad (1.2)$$

$$(s \in \mathbb{C}, d \in \mathbb{C}^* = \mathbb{C} - Z_0^- ;\ z \in \mathfrak{U}^*).$$

Setting $\mathcal{J}_{p,d}^{s}$ as follows

$$\mathcal{J}_{p,d}^{s}(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{k+p+d}{d}\right)^s z^k,$$

and

$$(\mathcal{J}_{p,d}^{s,\lambda} * \mathcal{J}_{p,d}^{s})(z) = \frac{1}{z^p(1-z)^\lambda}, \ (\lambda>0),$$

We get

$$\mathcal{J}_{p,d}^{s,\lambda}(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{d}{k+p+d}\right)^s \frac{(\lambda)_{k+p}}{(1)_{k+p}} z^k.$$

With the Hadamard product the operator $\mathcal{J}_{p,d}^{s,\lambda}$ defined by El-Ashwah and Hassan in [4] as follows

$$\mathcal{J}_{p,d}^{s,\lambda} f(z) = \mathcal{J}_{p,d}^{s,\lambda}(z) * f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{d}{k+p+d}\right)^s \frac{(\lambda)_{k+p}}{(1)_{k+p}} a_k z^k.$$

$$(\lambda>0, s \in \mathbb{C}, d \in \mathbb{C}^* = \mathbb{C} - Z_0^- ;\ z \in \mathfrak{U}^*),$$

The above relation can be written

$$\mathcal{J}_{p,d}^{s,\lambda} f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \left(\frac{d}{k+d}\right)^s \frac{(\lambda)_k}{(1)_k} a_{k-p} z^{k-p}, \qquad (1.3)$$

where $f \in \Sigma_P$ is in the form (1.1) and $(a)_k$ is the Pochammer symbol which is defined by

$$(a)_k = \begin{cases} 1 & for\ k = 0, a \in \mathbb{C} - \{0\}, \\ a(a+1)(a+2)\ldots\ldots(a+k-1) & for\ k \in \mathbb{N} = \{1,2,3\ldots.\}, \text{and } a \in \mathbb{C}. \end{cases}$$

Mishra and et al. [5] defined: For $f \in \Sigma_P$ of the form (1.1) set

$\mathcal{C}^0 f(z) = f(z),$

$\mathcal{C}^{(t,1)} f(z) = (1-t)f(z) + \frac{tz(-f(z))'}{p} = \frac{1}{z^p} + \sum_{k=1}^{\infty}\left(\frac{p-kt}{p}\right) a_{k-p}\, z^{k-p} = \mathcal{C}^t f(z)$, (t≥0),

for m = 2,3,4,…..

$$\mathcal{C}^{(t,m)} f(z) = \mathcal{C}^t\left(\mathcal{C}^{(t,m-1)} f(z)\right) = \frac{1}{z^p} + \sum_{k=1}^{\infty}\left(\frac{p-kt}{p}\right)^m a_{k-p} z^{k-p}, (z \in \mathfrak{U}^*). \quad (1.4)$$

From (1.3) and (1.4), we define the operator

$$\mathcal{T}_{p,d}^{s,n,m}(\lambda): \Sigma_P \to \Sigma_P$$

$$\mathcal{T}_{p,d}^{s,n,m}(\lambda) f(z) = \mathcal{J}_{p,d}^{s,\lambda}\mathcal{C}^{(t,m)} f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty}\left(\frac{(\lambda)_k}{(1)_k}\right)^n\left(\frac{d}{k+d}\right)^s\left(\frac{p-kt}{p}\right)^m a_{k-p} z^{k-p}, (z \in \mathfrak{U}^*) \quad (1.5)$$

where n=0,1,2….

Motivated by [5], we introduced this paper

By specializing parameters, we get the well-known operators introduced by several researchers as follows

1. For n =0, $s = 0$, the operator $\mathcal{T}_{p,d}^{0,0,m}(\lambda) = \mathcal{C}^{(t,m)}$ has been studied in [5].
2. Setting n=1and m =0, the operator $\mathcal{T}_{p,d}^{s,1,m}(\lambda) = \mathcal{J}_{p,d}^{s,\lambda}$ has been studied in [4].
3. Setting n=0,s=0and t =1, we get meromorphic similar of the salagean operator [6].
4. For n =0, s = 0, m =1, we have the identity operator and for n = 0, s= 0, m =1 and t =1, we obtain the Alexander transform for meromorphic function.
5. For n=1,s=1,m=0,λ=1 and d=μ,

   $\mathcal{T}_{p,d}^{0,0,m}(\lambda) f(z) = \mathcal{F}_\mu f(z) = \frac{\mu}{z^{\mu+p}}\int_0^z t^{\mu+p+1} f(t)dt, (\mu>0)$. (cf. [7])
6. For n=1,s=α,m=0,λ=1 and d=1

   $\mathcal{T}_{p,1}^{\alpha,1,0}(1) f(z) = p^\alpha f(z) = \frac{1}{z^p \Gamma(\alpha)}\int_0^z (\log\frac{z}{t})^{\alpha-1} t^p f(t)dt, (\alpha>0)$..[8]
7. For n=1, s=γ,m=0,λ=1 and d=α

   $\mathcal{T}_{p,\alpha}^{\gamma,1,0}(1) f(z) = \mathrm{J}_{p,\alpha}^{\gamma} f(z) = \frac{\alpha^\gamma}{z^{p+\alpha}\Gamma(\gamma)}\int_0^z (\log\frac{z}{t})^{\gamma-1} t^{p+\alpha-1} f(t)dt, (\alpha, \gamma>0)$.[9]
8. For n =1, m =0, λ=1 and p =1

   $\mathcal{T}_{1,d}^{s,1,0}(1) f(z) = \mathcal{L}_d^s f(z) = \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{d}{k+1+d}\right)^s a^{k-p} z^{k-p}$. [10]

From (1.5), we show that: For n=1, $\mathcal{T}_{p,d}^{s,1,m}(\lambda) = \mathcal{T}_{p,d}^{s,m}(\lambda)$.

Utilizing (1.5), we can get the identities relationships of the operator $\mathcal{T}_{p,d}^{s,n,m}(\lambda)$, which are necessary for our study

$$z\left(\mathcal{J}_{p,d}^{s,n,m}(\lambda)f(z)\right)' = \frac{p(1-t)}{t}\mathcal{J}_{p,d}^{s,n,m}(\lambda)f(z) - \frac{p}{t}\mathcal{J}_{p,d}^{s,n,m+1}(\lambda)f(z) \quad (1.6)$$

$$z\left(\mathcal{J}_{p,d}^{s+1,n,m}(\lambda)f(z)\right)' = d\mathcal{J}_{p,d}^{s,n,m}(\lambda)f(z) - (d+p)\mathcal{J}_{p,d}^{s+1,n,m}(\lambda)f(z) \quad (1.7)$$

$$z\left(\mathcal{J}_{p,d}^{s,n,m}(\lambda)f(z)\right)' = \lambda\mathcal{J}_{p,d}^{s,n,m}(\lambda+1)f(z) - (\lambda+p)\mathcal{J}_{p,d}^{s,n,m}(\lambda)f(z) \quad (1.8)$$

Here, we start with the following definition:

Definition (1.1). A function $f \in \Sigma_p$ is said to be in the class $\mathbb{H}_{p,d}^{s,n,m}(\lambda, \alpha; A, B)$ if it satisfies the condition as follow:

$$\frac{1}{p-\alpha}\left(\frac{-z\left(\mathcal{J}_{p,d}^{s,n,m}(\lambda)f(z)\right)'}{\mathcal{J}_{p,d}^{s,n,m}(\lambda)f(z)} - \alpha\right) \prec \frac{1+Az}{1+Bz}, (z \in \mathfrak{U}) \quad (1.9)$$

where A,B and α are fixed parameters such that $(-1 \leq B<A \leq 1), 0 \leq \alpha<p$.

In particular case when n=1, we have

$$\mathbb{H}_{p,d}^{s,1,m}(\lambda, \alpha; A, B) \coloneqq \mathbb{H}_{p,d}^{s,m}(\lambda, \alpha; A, B).$$

Note that in special cases, the following subclasses are obtained for the parameters A, B, n, t, and m.

i. For $n=0, m=0, s=0, A=1, B=-1$; $\mathbb{H}_{p,d}^{0,0,0}(\lambda, \alpha; 1, -1)$ is the class of p-valent meromorphic starlike functions of order α.

ii. For $n=0, m=1, s=0, A=1, B=-1$ and $t=1$; $\mathbb{H}_{p,d}^{0,0,1}(\lambda, \alpha; 1, -1)$ is the class of p-valent meromorphic convex functions of order α.

In section four, we introduce some enough conditions under which subordination outcomes of the following formula

$$\left[\frac{az^p\mathcal{J}_{p,d}^{s,n,m}(\lambda+1)f(z) + bz^p\mathcal{J}_{p,d}^{s,n,m}(\lambda)f(z)}{a+b}\right]^\eta \prec q(z)$$

satisfy for $f \in \sigma_P$ and appropriate univalent function q in $\mathfrak{U}$. In addition, *w*e examine the results of subordination and its applications in electromagnetic cloaking.

2. Preliminaries

In our present investigation, we will need the following lemmas:

Lemma (2.1)[7]. Let $A, B, \gamma, \beta \in \mathbb{C}, \beta \neq 0, |B| \leq 1, A \neq B$.We suppose that, these constants fulfill the following relations

$$Re[\beta(1-A)(1-\bar{B}) + \gamma|1-B|^2] > 0 \quad (2.1)$$

and

$$Re[\beta(1-A)(1-\bar{B}) + \gamma|1-B|^2]\, Re[\beta(1+A)(1+\bar{B}) + \gamma|1+B|^2] - Im[\beta(\bar{B}-A) + \gamma|\bar{B}-B|^2] \geq 0,$$

or

$$Re[\beta(1+A)(1+\bar{B}) + \gamma|1+B|^2] \geq 0 \quad (2.2)$$

And

$$Re[\beta(1-A)(1-\bar{B}) + \gamma|1-B|^2] = \Im[\beta(\bar{B}-A) + \gamma|\bar{B}-B|^2] = 0.$$

Then the following equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=\frac{1+Az}{1+Bz}$$

has univalent solution given by

$$q(z)=\begin{cases}\dfrac{z^{\beta+\gamma}(1+Bz)^{\frac{\beta(A-B)}{B}}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\frac{\beta(A-B)}{B}}dt}-\dfrac{\gamma}{\beta} & (B\neq 0)\\ \dfrac{z^{\beta+\gamma}e^{\beta Az}}{\beta\int_0^z t^{\beta+\gamma-1}e^{\beta At}dt}-\dfrac{\gamma}{\beta} & (B=0)\end{cases} \qquad (2.3)$$

If $\phi(z)$ is regular in $\mathfrak{U}$ and satisfies

$$\phi(z)+\frac{z\phi'(z)}{\beta\phi(z)+\gamma}=\frac{1+Az}{1+Bz},$$

then

$$\phi(z)\prec q(z)\prec\frac{1+Az}{1+Bz}$$

and $q(z)$ is the best dominant.

Lemma (2.2)[11]. Let v be positive measure on interval [0, 1]. Let $h(z,t)$ be a complex valued function defined on $\mathfrak{U}\times[0,1]$ such that $h(.,t)$ is regular in $\mathfrak{U}$ for every $t\in[0,1]$ and $h(z,.)$ is v-integrable on[0,1] for each $z\in\mathfrak{U}$.

Addition, suppose that $\mathrm{Re}\big(h(z,t)\big)>0, h(-r,t)$ is real and

$$\mathrm{Re}\left(\frac{1}{h(z,t)}\right)\geq\frac{1}{h(-r,t)}\quad(|z|\leq r<1, t\in[0,1]).$$

If the function $\mathcal{H}(z)$ is defined by

$$\mathcal{H}(z)=\int_0^1 h(z,t)dv(t)$$

$$Re\left(\frac{1}{\mathcal{H}(z)}\right)\geq\frac{1}{\mathcal{H}(-r)}\quad(|z|\leq r<1). \qquad (2.4)$$

Lemma (2.3)[12]. For real and complex numbers a, b,c ($c\notin\bar{Z}_0$), we have

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b\colon c;\ z) \qquad (2.5)$$

Where

$$\mathrm{Re}(c)>Re(b)>0, z\in\mathfrak{U}),$$

$${}_2F_1(a,b\colon c;\ z)={}_2F_1(b,a\colon c;\ z), \qquad (2.6)$$

$${}_2F_1(a,b\colon c;\ z)=(1-z)^{-a}\,{}_2F_1\left(a,c-b\colon c;\ \frac{z}{z-1}\right), \qquad (2.7)$$

$$(b+1)\,{}_2F_1(1,b\colon b+1;\ z)=(b+1)+bz\,{}_2F_1(1,b+1\colon b+2;\ z) \qquad (2.8)$$

Lemma (2.4)[7]. Let q(z) be univalent function in $\mathfrak{U}$ and let Φ and θ be regular in a domain $\mathcal{D}$ containing q($\mathfrak{U}$) and $\Phi(\omega)\neq 0$ when $\omega\in$ q($\mathfrak{U}$).

Set

$$\mathcal{Q}(z) = z\mathrm{q}'(z)\Phi(\mathrm{q}(z))$$

and

$$\mathrm{h}(z) = \theta(\mathrm{q}(z)) + \mathcal{Q}(z).$$

Suppose that

(i) $\mathcal{Q}(z)$ is univalent starlike function in $\mathfrak{U}$.

(ii) $\mathrm{Re}\left(\frac{zh'(z)}{\mathcal{Q}(z)}\right) > 0$ for $z \in \mathfrak{U}$. If p is a regular with p(0) = q(0), $\mathrm{p}(\mathfrak{U}) \subseteq \mathcal{D}$ and

$$\theta(\mathrm{p}(z)) + z\mathrm{p}'(z)\Phi(\mathrm{p}(z)) \prec \theta(\mathrm{q}(z)) + z\mathrm{q}'(z)\, \Phi(\mathrm{q}(z)), \qquad (2.9)$$

then

$$p(z) \prec \mathrm{q}(z) \quad (z \in \mathfrak{U})$$

and $\mathrm{q}(z)$ is the best dominant.

Lemma (2.5) [13]. Let q be a convex univalent function in $\mathfrak{U}$ and let $\Psi \in \mathbb{C}$, $\gamma \in \mathbb{C}^*$ where $\mathbb{C}^* = \mathbb{C} - \{0\}$ with

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\mathrm{Re}\left(\frac{\Psi}{\gamma}\right)\right\}.$$

If $\mathrm{p}(z)$ is a regular in $\mathfrak{U}$ with p(0) = q(0) and

$$\Psi\mathrm{p}(z) + \gamma\, z\mathrm{p}'(z) \prec \Psi\mathrm{q}(z) + \gamma\, z\mathrm{q}'(z), \qquad (2.10)$$

then

$$\mathrm{p}(z) \prec \mathrm{q}(z) \quad (z \in \mathfrak{U})$$

and q is the best dominant.

Next, we discuss the inclusion relations and some of its properties for the class $\mathbb{H}_{p,d}^{s,n,m}(\lambda, \alpha; A, B)$.

3. Inclusion Properties of the function class $\mathbb{H}_{p,d}^{s,n,m}(\lambda, \alpha; A, B)$

In this section, we find some inclusion relations for the class $\mathbb{H}_{p,d}^{s,n,m}(\lambda, \alpha; A, B)$ with relation to changes in the parameters λ,m and s . In particular, we show that increasing λ by one decreasing the size of the class $\mathbb{H}_{p,d}^{s,n,m}(\lambda,\alpha;A, B)$, but increasing the parameters s or m or λ by one decreasing its size. We start with inclusion relations related the parameter m of the class $\mathbb{H}_{p,d}^{s,n,m}(\lambda,\alpha;A, B)$.

We start with the following inclusion relation with regard to the parameter λ of the class $\mathbb{H}_{p,d}^{s,n,m}(\lambda; \alpha; A, B)$

Theorem (3.1) **(i)** If $f \in \mathbb{H}_{,p,d}^{s,n,m}(\lambda + 1; \alpha; A, B)$ and

$$Q(\mathrm{z}) = \begin{cases} \int_0^1 u^{\lambda+p-1}\left(\frac{1+Buz}{1+Bz}\right)^{\frac{-(P-\alpha)(A-B)}{B}} du & B = 0 \\ \int_0^1 u^{\lambda+p-1}\, e^{-(P-\alpha)A(u-1)z}\, du & B \neq 0 \end{cases} \quad (3.1)$$

$$(-1 \leq B < A \leq 1, 0 \leq \alpha < p, \lambda > -p) \quad (3.2)$$

then

$$\frac{1}{p-\alpha}\left(\frac{-\mathrm{z}\left(\mathcal{T}_{\lambda,p,t}^{n,m}(a,c,\delta)f(z)\right)'}{\mathcal{T}_{\lambda,p,t}^{n,m}(a,c,\delta)f(z)} - \alpha\right) \prec \frac{1}{p-\alpha}\left[\lambda + p - \alpha - \frac{1}{\mathrm{Q}(\mathrm{z})}\right] = \mathrm{q}_1(\mathrm{z}) \prec \frac{1+Az}{1+Bz}, (z \in \mathfrak{U}) \quad (3.3)$$

and $\mathrm{q}_1(\mathrm{z})$ is the best dominant of (3.3). As a result of this

$$\mathbb{H}_{,\mathrm{p,d}}^{\mathrm{s,n,m}}(\lambda + 1, \alpha; \mathrm{A}, \mathrm{B}) \subseteq \mathbb{H}_{,\mathrm{p,d}}^{\mathrm{s,n,m}}(\lambda, \alpha; \mathrm{A}, B) \quad (3.4)$$

(ii) Moreover if the additional restricts, 0< B <1 and

$$\lambda > \frac{(p-\alpha)(A-B)}{B} - 1 \quad (3.5)$$

are satisfied then

$$\frac{1-|A|}{1-|B|} < \frac{1}{p-\alpha}\left(-\mathrm{Re}\left(\frac{\mathrm{z}\left(\mathcal{T}_{\lambda,p,t}^{n,m}(a,c,\delta)f(z)\right)'}{\mathcal{T}_{\lambda,p,t}^{n,m}(a,c,\delta)f(z)}\right) - \alpha\right) < \mathrm{r}. \quad (3.6)$$

where

$$r_1 = \frac{1}{p-\alpha}\{\lambda + p - \alpha - \frac{\lambda+p}{2F1\left(1,\frac{(P-\alpha)(A-B)}{B};\lambda+p+1;\frac{B}{B-1}\right)}\} \quad (3.7)$$

Proof (i). Let $f \in \mathbb{H}_{,\mathrm{p,d}}^{\mathrm{s,n,m}}(\lambda + 1, \alpha; \mathrm{A}, B)$

Set

$$\Psi(z) = \frac{1}{p-\alpha}\left(\frac{-\mathrm{z}\left(\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)\right)'}{\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)} - \alpha\right) \quad (3.8)$$

It is easy to show that $\Psi(z)$ is regular in $\mathfrak{U}$ and $\psi(0) = 1$.

Using the identity (1.8) in (3.8), we get

$$\Psi(z) = \frac{1}{\mathrm{P}-\alpha}\left(\lambda + \mathrm{P} - \alpha - \frac{\lambda\mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z)}{\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)}\right)$$

or equivalently

$$(p-\alpha)\,\Psi(z) - \lambda - p + \alpha = -\frac{\lambda\mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z)}{\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)} \quad (3.9)$$

Utilization, the logarithmic differentiation respect to z together for two sides of the relation (3.9), we get

$$\frac{(p-\alpha)z\,\Psi'(z)}{-(p-\alpha)\,\Psi(z)-\lambda-p+\alpha}+(p-\alpha)\Psi(z)=\frac{-z\left(\mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z)\right)'}{\mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z)}-\alpha$$

Multiplying by $\frac{1}{p-\alpha}$, we have

$$\Psi(z)+\frac{z\Psi'(z)}{-(p-\alpha)\,\Psi(z)-\lambda-p+\alpha}=\frac{1}{p-\alpha}\left(\frac{-z\left(\mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z)\right)'}{\mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z)}-\alpha\right)\prec\frac{1+Az}{1+Bz},(z\in\mathfrak{U}). \qquad (3.10)$$

Combining relation (3.10) together with Lemma (2.1), for special case $\beta=-(p-\alpha)$, and $\gamma=\alpha-\lambda-p$, we get

$$\Psi(z)\prec q_1(z)\prec\frac{1+Az}{1+Bz}$$

where $q_1(z)$ is the best dominant to (3.10). The proof of part (i) of Theorem (3.1) is complete.

Proof (ii). From (1.9) in Definition (1.1) , we observe that

$$\frac{1-|A|}{1-|B|}<\frac{1}{p-\alpha}\left(-\mathrm{Re}\left(\frac{z\left(\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)\right)'}{\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)}\right)-\alpha\right)$$

In order to prove the left side of (3.6), we note that an application of subordination concept in (1.9) yields

$$\frac{1}{p-\alpha}\left(-\mathrm{Re}\left(\frac{z\left(\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)\right)'}{\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)}\right)-\alpha\right)\leq \mathrm{Sup\,Re}\,\{q_1(z)\}$$

$$=\sup\left[\frac{1}{p-\alpha}\left(\lambda+p-\alpha-Re\left(\frac{1}{Q(z)}\right)\right)\right]$$

$$=\frac{1}{p-\alpha}\left(\lambda+p-\alpha-\inf Re\left(\frac{1}{Q(z)}\right)\right) \qquad (3.11)$$

In this case, we shall to compute only $\inf_{z\in\mathfrak{U}}\mathrm{Re}\left(\frac{1}{Q(z)}\right)$. We have $B\neq 0$, therefore by (3.1), we get

$$Q_1(z)=(1+B\,z)^{\eta}\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1+Buz)^{-\eta}\,du, \qquad (z\in\mathfrak{U})$$

where

$\eta = \frac{(P-\alpha)(A-B)}{B}$, $\beta = \lambda$ and $\gamma = \beta + 1$.

Also since $\gamma > \beta > 0$, by successively utilizing (2.5)-(2.7) of Lemma (2. 3), we have

$$Q_1(z) = \frac{\Gamma(\beta)}{\Gamma(\gamma)}\, {}_2F_1\left(1, \eta, \gamma, \frac{Bz}{Bz+1}\right) \tag{3.12}$$

Moreover the condition $\lambda > \frac{(P-\alpha)(A-B)}{B} - 1$, with $0 < B < 1$, implies that $\gamma > \eta > 0$.
Other application of (2.5) in Lemma (2.3) to (3.12), gives

$$Q_1(z) = \int_0^1 h(z,u)\,dv(u),$$

such that

$$h(z,u) = \frac{1+Bz}{1+(1-u)Bz} \quad (0 \leq u \leq 1)$$

and

$$dv(u) = \frac{\Gamma(\beta)}{\Gamma(\eta)\Gamma(\gamma-\eta)} u^{\eta-1} (1-u)^{\gamma-\eta-1} du.$$

is positive measure on $u \in [0,1]$. We note that Re $[h(z,u)] > 0$ and $h(\rho, u)$ is real for $0 \leq \rho < 1$ and $u \in [0,1]$. Hence, by Lemma(2. 2).

$$Re\left\{\frac{1}{Q(z)}\right\} \geq \frac{1}{Q(-\rho)}, \; (|z| \leq \rho < 1)$$

and

$$\inf_{z \in \mathfrak{U}} \left\{\frac{1}{Q(z)}\right\} = \inf_{z \in \mathfrak{U}} \frac{1}{Q(-\rho)} = \frac{1}{\int_0^1 h(-1,u)du} = \frac{1}{Q(-1)}$$

$$= \frac{\lambda+p}{2F1\left(1, \frac{(P-\alpha)(A-B)}{B}; \lambda+1; \frac{B}{B-1}\right)} \quad . \tag{3.13}$$

The right hand of inequality (3.6) follows from (3.13).The bound r_1 is sharp by principle subordination. The proof is complete.

The following theorem gives inclusion relationship with regard to the parameter m of the class $\mathbb{H}_{p,d}^{n,m}(\lambda, \alpha; A, B)$.

Theorem (3.2) Suppose that

$$\lambda \geq -p, \; 0 \leq \alpha < p, \; -1 \leq B < A \leq 1$$

and, A, B, t, p, α , satisfy

$$\frac{1+A}{1+B} < \frac{\alpha - \frac{p(1-t)}{t}}{p-\alpha}, \; t > 0. \tag{3.14}$$

If $f(z) \in \mathbb{H}_{p,d}^{n,m}(\lambda, \alpha; A, B)$ and the function Q defined on $\mathfrak{U}$ as following

$$. \; Q(z) = \begin{cases} \int_0^1 u^{\frac{-p}{t}-1} \left(\frac{1+Bzu}{1+Bz}\right)^{\frac{-(P-\alpha)(A-B)}{B}} du & B \neq 0 \\ \int_0^1 u^{\frac{-p}{t}-1} e^{-(P-\alpha)A(u-1)z}\, du & B = 0 \end{cases} \tag{3.15}$$

then

$$\frac{1}{p-\alpha}\left(\frac{-z(\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z))'}{\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)}-\alpha\right)\prec\frac{1}{p-\alpha}\left[\left(\alpha-\frac{p(1-t)}{t}\right)-\frac{1}{Q(z)}\right]:=q_2(z)\prec\frac{1+Az}{1+Bz},(z\in\mathfrak{U}) \quad (3.16)$$

and $q_2(z)$ is the best dominant of (3. 16). Therefore

$$\mathbb{H}_{p,d}^{s,n,m+1}(\lambda;\alpha;A,B)\subset\mathbb{H}_{p,d}^{s,n,m}(\lambda;\alpha;A,B) \quad (3.17)$$

Proof. Let $f\in\mathbb{H}_{p,d}^{s,n,m+1}(\lambda;\alpha;A,B)$

Suppose that

$$\psi(z)=\frac{1}{p-\alpha}\left(\frac{-z(\mathcal{T}_{\lambda,p,t}^{n,m+1}(a,c,\delta)f(z))'}{\mathcal{T}_{\lambda,p,t}^{n,m+1}(a,c,\delta)f(z)}-\alpha\right),(z\in\mathfrak{U}), \quad (3.18)$$

Using the relation (1.6) in (3.18), we obtain

$$\psi(z)=\frac{1}{p-\alpha}\left(\frac{\frac{t+\delta p}{\delta}\mathcal{T}_{\lambda,p,t}^{n,m+1}(a,c,\delta)f(z)-\frac{t}{\delta}\mathcal{T}_{\lambda,p,t}^{n,m}(a,c,\delta)f(z)}{\mathcal{T}_{\lambda,p,t}^{n,m}(a,c,\delta)f(z)}-\alpha\right)$$

or equivalently

$$(p-\alpha)\psi(z)-\frac{t+\delta p}{\delta}+\alpha=\frac{-\frac{t}{\delta}\mathcal{T}_{\lambda,p,t}^{n,m}(a,c,\delta)f(z)}{\mathcal{T}_{\lambda,p,t}^{n,m+1}(a,c,\delta)f(z)} \quad (3.19)$$

Utilizing the logarithmic differentiation of both sides of (3.19) with regard to z, we get

$$\frac{(p-\alpha)\,\psi'(z)}{(p-\alpha)\,\psi(z)-\frac{t+\delta p}{\delta}+\alpha}=\frac{(\mathcal{T}_{\lambda,p,t}^{n,m}(a,c,\delta)f(z))'}{\mathcal{T}_{\lambda,p,t}^{n,m}(a,c,\delta)f(z)}-\frac{-z(\mathcal{T}_{\lambda,p,t}^{n,m+1}(a,c,\delta)f(z))'}{\mathcal{T}_{\lambda,p,t}^{n,m+1}(a,c,\delta)f(z)} \quad (3.20)$$

Put (3.8) in (3.20), we obtain

$$\psi(z)+\frac{\psi'(z)}{-(p-\alpha)\psi(z)+\frac{t+\delta p}{\delta}-\alpha}=\frac{1}{p-\alpha}\left(\frac{-z(\mathcal{T}_{\lambda,p,t}^{n,m}(a,c,\delta)f(z))'}{\mathcal{T}_{\lambda,p,t}^{n,m}(a,c,\delta)f(z)}-\alpha\right)\prec\frac{1+Az}{1+Bz},(z\in\mathfrak{U}).$$

Put $\beta=-(p-\alpha)$, $\gamma=\frac{t+\delta p}{\delta}-\alpha$ and implicate Lemma (2.1), we obtain

$$\psi(z)\prec q_2(z)\prec\frac{1+Az}{1+Bz}$$

where the best dominant $q_2(z)$ is defined by (3.16). The proof is complete.

Next theorem gives the corresponding outcomes due to the parameter s.

Theorem (3.3) **(i)** If $f \in \mathbb{H}^{s,n,m}_{,p,d}(\lambda;\alpha;A,B)$ and

$$\frac{1+A}{1+B}<\frac{d-p+\alpha}{p-\alpha}, \tag{3.21}$$

then

$$\frac{1}{p-\alpha}\left(\frac{-z(\mathcal{J}^{n,m}_{\lambda,p,t}(a,c,\delta)f(z))'}{\mathcal{J}^{n,m}_{\lambda,p,t}(a,c,\delta)f(z)}-\alpha\right)\prec\frac{1}{p-\alpha}\left[d-p+\alpha-\frac{1}{Q(z)}\right]=q_3(z)\prec\frac{1+Az}{1+Bz},\ (z\in\mathfrak{U}) \tag{3.22}$$

Where

$$Q(z)=\begin{cases}\int_0^1 u^{\lambda+p-1}(\frac{1+Buz}{1+Bz})^{\frac{-(P-\alpha)(A-B)}{B}}du & B=0\\ \int_0^1 u^{\lambda+p-1}\,e^{-(P-\alpha)A(u-1)z}\,du & B\neq 0\end{cases} \tag{3.23}$$

$$(-1\le B<A\le 1,\ 0\le\alpha<p,\ \lambda>-p),$$

and $q_3(z)$ is the best dominant of (3.22). As a result of this

$$\mathbb{H}^{s,n,m}_{,p,d}(\lambda,\alpha;A,B)\subseteq\mathbb{H}^{s+1,n,m}_{,p,d}(\lambda,\alpha;A,B) \tag{3.24}$$

(ii) Moreover if the additional restrict, $0<B<1$ and

$$d-p+1>\frac{(p-\alpha)(A-B)}{B} \tag{3.25}$$

are satisfied then

$$\frac{1-|A|}{1-|B|}<\frac{1}{p-\alpha}\left(-\mathrm{Re}\left(\frac{z(\mathcal{J}^{s,n,m}_{p,d}(\lambda)f(z))'}{\mathcal{J}^{s,n,m}_{p,d}(\lambda)f(z)}\right)-\alpha\right)<r_3\,. \tag{3.26}$$

where

$$r_3=\frac{1}{p-\alpha}\left\{\alpha-(p+d)-\frac{d+p}{2F1\left(1,\frac{(P-\alpha)(A-B)}{B};1-p-d;\frac{B}{B-1}\right)}\right\}. \tag{3.27}$$

The bound $r_3$is the best possible.

Proof (i). Let $f\in\mathbb{H}^{s,n,m}_{,p,d}(\lambda,\alpha;A,B)$

Set

$$\chi(z)=\frac{1}{p-\alpha}\left(\frac{-z(\mathcal{J}^{s,n,m}_{p,d}(\lambda)f(z))'}{\mathcal{J}^{s,n,m}_{p,d}(\lambda)f(z)}-\alpha\right) \tag{3.28}$$

It is easy to show that $\chi(z)$ is regular in$\mathfrak{U}$ and $\chi(0)=1$.

Using the identity (1.7) in (3.28), we get

$$(p-\alpha)\chi(z)-(d+p)+\alpha = -\frac{d\mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z)}{\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)} \qquad (3.29)$$

Utilization, the logarithmic differentiation respect to z together for two sides of the relation (3.29), we get

$$\frac{(p-\alpha)z\,\chi'(z)}{-(p-\alpha)\,\chi(z)-(d-p)+\alpha}+(p-\alpha)\chi(z)=\frac{-z(\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z))'}{\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)}-\alpha$$

Multiplying by $\frac{1}{p-\alpha}$, we have

$$\chi(z)+\frac{z\chi'(z)}{-(p-\alpha)\,\chi(z)-(d+p)+\alpha}=\frac{1}{p-\alpha}\left(\frac{-z(\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z))'}{\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)}-\alpha\right)\prec\frac{1+Az}{1+Bz},(z\in\mathfrak{U}). \quad (3.30)$$

Combining relation (3.30) with Lemma (2.1), for special case $\beta=-(p-\alpha)$, and $\gamma=\alpha-(d+p)$, we get

$$\chi(z)\prec q_3(z)\prec\frac{1+Az}{1+Bz},\ (z\in\mathfrak{U}).$$

where q_3 is the best dominant to (3.30). The proof of part (i) of Theorem (3.4) is complete.

Proof (ii). For the purpose to prove (3.26), we utilize the similar technique used before. Write

$$Q_3(z)=(1+Bz)^{\eta}\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1+Buz)^{-\eta}\,du, \qquad (z\in\mathfrak{U})$$

$$=\frac{\Gamma(\beta)}{\Gamma(\gamma)}\,{}_2F_1\left(1,\eta,\gamma,\frac{BZ}{Bz+1}\right), \qquad (3.31)$$

where

$$\eta=\frac{(P-\alpha)(A-B)}{B},\ \beta=-d-p \text{ and } \gamma=\beta+1.$$

Moreover the condition d> $\frac{(P-\alpha)(A-B)}{B}$ +p-1, with 0<B<1, implies that $\gamma>\eta>0$.

Again application of (2.7) in Lemma (2.3) to (3.31), gives

$$Q_3(z)=\int_0^1 h(z,u)dv(u),$$

such that

$$h(z,u)=\frac{1+Bz}{1+(1-u)Bz} \quad (0\le u\le 1)$$

and

$$dv(u)=\frac{\Gamma(\beta)}{\Gamma(\eta)\Gamma(\gamma-\eta)}u^{\eta-1}(1-u)^{\gamma-\eta-1}du.$$

Hence, by Lemma (2.2)

$$\text{Re}\left\{\frac{1}{Q_3(z)}\right\}\ge\frac{1}{Q_3(-\rho)} \quad (|z|\le\rho<1)$$

and

$$\inf_{z\in\mathfrak{U}} \mathrm{Re}\{\frac{1}{Q_3(z)}\} = \inf_{z\in U} \frac{1}{Q_3\,(-r)} = \frac{1}{\int_0^1 h(-1,u)du} = \frac{1}{Q_3\,(-1)}$$

$$= \frac{-d-p}{2F1\left(1,\frac{(P-\alpha)(A-B)}{B};-d-p+1;\frac{B}{B-1}\right)} \quad . \qquad (3.32)$$

The right hand of inequality (3.26), follows from (3. 32).The bound r_3 is sharp by principle subordination. The proof of Theorem (3.4) is complete.

In this section, we derive some subordination results involving the operator $\mathcal{T}_{p,d}^{s,n,m}(\lambda)$

4. Subordination results

Theorem (4.1) Let $r \in \mathbb{C}^*$. Let q be convex univalent function in $\mathfrak{U}$ such that $q(0) = 1$, with

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -p\lambda Re\left(\frac{1}{r}\right)\right\}, r \neq 0. \qquad (4.1)$$

If $f\epsilon\sigma_p$ satisfies the subordination

$$\frac{r}{p} z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z) + \frac{p-r}{p} z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z) \prec q(z) + \frac{r}{p\lambda} zq'(z), \qquad (4.2)$$

then

$$z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z) \prec q(z) \qquad (4.3)$$

Proof: Define the function G by

$$G(z) := z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z) \qquad (4.4)$$

Differentiating (4.4) with respect to z, we get

$$zG'(z) = z^p z(\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z))' + P\, z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z). \qquad (4.5)$$

It follows from (1.8) and (4.5), that

$$zG'(z) = z^p \lambda \mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z) - \lambda\, z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z). \qquad (4.6)$$

It follows from (4.4) and (4.6), that

$$z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z) = \frac{1}{\lambda} zG'(z) + G(z).$$

From the subordination condition of (4.2), we have

$$\frac{r}{p\lambda} zG'(z) + G(z) \prec q(z) + \frac{r}{p\lambda} zq'(z) \qquad (4.7)$$

An applying of Lemma (2.5) to (4.7), with $\gamma = \frac{r}{p\lambda}$ and $\psi = 1$, leads to (4.3).

By specializing function for q (z), we obtain the following result.

Corollary (4.2) Let $r \in \mathbb{C}^*, -1 \leq B < A \leq 1$ and

$$\frac{|B|-1}{|B|+1} < P\lambda Re\left(\frac{1}{r}\right). \qquad (4.8)$$

$$\frac{r}{p} z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z) + \frac{p-r}{p} z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z) \prec \frac{1+Az}{1+Bz} + \frac{r}{p\lambda}\frac{(A-B)z}{(1+Bz)^2}, \qquad (4.9)$$

then

$$z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda) f(z) \prec \frac{1+Az}{1+Bz}. \quad (4, 10)$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (4.10) ■

Proof: Let $q(z) = \frac{1+Az}{1+Bz}$, we see that

.

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \frac{1-|B|}{1+|B|}.$$

Consequently

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0, P\lambda Re\left(\frac{1}{r}\right)\}.$$

By Theorem (4.1), we get

$$z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda) f(z) \prec \frac{1+Az}{1+Bz} \blacksquare$$

Thus the following Corollary would be thru, when we suppose that $q(z) = \frac{1+z}{1-z}$.

Corollary (4.3) Let $r \in \mathbb{C}^*, p = A = 1, B = -1$ and

$$Re\left(\frac{1}{r}\right) > 0$$

If $f \epsilon \sigma_1$ satisfies the subordination

$$rz\mathcal{T}_{p,d}^{s,n,m}(\lambda+1) f(z) + (1-r)z\mathcal{T}_{p,d}^{s,n,m}(\lambda) f(z) \prec \frac{1+z}{1-z} + \frac{r}{\lambda}\frac{2z}{(1-z)^2},$$

then

$$z\mathcal{T}_{p,d}^{s,n,m}(\lambda) f(z) \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant ■

Theorem (4.4) Let $r \in \mathbb{C}^*$. Let q be convex univalent function in $\mathfrak{U}$ such that $q(0) = 1$, with

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, \frac{1}{pd+d-rp} \quad Re\left(\frac{1}{r}\right)\right\}, r \neq 0. \quad (4.11)$$

If $f \epsilon \sigma_p$ satisfies the subordination

$$rd(pd+d-rp)z^p\mathcal{T}_{p,d}^{s-1,n,m}(\lambda) f(z) + \{1 - rd(pd+d-rp)z^p\mathcal{T}_{p,d}^{s,n,m}(\lambda) f(z) \prec q(z) + r(pd+d-\mathrm{rp})zq'(z), \quad (4.12)$$

then

$$z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z) \prec q(z) \quad (4.13)$$

Proof: Define the function $\mathfrak{p}$ by

$$\mathfrak{p}(z) := z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z) \quad (4.14)$$

Differentiating (4.14) with respect to z, we get

$$z\mathfrak{p}'(z) = z^p z(\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z))' + Pz^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z). \quad (4.15)$$

It follows from (1.7) and (4.15) and subordination condition of (4.12), we have

$$\mathfrak{p}(z) + r(pd + d - \mathrm{rp})\, z\mathfrak{p}'(z) \prec q(z) + r(pd + d - \mathrm{rp})\, zq'(z) \quad (4.16)$$

An applying of Lemma (2.5) to (4.16), with $\gamma = \frac{r}{d}$ and $\psi = 1$, leads to (4.13).

Taking q(z)=$\frac{1+Az}{1+Bz}$, in Theorem (4.4), we obtain the following Corollary.

Corollary (4.5) Let $r \in \mathbb{C}^*, -1 \le B < A \le 1$ and

$$\frac{|B|-1}{|B|+1} < \frac{1}{pd+d-rp} Re\left(\frac{1}{r}\right). \quad (4.17)$$

If $f \epsilon \sigma_p$ satisfies the subordination

$$rd(pd + d - rp)z^p \mathcal{T}_{p,d}^{s-1,n,m}(\lambda)f(z) + \{1 - rd(pd + d - \mathrm{rp})\}z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z) \prec$$

$$\frac{1+Az}{1+Bz} + r(pd + d - \mathrm{rp})\frac{(A-B)z}{(1+Bz)^2} \quad (4.18)$$

then

$$z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z) \prec \frac{1+Az}{1+Bz}. \quad (4.19)$$

and $\frac{1+Az}{1+Bz}$ is the best dominant ■

Corollary (4.6) Let $r \in \mathbb{C}^*, p = A = 1, B = -1$ and $\left(\frac{1}{r}\right) > 0$.

If $f \epsilon \sigma_1$ satisfies the subordination

$$rd(2d - r)z\mathcal{T}_{1,d}^{s,n,m}(\lambda)f(z) + \{1 - rd(2d - \mathrm{r})\}z\mathcal{T}_{1,d}^{s,n,m}(\lambda)f(z) \prec$$

$$\frac{1+z}{1-z} + r(2d - r)\frac{2z}{(1-z)^2},$$

then

$$z\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z) \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant ■

Theorem (4.7) Let q$(z) \neq 0$ be univalent function in $\mathfrak{U}$ such that $q(0) = 1$. Le t $\eta \in \mathbb{C}^*$, $a, b \epsilon \mathbb{C}$ $and\ a + b \neq 0$. with

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0, z \in \mathfrak{U}. \quad (4.20)$$

If $f \epsilon \sigma_p$ satisfies the next conditions

$$\frac{az^p \mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z)+bz^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)}{a+b} \neq 0, z \in \mathfrak{U}.$$

$$1+\eta\left[p+\frac{az(\mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z))'+bz(\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z))'}{a\mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z)+b\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)}\right] \prec 1+\frac{zq'(z)}{q(z)}, \qquad (4.21)$$

then

$$[\frac{az^p \mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z)+bz^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)}{a+b}]^\eta \prec q(z), \qquad (4.22)$$

and q(z) is the best dominant.

Proof: Let us consider function $\mathcal{P}$ defined by

$$\mathcal{P}(z) := [\frac{az^p \mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z)+bz^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)}{a+b}]^\eta, \ \eta \neq 0, a+b \neq 0 \qquad (4.23)$$

Then $\mathcal{P}$ is a regular in$\mathfrak{U}$, $\mathcal{P}(0) = q(0) = 1$.

Differentiating (4.23) logarithmically, we obtain

$$\frac{z\mathcal{P}'(z)}{\mathcal{P}(z)} = \eta\left[p+\frac{az(\mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z))'+bz(\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z))'}{a\mathcal{T}_{p,d}^{s,n,m}(\lambda+1)f(z)+b\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)}\right]. \qquad (4.24)$$

Setting

$$\phi(w) = 1 \ and \ \psi(w) = \frac{1}{w}$$

By observing that $\phi(w)$ is a regular in $\mathbb{C}$ and $\psi(w) \neq 0$ is regular in $\mathbb{C} - \{0\}$.

Moreover, we let

$$\mathrm{Q}(\mathrm{z}) := \mathrm{zq}'(\mathrm{z})\, \psi(q(z)) = \frac{\mathrm{zq}'(\mathrm{z})}{q(z)}$$

and

$$h(z) := \phi(q(z)) + \mathrm{Q}(\mathrm{z}) = 1 + \mathrm{Q}(\mathrm{z}).$$

From (4.20), we see that $\mathrm{Q}(\mathrm{z})$ is starlike univalent in$\mathfrak{U}$, and

$$\mathrm{Re}\left(\frac{\mathrm{zh}'(\mathrm{z})}{\mathrm{Q}(\mathrm{z})}\right) = \mathrm{Re}\left(\frac{\mathrm{zQ}'(\mathrm{z})}{\mathrm{Q}(\mathrm{z})}\right) = \mathrm{Re}\left\{1+\frac{zq''(z)}{\mathrm{q}'(\mathrm{z})} - \frac{zq'(z)}{\mathrm{q}(\mathrm{z})}\right\} > 0, z \in \mathfrak{U}.$$

Using (4.24) in (4.21), we get

$$1+\frac{z\mathcal{P}'(z)}{\mathcal{P}(z)} \prec 1+\frac{zq'(z)}{q(z)}.$$

It is equivalent to

$$\phi(\mathcal{P}(z)) + z\mathcal{P}'(z)\psi(\mathcal{P}(z)) \prec \phi(q(z)) + zq'(z)\psi(q(z)).$$

Hence by Lemma (2.5), we obtain

$$\mathcal{P}(z) \prec q(z), (z \in \mathfrak{U}).$$

And q(z) is the best dominant. The proof of Theorem (4.7) is complete.

We obtain the following corollary by taking $a = 0, b = 1$ and $q(z) = \frac{1+Az}{1+Bz}$, in Theorem (4.7).

Corollary (4.8) Let , $-1 \le B < A \le 1$ and $\eta \in \mathbb{C}^*$. Let $f \epsilon \sigma_p$ and suppose that

$$z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z) \neq 0$$

If

$$1 + \eta\left[p + \frac{z(\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z))'}{\mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)}\right] \prec 1 + \frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$[z^p \mathcal{T}_{p,d}^{s,n,m}(\lambda)f(z)]^\eta \prec \frac{1+Az}{1+Bz},$$

and $\frac{1+Az}{1+Bz}$ is the best dominant ■

Next, we choose the following special function

$$g_{\alpha,\beta}(z) = \frac{z}{k_{\alpha,\beta}(z)},$$

where

$$k_{\alpha,\beta}(z) = z(1-\alpha z)^{-2(1-\beta)}, \ z \in \mathfrak{U}, 0 < \beta \le 1, 0 \le \alpha < 1.$$

We will therefore take q(z) as the following form

$$q(z) = (1-\alpha z)^{2(1-\beta)}.$$

Corollary (4.9). Let $f(z)$ be univalent meromorphic starlike function in $\mathfrak{U}^*$ with $0 \le \alpha < 1$ and $0 < \beta \le 1$, then

$$[z\,f(z)]^\eta \prec (1-\alpha z)^{2(1-\beta)} \qquad (4.25)$$

Proof: From (4.21) in Theorem (4.7), we get

$$1 + \eta\left[1 + \frac{zf'(z)}{f(z)}\right] \prec 1 - 2\alpha(1-\beta)\frac{z}{1-\alpha z},$$

Hence, putting $a = 0, b = 1, \lambda = p = 1, s = m = 0$ and $q(z) = (1-\alpha z)^{2(1-\beta)}$, in Theorem (4.7), we get (4.25). The proof is complete■

Corollary (4.10). Let $f(z)$ be univalent meromorphic starlike function in $\mathfrak{U}^*$ with $|\tau| \le 1$, then

$$[z\,f(z)]^\eta \prec e^{\tau z}, \qquad (4.26)$$

Proof: Put $a = 0, b = 1, \lambda = p = 1, s = m = 0$ and $q(z) = e^{\tau z}$ in (4.21) of Theorem (4.7), to obtain

$$1 + \eta\left[1 + \frac{zf'(z)}{f(z)}\right] \prec 1 + e^{\tau z},$$

which is turn to give the target in (4.26). The proof is complete ■

5. Subordination and Electromagnetic Cloaking

In the field of defense and its applications, it is a matter of concealing things from the most important issues and preventing the response of the electromagnetic spectrum, which leads to unclear identity of the hidden body. Recent RCS studies discuss the hidden body's response to electromagnetic radiation. Electromagnetic cloaking has aroused increasing interest in the scientific community, especially amongst researchers who are developing so-called metamaterials - artificial composites having exotic electromagnetic properties.

In the mathematical sense, the two-dimensional cloak and cloaked object can be considered as simple connected regions in complex plane. Both regions are equivalent to conformal maps of the unit circle according to Riemann Mapping Theorem. Let the function $g(z)$ denote to the cloaked object and by the function $q(z)$ to the cloak then, we obtain

$$g(z) \prec q(z).$$

In Theorem (4.1), we consider the cloak function $q(z)$ is a regular univalent convex function. We know that a regular function $q(z)$ maps open unit disk $\mathfrak{U}$ on to convex region if and only if

$$\mathrm{Re}\left\{1+\frac{zq''(z)}{q'(z)}\right\}>0, (z\in\mathfrak{U}). \text{ [14]} \qquad (5.1)$$

In formula (1.5), the rate of the change of the angle between the polar axis and tangent vector at $z=re^{i\theta}, 0<r<1,$ on $f(|z|=r),$ as $0<\theta<2\pi.$

As per condition (4.2) in Theorem (4.1), and (4.12) in Theorem (4.4), we have the smallest possible cloak $q(z)$ for the cloak object g(z). Take a special cases of the Theorem (4.7), given in the Corollary (4.9) and Corollary (4.10). Through, the data of function

$$q_{\alpha,\beta}(z) = (1-\alpha z)^{2(1-\beta)}$$

and give special values for the parameters α *and* β ,we see that the geometric properties as the following below:

Alpha (α)	**Beta (β)**	**q(z)**	**Figure No.**
$\frac{2}{3}$	$\frac{2}{3}$	$\left[1-\frac{2}{3}cos\theta - i\left(\frac{2}{3}sin\theta\right)\right]^{\frac{2}{3}}$	5.1
$\frac{2}{3}$	$\frac{4}{5}$	$\left[1-\frac{2}{3}cos\theta - i\left(\frac{2}{3}sin\theta\right)\right]^{\frac{2}{5}}$	5.2
$\frac{5}{6}$	$\frac{4}{5}$	$\left[1-\frac{5}{6}cos\theta - i\left(\frac{5}{6}sin\theta\right)\right]^{\frac{2}{5}}$	5.3

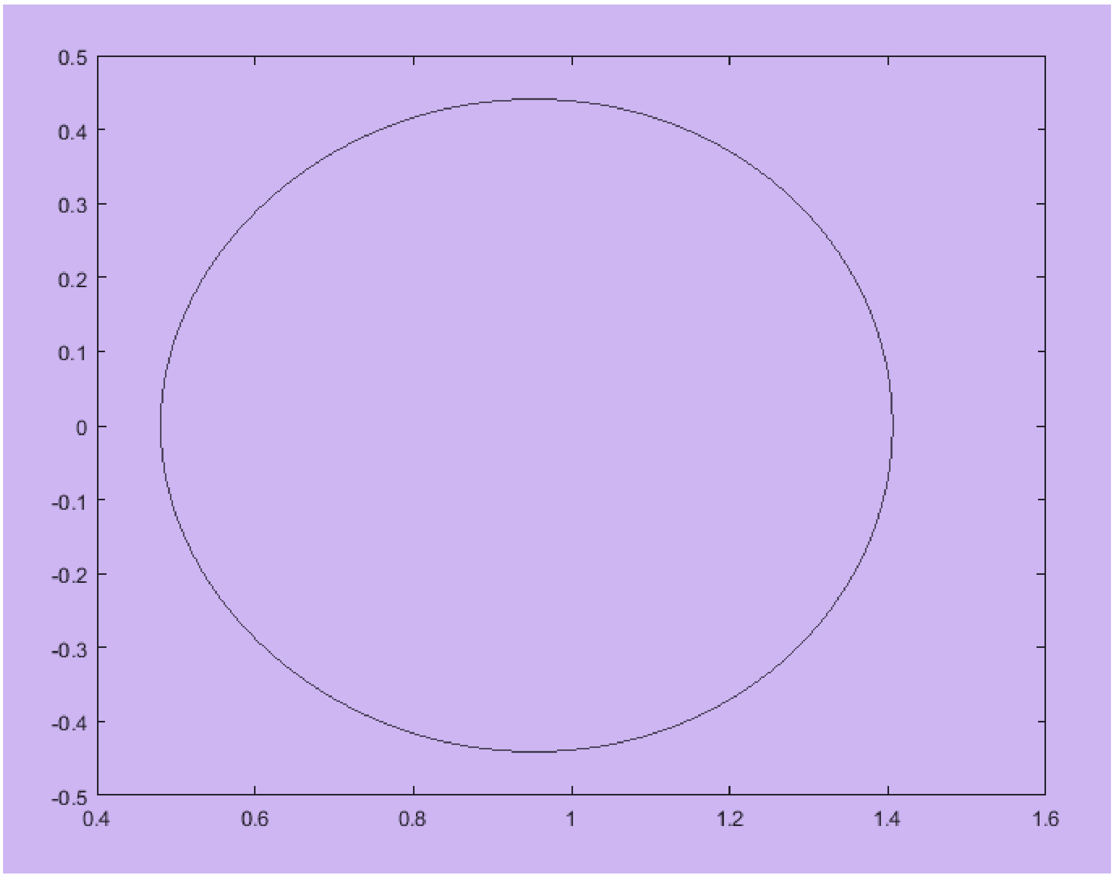

Figure 5.1

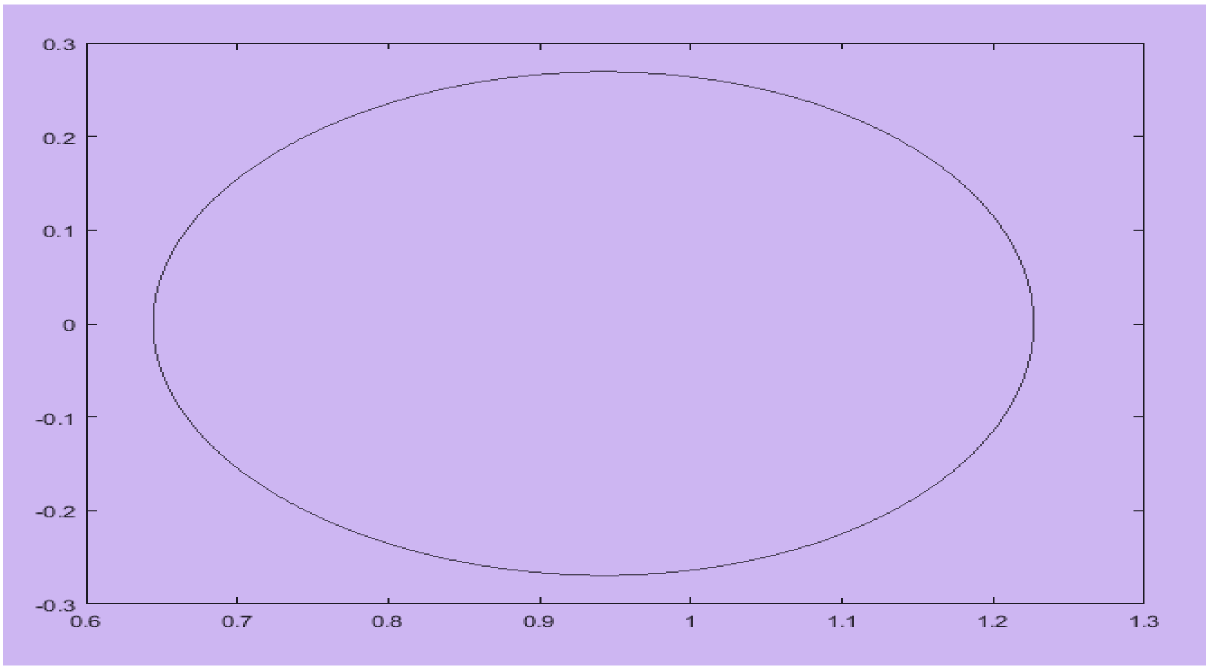

Figure 5.2

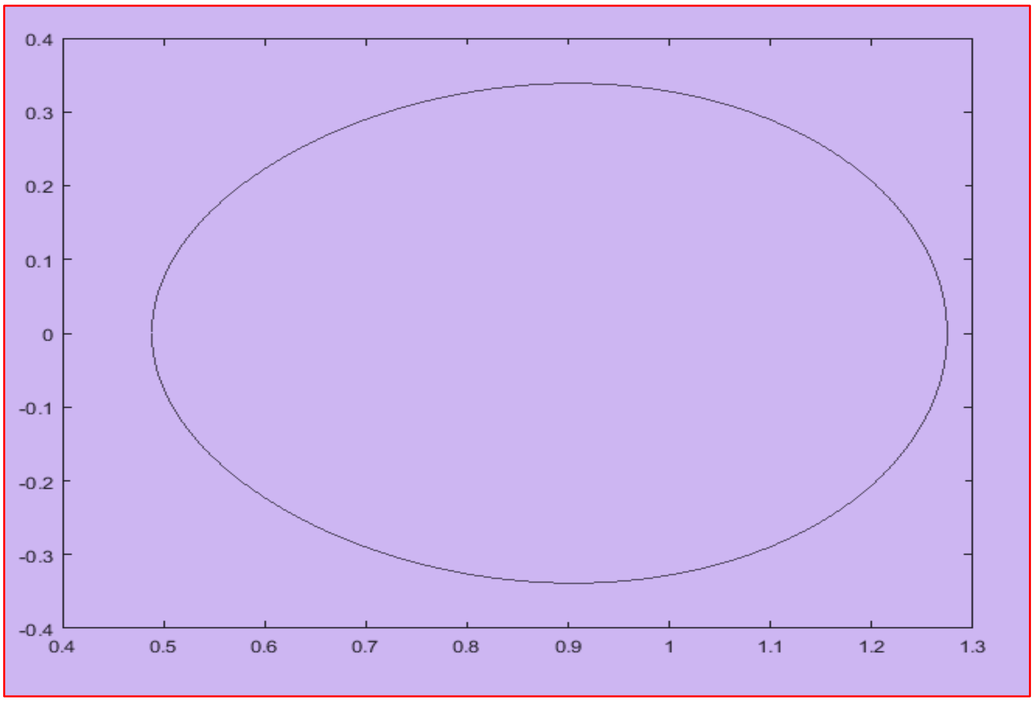

Figure 5.3

In Corollary (4.9), put $\eta = 1$and $\beta = 0$, we get the following cloak function

$$q(z)= (1-\alpha z\ )^2,$$

We can write the previous relation in another method

$$\mathrm{q(z)} = \ (1-\alpha e^{i\theta}\ )^2$$

$$= 1+\alpha^2 cos2\theta - 2\alpha sin\theta + i(\alpha^2 sin2\theta - 2\alpha sin\theta)$$

$$u_\alpha = 1+\alpha^2 cos2\theta - 2\alpha cos\theta,$$

$$v_\alpha = 2\alpha^2 cos\theta sin\theta - 2\alpha sin\theta.$$

Thus the enclosed region $(u_\alpha(\theta),\ v_\alpha(\theta))$ in this case represents full cardioid symmetric with respect to the real axis which means that the cloak is not a convex region.
Therefore, the function $g(z)$ is a candidate for the representation of the hidden object and $q(z)$= $(1-\alpha z\ )^2$ represents to smallest cloak function. These functions represent the following figure.

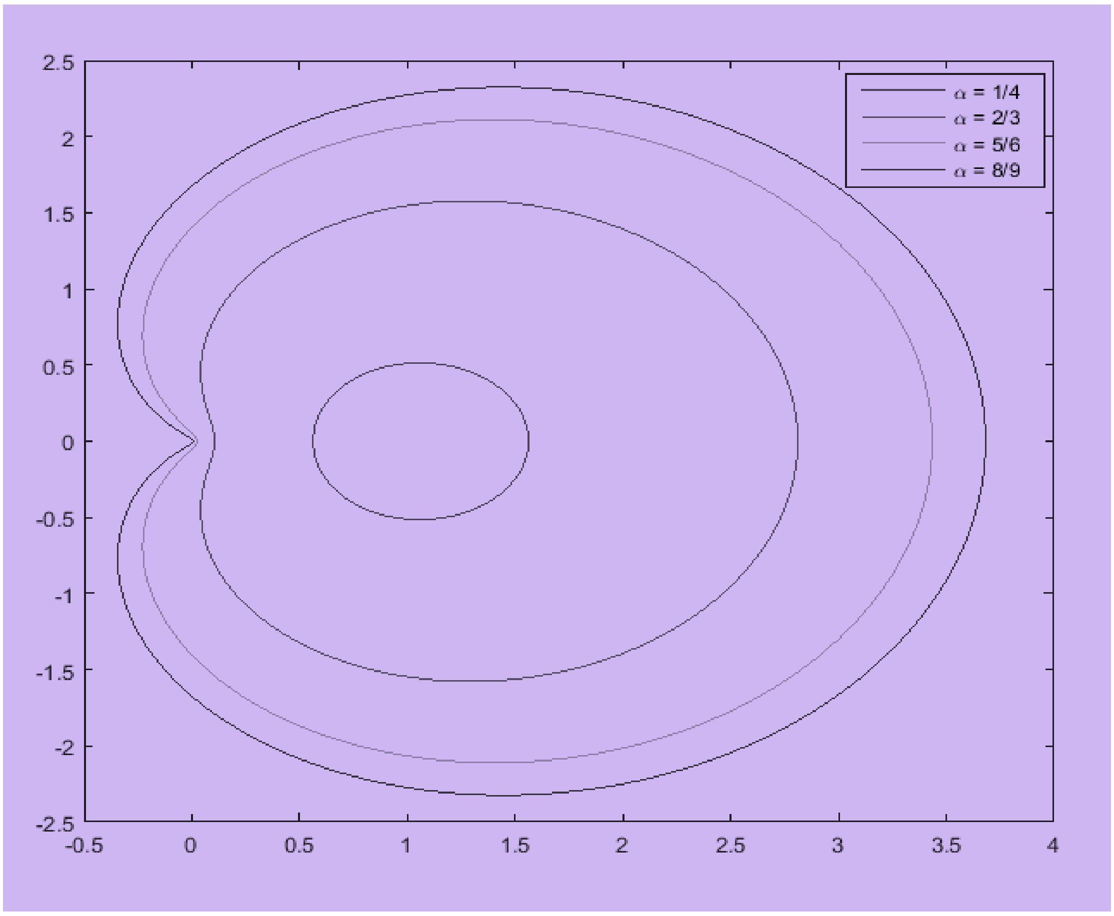

Figure 5.4

Finally, since the cloak depends on the hidden body and the rays reflected by the body may be cloak for not containing all reflected rays so it is better to be a three-dimensional cloak.

References:

[1] P.L. Duren, 1983, Univalent Functions, in: Graduate Texts in Mathematics, vol. 259, Springer-Verlag, New York.

[2] R. M. El-Ashwah and T. Bulboaca, Sandwich results for p-valent meromorphic functions associated with Hurwitz-Lerech Zeta function, (Submitted).

[3] A.R.S. Juma and M.H. Saloomi, 2018, Quasi-subordination conditions on bi-univalent functions involving Hurwitz-Lerch zeta functions, Science International, 30(1),127-131.

[4] R. M. El-Ashwah and H. Hassan, Some Third-Order Differential Subordination And Superordination Results of Some Meromorphic Functions using Hurwitz-Lerech Zeta Type operator, 2015, Ilirias Journal of mathematics, Volume 4, 1-15.

[5] A. K. Mishra, T. Panigrahi and R. K. Mishra, Subordination and inclusion theorems for subclasses of meromorphic functions with applications to electromagnetic cloaking, 2013,Math. Comput. Modelling, 57(3), 945-962.

[6] M.K. Aouf, T. Bulboacă, A.O. Mostafa, Subordination properties of p-valent functions defined by Sălăgean operator, 2010, Complex Var. Elliptic Equ. 55 ,185–199.

[7] S.S. Miller, P.T. Mocanu, 2000,Differential Subordinations: Theory and Applications, in: Monographs and Text books in Pure and Applied Mathematics, vol. 225, Marcel Dekker, New York.

[8] E. Aqlan, J. M. Jahangiri and S. R. Kulkarni, 2003, Certain integral operators applied to p-valent functions, J. Nat. Geom, 24, 111-120.

[9] R. M. El-Ashwah, M. K. Aouf, Applications of differential subordination on certain class of meromorphic p-valent functions associated with certain integral operator, 2012, Acta Univ. Apulensis, 31, 53–64.

[10] R. M. El-Ashwah, 2013, Inclusion relationships properties for certain subclasses of meromorphic functions associated with Hurwitz-Lerech Zeta function, 1980, Acta Univ. Apulensis 34, 191–205.

[11] D.R. Wilken, J. Feng, A remark on convex and starlike functions, J. London Math. Soc. (2) 21, 287–290.

[12] E.T. Whittaker, G.N. Watson,1927, A Course on Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With An Account of the Principal Transcendental Functions, Fourth ED., Cambridge University Press, Cambridge,.

[13] T.N. Shanmugam, S. Sivasubramanian, H.M. Srivastava, 2006, Differential sandwich theorems for certain subclasses of analytic functions involving multiplier transformations, Integral Transforms Spec. Funct. 17 (12) ,889–899.