Generating Exact Solutions for Focusing-Defocusing NLS System by Using First Integral Method

A. S. J. Al-Saif
sattaralsaif@yahoo.com
Ammar A.Al-salih
ammarabdalhussien@yahoo.com
Department of Mathematics, College of Education for Pure Sciences, Basrah University, Basrah, Iraq

Abstract

The aim of the study is to find exact solutions for focusing-defocusing NLS system by using the first integral method. This method depends on reciprocal algebra theory in getting an exact solution for nonlinear equations. Results show that the proposed method is effective and general.

المستخلص

الهدف من دراستنا هو ايجاد الحلول الدقيقة لـ (focusing-defocusing NLS system) باستخدام طريقة التكامل الاول. هذه الطريقة تعتمد على نظرية الجبر في الحصول على حل دقيق للمعادلات غير الخطية. النتائج بينت ان هذه الطريقة كفؤة و عامة.

Key words: Exact solution; first integral method; theory of commutative algebra; focusing-defocusing NLS system.

1.Introduction

Most of the phenomena that arise in real world are described by nonlinear differential and integral equations. Nonlinear equations are widely used to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma wave and chemical physics. Nonlinear equations also cover the cases of the following types: surface waves incompressible fluids, hydro magnetic waves in cold plasma, acoustic waves in inharmonic crystal, etc. However, they are usually very difficult to solve, either numerically or theoretically. An effective method is required to analyze the mathematical model that provides solutions conforming to physical reality. Wide variety of the powerful and direct methods to find all kinds of analysis solutions of nonlinear evolution equations have been developed such as the tanh function method [9], the extended tanh function method [10], the Jacobi elliptic function expansion method [7], the F-expansion method [4] and so on. The first integral method is one of many methods that used to find the exact solution for some partial differential equation and this method is proposed by Feng [5] to obtain new exact solution for some of these equation such as [2, 3,8].

The main aim of this paper is to apply the first integral method to solve the focusing-defocusing NLS system of nonlinear coupled one-dimensional partial differential equations. This system and the close form to it appear in many studies such as [1,6] and the references there in. An important goal of the present work is to show that the efficiency and ability of the first integral method for finding new forms for the exact solutions of nonlinear coupled one-dimensional partial differential equations of focusing-defocusing NLS system.

2.The first integral method

Let us consider the nonlinear system of partial differential equations with independent variables x and t and dependent variables u and v :

$R_1(u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{xx}, v_{xx}, \dots) = 0,$

$R_2(u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{xx}, v_{xx}, \dots) = 0.$ (1)

Applying the travelling wave solution transformation $u(x,t) = f(\xi)$ and $v(x,t) = g(\xi)$, where $\xi = x - ct$, now, by using chain rule we get:

$$\frac{\partial}{\partial t}(.) = -c\frac{d}{d\xi}(.), \frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.), \frac{\partial^2}{\partial x^2}(.) = \frac{d^2}{d\xi^2}(.), \dots (2)$$

Using (2) into (1) to transfer the partial differential equations into ordinary differential equations (ODEs) as follows:

$T_1(f, g, f', g', \dots\dots) = 0$,

$T_2(f, g, f', g', \dots\dots) = 0$ (3)

Where prime denotes the derivative with respect to the same variable ξ.

Make this system a single equation with one dependent variable of the second order, using the integration we have the equation as follows:

$H(f, f', f'') = 0$ (4)

We define new independent variables:

$X(\xi) = f(\xi),\ Y(\xi) = f'(\xi)$ (5)

This leads to a system of ordinary differential equations:

$$\begin{cases} X'(\xi) = Y(\xi), \\ Y'(\xi) = F(X(\xi), Y(\xi)) \end{cases} \quad (6)$$

Now, the division theorem which is based on ring theory of commutative algebra is adopted to obtain one first integral to(6), which reduces (4) to a first-order integrable ordinary differential equation. Finally, an exact solution to (1) is established, through solving the resulting first-order integrable differential equation.

Division Theorem: Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials of two variables w and z in complex domain $C[w,z]$ and $P(w,z)$ is an irreducible polynomial in $C[w,z]$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C[w,z]$ such as:

$$Q(w,z) = P(w,z)G(w,z)$$

3. Exact solution for focusing-defocusing NLS system

Let us consider the focusing-defocusing NLS system [6] as follows:

$$iu_t + u_{xx} + (|u|^2 - |v|^2)u = 0\ , \quad (7a)$$

$$iv_t + v_{xx} + (|u|^2 - |v|^2)v = 0 \quad (7b)$$

Assume that eq. (7) has traveling wave solutions in the form:

$$u(x,t) = f(\xi)\ , \quad v(x,t) = g(\xi)$$

where $\xi = x - ct$ and c is a constant. By using (2) equations (7a) and (7b) become:

$$-icf'(\xi) + f''(\xi) + \left(f^2(\xi) - g^2(\xi)\right)f(\xi) = 0 \quad (8a)$$

$$-icg'(\xi) + g''(\xi) + \left(f^2(\xi) - g^2(\xi)\right)g(\xi) = 0 \quad (8b)$$

Suppose $\alpha = -(f^2(\xi) - g^2(\xi))$, this implies that:

$g(\xi) = \mp\sqrt{f^2(\xi) + \alpha}$ (9)

Then, equation (8a) becomes:

$$f''(\xi) = icf'(\xi) + \alpha f(\xi) \quad (10)$$

Using (5) in (10), we obtain:

$$X'(\xi) = Y(\xi) \quad (11a)$$

$$Y'(\xi) = icY(\xi) + \alpha X(\xi) \quad (11b)$$

Now, we are applying the division theorem to seek the first integral method to equation (7). Suppose that $X(\xi)$ and $Y(\xi)$ are the nontrivial solutions to equations (11a) and (11b), and

$$q(X, Y) = \sum_{i=0}^{m} a_i(X)\, Y^i = 0$$

is an irreducible polynomial in the complex domain $C[X, Y]$ such as:

$$q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X) Y^i = 0 \quad (12)$$

where $a_i(x)(i = 0,1, \dots, m)$ are polynomials of X and all relatively prime in $C[X, Y]$, $a_m(X) \neq 0$. Equation (12) is also called the first integral to (11). Assuming that $m = 2$, note that $\frac{dq}{d\xi}$ is a polynomial in X and Y, and $q[X(\xi), Y(\xi)] = 0$ implies $\frac{dq}{d\xi} = 0$, due to the division theorem, there exists a polynomial $(g(X) + h(X)Y)$ in $C[X, Y]$ such as:

$$\frac{dq}{d\xi} = \frac{\partial q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial q}{\partial Y}\frac{\partial Y}{\partial \xi} = (g(X) + h(X)Y) \sum_{i=0}^{2} a_i(X)\, Y^i \quad (13)$$

Comparing the coefficients of $Y^i(i = 3,2,1,0)$ on both sides of equation (13), we get:

$a_2'(X) = a_2(X)h(X)$ (14a)

$a_1'(X) = a_2(X)g(X) + a_1(X)h(X) - 2ica_2(X)$ (14b)

$a_0'(X) = a_1(X)g(X) + a_0(X)h(X) - ica_1(X) - 2\alpha a_2(X)X$ (14c)

$$\alpha a_1(X)X = a_0(X)g(X) \quad (14d)$$

Since $a_2(X)$ is polynomial of X, then from (14a) we conclude that $a_2(X)$ is constant and $h(X) = 0$. To simplify, we take $a_2(X) = 1$, and balance the degrees of $g(X)$, $a_1(X)$ and $a_0(X)$, we conclude that $\deg g(X) = 0$ only. Now, we discuss this case:

If $\deg g(X) = 0$, suppose that $g(X) = A_1$, then we can find $a_1(X)$ and $a_0(X)$:

$a_1(x) = (A_1 - 2ic)X + B_0$ (15)

$$a_0(X) = d + (A_1 B_0 - icB_0)X + \frac{A_1^2 - 3icA_1 - 2c^2 - 2\alpha}{2} X^2 \quad (16)$$

where B_0 are arbitrary integration constants. By substituting $a_0(X)$ and $g(X)$ in (14d) and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations:

$$2\alpha A_1 - 2ic\alpha = \frac{A_1^3}{2} - \frac{3icA_1^2}{2} - A_1 c^2 \quad (17a)$$

$$\alpha B_0 = A_1^2\, B_0 - icB_0 A_1 \quad (17b)$$

$$A_1 d = 0 \quad (17c)$$

By solving the last algebraic equations, we obtain:

$$A_1 = 0\ , \qquad \alpha = 0 \qquad (18a)$$
$$A_1 = ic\ , \quad d = 0, \quad \alpha = 0 \qquad (18b)$$

By using the condition (18a) into (12), we obtain:

$$Y(\xi) = icX(\xi) + \frac{\sqrt{B_0^2 - 4d} - B_0}{2} \qquad (19a)$$
$$Y(\xi) = icX(\xi) - \frac{\sqrt{B_0^2 - 4d} + B_0}{2} \qquad (19b)$$

By combining (19a) with (11), we obtain the exact solution to (10) as follows:

$$f(\xi) = e^{ic(\xi+\xi_0)} + \frac{i\sqrt{B_0^2 - 4d} - iB_0}{2c} \qquad (20)$$

By substituting (20) in (9), we get:

$$g(\xi) = \mp\left(e^{ic(\xi+\xi_0)} + \frac{i\sqrt{B_0^2 - 4d} - iB_0}{2c}\right) \ (21)$$

where ξ_0 is an arbitrary constant. Then the exact solution to eq. (7) can be written as:

$$u(x,t) = e^{ic(x-ct+\xi_0)} + \frac{i\sqrt{B_0^2 - 4d} - iB_0}{2c} \qquad (22)$$
$$v(x,t) = \mp\left(e^{ic(x-ct+\xi_0)} + \frac{i\sqrt{B_0^2 - 4d} - iB_0}{2c}\right) \quad (23)$$

Similarly, for (19b), the exact solution to (10) is:

$$f(\xi) = e^{ic(\xi+\xi_0)} - \frac{iB_0 + i\sqrt{B_0^2 - 4d}}{2c} \qquad (24)$$

By substituting (24) in (9), we get:

$$g(\xi) = \mp\left(e^{ic(\xi+\xi_0)} - \frac{iB_0 + i\sqrt{B_0^2 - 4d}}{2c}\right) \ (25)$$

Then, the exact solution to eq. (7) can be written as:

$$u(x,t) = e^{ic(x-ct+\xi_0)} - \frac{iB_0 + i\sqrt{B_0^2 - 4d}}{2c} \ (26)$$
$$v(x,t) = \mp\left(e^{ic(x-ct+\xi_0)} - \frac{iB_0 + i\sqrt{B_0^2 - 4d}}{2c}\right) \qquad (27)$$

By using the condition (18b) in (12), we get:

$$Y(\xi) = icX(\xi) - B_0 \qquad (28)$$

By combining (28) with (13), we obtain the exact solution to (10) as follows:

$$f(\xi) = e^{ic(\xi+\xi_0)} - \frac{iB_0}{c} \qquad (29)$$

By substituting (29) in (9), we get:

$$g(\xi) = \mp\left(e^{ic(\xi+\xi_0)} - \frac{iB_0}{c}\right) \qquad (30)$$

Then, the exact solution to eq. (8) can be written as:

$$u(x,t) = e^{ic(x-ct+\xi_0)} - \frac{iB_0}{c} \qquad (31)$$
$$v(x,t) = \mp\left(e^{ic(x-ct+\xi_0)} - \frac{iB_0}{c}\right) \qquad (32)$$

Conclusions:

In this paper we introduce some new exact solutions to the focusing-defocusing NLS system in one dimension which obtained by using the first integral method and we can apply it to solve some nonlinear evolution equations. The key idea is using the travelling wave solution and by using the division theorem to solve nonlinear equations by using this method. It is worthwhile to mention that the method can be applied to solve some different phenomena and this method is very useful and important and easy at application.

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