A model for non-competitive substrate inhibition in a Bioreactor

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Abstract

The objective of this study is to learn how to analyse a system of nonlinear differential equations through the techniques of finding their steady states and determining their stability. Also,we learn how to identify the dimensionless groupings in the model in terms of dimensional parameters. In addition, use Maple to solve some problems and plotting the figures, it is very useful. Finally,the objective of this study is to investigate the quality of the effluent leaving the reactor and to investigate the biological reaction of the interaction.

Keywords: nonlinear differential equations, steady states, stability,A bioreactor,invariant, biological reaction.

1. Introduction

A bioreactor can be defined as a system in which a chemical process is carried out usingorganisms or biochemically active substance. In our model, the state of a continuous flow bioreactor is described in terms of two variables, the concentrations of a microorganism and a growth limiting substrate. The bioreactor is a well-mixed vessel containing microorganisms (X) through which a substrate(S) flows at acontinuous rate (F).An equation for the product is not required unless the concentration of the product species appears in the growth rate law .We write down the model equations for non-competitive substrate inhibition. This model arises in a multiplicity of applications.

1.1The dimensional model

The model equations for the dimensional model are [Nelson,2008]

$$V\ \frac{dS}{dt}=F(S_0-S)-\frac{\mu(S)}{\alpha_s}\ VX,(1)$$

$$V\ \frac{dX}{dt}=\beta F(X_0-X)+\ \mu(S).VX-Vk_dX.(2)$$

The specific growth rate is given by

$$\mu(S)=\frac{\mu_m S}{K_s+S+\frac{S^2}{K_i}}\ . \qquad (3)$$

Residence time $\tau=\frac{V}{F}.(4)$

The variables S and X denote the concentration of the substrate and the microorganisms respectively. The initial conditions, S (0) and X (0) must be non-negative. We denote the concentration of the substrate (S) and the microorganism (X) by $|S|$ and $|X|$ respectively.

In these equations F is the flow rate through the bioreactor (dm^3hr^{-1}), K_i is the substrate inhibition constant $(|S|)$, S is the substrate concentration within the bioreactor $(|S|)$, S_0 is the concentration of substrate flowing into the reactor $(|S|)$, Vis the volume of the bioreactor$(dm)^3$, X is the concentration of microorganisms within the bioreactor $(|X|)X_0$is the concentration of microorganisms flowing into the reactor $(|X|)$, k_d is the decay, or death, coefficient (hr^{-1}), t is the time (hr^{-1}), α is the yield factor $(|X||S|^{-1})$, $\mu$is the specific growth rate model, μ_m is the maximum specific growth rate (hr^{-1}), and (τ) is the residence time(hr).

For a specific waste-water, a given biological community and a particular set of environmental conditions the parameters K_i, K_s, k_d, α and μ_{max} are fixed. The parameters that can be varied are S_0, X_0 and τ. In equations (1) & (2) the main experimental control parameter, i.e. the primary bifurcation parameter, is the residence time (τ).

<u>2. Preliminary calculations</u> ***:*** In this section we scale the equations and then find a positively invariant region. Then we show that if the dimensionless death-rate is too high that all the micro-organisms must die $(\lim_{t\to\infty} X^*(t^*) = 0)$.

<u>2.1Scaling the equations:</u>By introducing dimensionless variables for the substrate concentration ($S^*=\frac{S}{K_s}$), the cell mass concentration $((X^*) =$X$/(\alpha_s K_s))$ and time $(t^* = \mu_m t)$, the system of differential equations (1) & (2) can be written in the dimensionless form

$$\frac{dS^*}{dt^*} = \frac{1}{\tau^*}(s_0{}^* - s^*) - \frac{X^*S^*}{1+S^*+\gamma S^{*2}}, \quad (5)$$

$$\frac{dX^*}{dt^*} = \frac{\beta}{\tau^*}(X_0{}^* - X^*) - \frac{X^*S^*}{1+S^*+\gamma S^{*2}} - k_d{}^* X^*. \quad (6)$$

Consider equation (1).Since $S^* = \frac{S}{K_s} \Rightarrow S = S^* K_s \cdot$

Thus $\frac{dS}{dt} = \frac{d}{dt}(S^* K_s) \Rightarrow \frac{ds}{dt} = K_s \frac{dS^*}{dt} \cdot$

In the originalequations Sis a function of (t) and in the scaled equations S^*will be a function of t^*.Thus the last equation is really

$$\frac{ds}{dt} = K_s \frac{dS^*(t^*)}{dt} .$$

Note that S^* $is\ a\ function\ of\ t^*$. It is not a function of (t).Thus using the chain rule we obtain

$$\frac{ds}{dt} = K_s \frac{dS^*}{dt^*} \frac{dt^*}{dt},$$

$$\frac{ds}{dt} = K_s \frac{dS^*}{dt^*} \frac{d\mu_m t}{dt},$$

$$\frac{ds}{dt} = K_s . \mu_m . \frac{dS^*}{dt^*} \cdot$$

Consider equation (2). Since $X^* = \frac{X}{(\alpha_s K_s)} \Rightarrow X = X^* . (\alpha_s K_s)$

$$\frac{dX}{dt} = \frac{d}{dt}(\alpha_s K_s X^*).$$

$$= \alpha_s K_s \frac{dX^*}{dt} .$$

In the original equations X is a function of (t) and in the scaled equations $X^*$is a function oft^*.
Thus the last equation is real $\frac{dX}{dt} = \alpha_s K_s \frac{dX^*(t^*)}{dt}$.

Note that X^* $is\ a\ function\ of\ t^*$. It is not a function of (t).Thus using the chain rule we obtain

$$\frac{dX}{dt} = \alpha_s K_s \frac{dX^*}{dt^*} \frac{dt^*}{dt},$$

$$=\alpha_s K_s \frac{dX^*}{dt^*}\frac{dt^*}{dt},$$

$$\frac{dX}{dt} = \alpha_s K_s \frac{dX^*}{dt^*}\frac{d\mu_m t}{dt},$$

$$=(\alpha_s K_s)(\ \mu_m)\ \frac{dX^*}{dt^*}.$$

Now, we substitute equation (3) into equation (1), to obtain

$$\frac{dS}{dt} = \frac{F}{V}(S_0 - S) - \frac{\mu_m\, S\, X}{\alpha_s(K_s + S + \frac{S^2}{K_i})}.$$

We substitute $\frac{ds}{dt} = K_s.\mu_m.\frac{dS^*}{dt^*}$ into this equation.

We obtain $K_s.\mu_m.\frac{dS^*}{dt^*} = \frac{F}{V}(S_0 - S^* K_s) - \frac{K_s S^* X^* \alpha_s\ K_s \mu_m}{\alpha_s(K_s + S^* K_s + \frac{(S^*\ K_s)^2}{K_i})}$

$$\Rightarrow \quad K_s.\mu_m \frac{dS^*}{dt^*} = \frac{(S_0 - S^* K_s)}{\tau} - \frac{S^* X^*\ \ K_s \mu_m}{(1 + S^* + \frac{K_s {S^*}^2}{K_i})}$$

$$\Rightarrow \qquad \frac{dS^*}{dt^*} = \frac{(S_0 - S^* K_s)}{K_s.\mu_m \tau} - \frac{S^* X^*\ \ K_s \mu_m}{K_s.\mu_m(1 + S^* + \frac{K_s {S^*}^2}{K_i})}$$

$$\frac{dS^*}{dt^*} = \frac{K_s(\frac{S_0}{K_s} - S^*)}{K_s.\mu_m \tau} - \frac{S^* X^*\ \ K_s \mu_m}{K_s.\mu_m(1 + S^* + \frac{K_s {S^*}^2}{K_i})},$$

$$\frac{dS^*}{dt^*} = \frac{(\frac{S_0}{K_s} - S^*)}{\mu_m \tau} - \frac{S^* X^*}{(1 + S^* + \frac{K_s {S^*}^2}{K_i})}, \text{(a)}$$

$$\frac{dS^*}{dt^*} = \frac{1}{\tau^*}({s_0}^* - s^*) - \frac{X^* S^*}{1 + S^* + \gamma {S^*}^2} \cdot \text{(b)}$$

Comparing equation (a) and equation (b), we find that$s_0{}^* = \frac{s_0}{K_s}$, $\tau^* = \mu_m \tau$,

$$\gamma = \frac{K_s}{K_i} \cdot$$

Now, we substitute equation (3) into equation (2), to obtain

$$\frac{dX}{dt} = \frac{\text{F}\,\beta}{V\,K_s}(X_0 - X) + \frac{\mu_m S X}{K_s + S + \frac{S^2}{K_i}} - V k_d X \cdot$$

We now substitute $\frac{dX}{dt}$ =$(\alpha_s K_s)(\,\mu_m)$ $\frac{dX^*}{dt^*}$ into the above equation .We obtain

$$K_s.\mu_m.\alpha_s \frac{dX^*}{dt^*} = \frac{\beta}{\tau}(X_0 - X\alpha_s K_s) - \frac{K_s S^* X^* \alpha_s\ K_s \mu_m}{\alpha_s(K_s + S^* K_s + \frac{(S^*\ K_s)^2}{K_i})} \text{-} k_d \alpha_s K_s X^*,$$

$$\frac{dX^*}{dt^*} = \frac{\beta}{\tau\,\mu_m}(\frac{X_0}{\alpha_s K_s} - X^*) + \frac{S^* X^*}{(1 + S^* + \frac{K_s {S^*}^2}{K_i})} - \frac{k_d}{K_i} X^*, \text{(c)}$$

$$\frac{dX^*}{dt^*} = \frac{\beta}{\tau^*}(X_0{}^* - X^*) - \frac{X^* S^*}{1 + S^* + \gamma {S^*}^2} - k_d{}^* X^*. \text{(d)}$$

Comparing equation(c) and equation (d), we find that

$$X_0{}^* = \frac{X_0}{\alpha_s K_s}\ , k_d{}^* = \frac{k_d}{K_i}.$$

In these equations $S_0{}^*$is the dimensionless substrate concentration in the feed [$s_0{}^* = \frac{s_0}{K_s}$] ,$X_0{}^*$ is the dimensionless cell mass concentration in the feed [$X_0{}^* = \frac{X_0}{\alpha_s K_s}$],$k_d{}^*$is the dimensionless decay coefficient [$k_d{}^* = \frac{k_d}{K_i}$], γ is the dimensionless substrate inhibition constant [$\gamma = \frac{K_s}{K_i}$], and $\tau^*$is the dimensionless residence time [$\tau^* = \mu_m \tau$] .

The main experimentally controllable parameter is the dimensionless residence time. From now on we make a standard assumption that the growth medium fed into the bioreactor is sterile, i.e. there are no microorganisms in the influent ($X^* = X_0{}^*$=0).

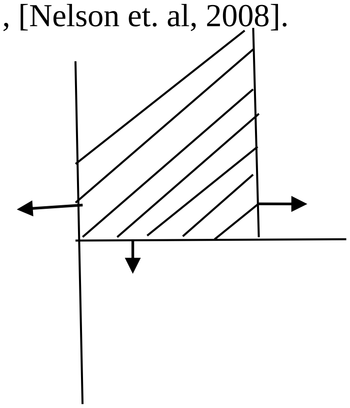
2.2 <u>An invariant region for the model</u>

<u>Definition</u>: A domain Σ is called invariant for a system of differential equations if for each x in Σ, the solution of the differential equation with initial condition $x(0) = x$ is defined and remains in the domain Σ for all time $t \geq 0$,[Britton, 1986], [Nelson et. al, 2008].

The region (*R*)bounded by

$$0\leq S^* \leq S_0{}^*, X^* \; S_0{}^*$$

$0\leq X^*, S^*$



is(positively) invariant for system 5.

We must prove confirm that f.n=$\dot{u}$ .n <0 along the three sides of the invariant region and check the direction of the vector at the two corner points of the invariant region

Edge (1): Consider the edge $S^* = 0$with $X^* > 0$(The case (S^*,X^*) = (0, 0) is point one). The unit outward normal is the vector (-1, 0) and we have

$f(S^*,X^*).n(S^*,X^*) = (\frac{dS^*}{dt^*}, \frac{dX^*}{dt^*}) \, . (-1,0),$

$$= -\left(\frac{dS^*}{dt^*}\right),$$

$= -\frac{1}{\tau^*}({s_0}^* - s^*) + \frac{X^*S^*}{1+S^*+\gamma {S^*}^2}$,

$= -\frac{{s_0}^*}{\tau^*}$ as S^*=0 along edge 1,

< 0, As ${S_0}^*$>o and $\tau^* > 0$.

Edge (2):Consider the edge $X^* = 0$ with 0≤ $S^* \leq {S_0}^*$(the cases (S^*,X^*) = (0, 0) and (S^*,X^*) =(${S_0}^*$, 0) correspond to points one and two respectively). The unit outward normal is the vector (0,-1) and we have

$f(S^*,X^*).n(S^*,X^*)= (\frac{dS^*}{dt^*}, \frac{dX^*}{dt^*}).(0,-1)$,

$= -\frac{dX^*}{dt^*}$,

$= -\frac{\beta}{\tau^*}(-X^*) + \frac{X^*S^*}{1+S^*+\gamma {S^*}^2} + {k_d}^* X^*$,

$= \frac{\beta X^*}{\tau^*} + \frac{X^*S^*}{1+S^*+\gamma {S^*}^2} + {k_d}^* X^*$,

= 0,as (X^* =0 along edge 1).

Note that we cannot apply the lemma because $f \,.\, n^{\wedge} = 0$.

Note, when X^* =0 system (5) reduces to

$\frac{dS^*}{dt^*} = \frac{1}{\tau^*}({s_0}^* - s^*)$ and $\frac{dX^*}{dt^*}$ =0.

The solution of this system of differential equations moves along the edge X^* =0 to the steady – state solution $(S^*,X^*) = ({S_0}^*, 0)$ without leaving the invariant region .Note that the line X^* =0 is itself invariant.

Edge (3): Consider the edge $S^*={S_0}^*$with $X^*>0$ (the case $(S^*,X^*)=({S_0}^*, 0)$ is point two).

The unit outward normal is the vector (1, 0) and we have

$$f(S^*,X^*).n(S^*,X^*)= (\frac{dS^*}{dt^*},\frac{dX^*}{dt^*}).(1,0),$$

$$=\frac{dS^*}{dt^*},$$

$$= \frac{1}{\tau^*}({s_0}^* - s^*) - \frac{X^*S^*}{1+S^*+\gamma {S^*}^2},$$

$$=-\frac{X^*S^*}{1+S^*+\gamma {S^*}^2}, \qquad \text{as } {S_0}^* = S^* \text{ along edge 3,}$$

<0 , as ${S_0}^*>0$ and $X^* > 0$along edge 3.

Point (1): Aunit outward normal along the invariant region is not defined at the point (S^*,X^*) = (0, 0). At this point system (5) reduces to

$$\frac{dS^*}{dt^*} = \frac{{s_0}^*}{\tau^*} > 0 \qquad ; \qquad \frac{dX^*}{dt^*} = 0 .$$

The solution trajectory at the point (S^*,X^*) = (0, 0) therefore points along the invariant region.

Point (2): A unit outward normal along the invariant region is not defined at the point at the point $(S^*,X^*) = ({S_0}^*, 0)$.This point is a steady- state solution of system (5). Consequently no solution can leave the invariant region through this point.We have therefore shown that the invariant region (R) is (positively) invariant.

2.3 Death takes us all

We show that the solution of system (5) converges to the steady-state (S^*,X^*) = (${S_0}^*$, 0) if ${k_d}^*$≥1.

Proof:

Let$X^* = {X_1}^*$.

Then $\frac{d{X_1}^*}{dt^*} = (-\frac{\beta {X_1}^*}{\tau^*}) + \frac{{X_1}^* S^*}{1+S^*+\gamma {S^*}^2} - {k_d}^* {X_1}^*$.

If ${k_d}^*$≥1 then $\mathcal{S} = k_d - \frac{S*}{1+S^*+\gamma {S^*}^2} > 0$.

Since $\frac{d{X_1}^*}{dt^*} < -\frac{\beta {X_1}^*}{\tau^*} - \mathcal{S}{X_1}^*$

$= -\gamma {X_1}^*$,

where $\gamma = \frac{\beta}{\tau^*} + \mathcal{S} > 0$.

Hence ${X_1}^*(t^*) \leq X^{\wedge *} e^{-\gamma t}$,

where $X^{\wedge *}$ is the initial condition.

Hence as t →∞ ⇨${X_1}^*(t^*)$ →0.

If X^* →0 for t^*»1

then $\frac{dS^*}{dt^*} \cong \frac{1}{\tau^*}({s_0}^* - s^*)$⇨$s^*$→${s_0}^*$.

This solution (S^*,X^*) = (${S_0}^*$, 0) is known as the (washout steady-state solution) because the steady-state value of the microorganism concentration is zero. As microorganisms do not flow into the reactor (${X_0}^*$=0) this means that all the original microorganisms present in the system have been literally “washed-out” .In practical applications, we want to avoid this solution.

3. Steady-state solutions

In this section we find the physically meaningful steady-state solutions of the model.

The steady state solutions of (5) and (6) are found by putting the time derivatives equal to zero and solving the resulting set of equations,

$$\frac{1}{\tau^*}({S_0}^* - S^*) - \frac{X^* S^*}{1+S^*+\gamma {S^*}^2} = 0 \qquad (7)$$

$$\frac{\beta}{\tau^*}({X_0}^* - X^*) - \frac{X^* S^*}{1+S^*+\gamma {S^*}^2} - {k_d}^* X^* = 0 \qquad (8)$$

$$\Rightarrow X^*\left[-\frac{\beta}{\tau^*} - \frac{S^*}{1+S^*+\gamma {S^*}^2} - {k_d}^*\right] = 0 \qquad (9)$$

Looking at equation (9) we find that there two solution branches. One of these corresponds to X^* = 0 whilst the other corresponds to $\frac{\beta}{\tau^*} - \frac{S^*}{1+S^*+\gamma {S^*}^2} - {k_d}^*$=0. We consider these two cases separately. First, we consider the case X^* = 0, which is known as the “washout branch”.

3.1 Washout branch

Let X^* = 0. Therefore, from equation (7) we have

$$\frac{1}{\tau^*}({S_0}^* - S^*) = 0,$$

$\Rightarrow$ $({S_0}^* - S^*) = 0,$

$\Rightarrow S^* = {S_0}^*.$

Therefore, the steady-state solution for the washout branch is

$(S^*, X^*) = ({S_0}^*, 0)$ (10)

The "washout" state is so-named because the steady-state value of the cell mass concentration is zero. As new cell mass does not flow into the reactor$({X_0}^*) = 0$, this means that all the original cell mass present in the system has been literally "washed-out" of the reactor.

3.2 No-washout branch

Now, we consider the second case. This is known as "no-washout solution". This corresponds to

$\frac{\beta}{\tau^*} - \frac{S^*}{1+S^*+\gamma {S^*}^2} - {k_d}^* = 0,$

$\Rightarrow \frac{S^*}{1+S^*+\gamma {S^*}^2} = \frac{\beta}{\tau^*} + {k_d}^*,$

$\Rightarrow S^* = (1 + S^* + \gamma {S^*}^2)(\frac{\beta}{\tau^*} + {k_d}^*),$

$\Rightarrow {S^*}^2 [\gamma({k_d}^* \tau^* + \beta)] - S^*([{k_d}^* - 1]\tau^* - \beta) + ({k_d}^* \tau^* + \beta) = 0.$

Let $G(s^{\wedge *}) = a s^{\wedge *2} + b s^{\wedge *} + c = 0.$

Where $s^{\wedge *}$ is a root of the equation (G).

The coefficients a, b and c are defined by

a = $\gamma({k_d}^{*}\tau^{*}+\beta)$,

b=$-[(1-{k_d}^{*})\,\tau^{*}-\beta]$,

c =$({k_d}^{*}\tau^{*}+\beta)$.

So, $s^{\wedge *}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

From equation (7), we have that $\frac{1}{\tau^*}({s_0}^{*}-\ s^{*})=\frac{X^*S^*}{1+S^*+\gamma {S^*}^2}$ (11)

From equation (8), we have that $\frac{\beta}{\tau^*}X^*+{k_d}^{*}X^*=\frac{X^*S^*}{1+S^*+\gamma {S^*}^2}$ ·(12)

By equating (11) and (12) we have that

$\frac{1}{\tau^*}({S_0}^{*}-\ S^{*})=\frac{\beta}{\tau^*}X^*+{k_d}^{*}X^*$,

$\Rightarrow x^*=\frac{S_0^*-S^*}{\beta+\tau^*kd^*}$.

Now, we substitute ${S^*}^{\wedge}=S^*$, we have that

$x^*=\frac{S_0^*-{S^*}^{\wedge}}{\beta+\tau^*kd^*}$.

Therefore, the steady-state solution for “no-washout branch “is

$$(s^*,x^*)=\left({s^*}^{\wedge},\frac{s_0^*-{s^*}^{\wedge}}{\beta+\tau^*kd^*}\right)\cdot$$

Note: The no-washout solution branch will not exist for all values of kd^* and all values of τ^*because the no-washout branch is physically meaningful only when the substrate and cell mass concentrations are positive.We want $s^{\wedge *} > 0$.

We consider the case $\square = 1$.

Recall that $\square^{\wedge *} = \frac{-\square \pm \sqrt{\square^2 - 4\square\square}}{2\square}$,

since$\square > 0, \ \square > 0 \quad , \square < 0$

where a = $\square(\square_{\square}{}^{*}\square^{*} + 1)$,

b = $-[(1 - \square_{\square}{}^{*})\ \square^{*} - 1]$,

c =$(\square_{\square}{}^{*}\square^{*} + 1)$.

The no-washout branch will be not be physically meaningful when

$\square^2 - 4\square\square < 0$(*)

By substituting a, b, c into (*) we obtain

$$\square = \square^2 - 4\square\square = \square_1\square^{*2} + \square_1\square^{*} + \square_1$$

Where $\square_1 = \left[1 - 2\square_{\square}{}^{*} + \square_{\square}{}^{*2}(1 - 4\square)\right]$,

$$\square_1 = 2[\square_{\square}{}^{*}(1 - 4\square) - 1],$$

$$\square_1 = (1 - 4\square)$$

By solving the(H) equation, we find the roots$\tau^* = \frac{-□_1 \pm \sqrt{□_1{}^2 - 4□_1□_1}}{2□_1}$where H=0.

After some algebra we find that $\sqrt{□_1{}^2 - 4□_1□_1} = 4\sqrt{□}$.

So,$□^* = \frac{-□_1 \pm 4\sqrt{□}}{2□_1}$.

Hence, $\frac{-□_1 - 4\sqrt{□}}{2□_1} < □^* < \frac{-□_1 + 4\sqrt{□}}{2□_1} \Rightarrow □^*_- < □^* < \tau^*_+$

So, $0 < □^* < □^*_+$ with $0 < □_□{}^* < 1.$

The solution is physically meaningful when $□^* > □^*_+$.

<u>**The limit point function:**</u> At a limit point bifurcation of the singularity function (□) satisfies the set of the equations $□ = □_□ = 0,$

and the inequalities $□_{□□} \neq 0$, $□_□ \neq 0$.

Since $□ = □□^{\wedge *2} + □\, □^{\wedge *} + □ = 0,$

We have $□_□ = 2□□^{\wedge *} + □.$

Then $□_□ = 0 \Rightarrow □^{\wedge *} = \frac{-□}{2□}$.Hence $□ = 0 \Rightarrow □_□ = □^{\wedge *2}□_□{}^* - □^{\wedge *}(1 - □_□{}^*)$, with $□^{\wedge *} = \frac{-b}{2□}$

Since $□^{\wedge *} > 0,\ 0 < □_□{}^* < 1 \Rightarrow □_□ \neq 0,$

By substituting $□^{\wedge *} = \frac{-□}{2□}$ into (□) we find $□ = \frac{-□^2}{4□} + □,$

Thus there is a limit point bifurcation at the point (□^∗, □∗+).

Figure (1):

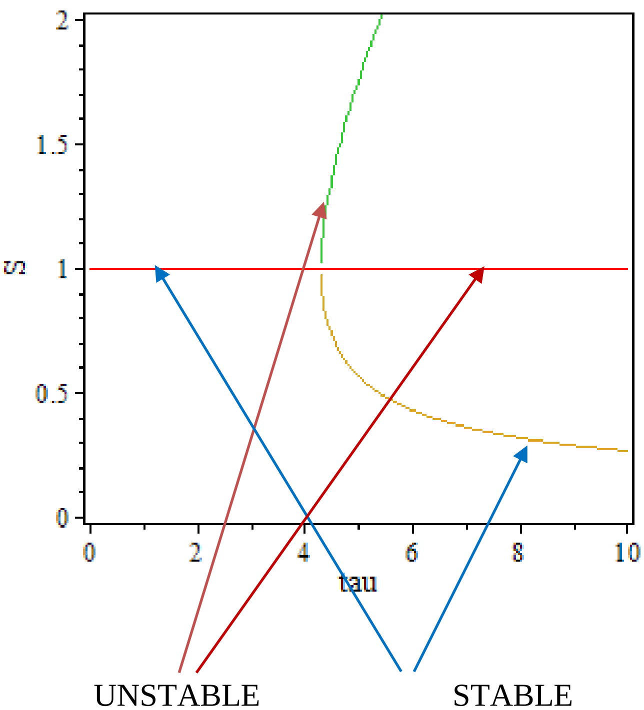


Figure (1): Substrance concentration in an ideal bioreactor as a function of residence time. Parameter values: $□_0^* = 1$, $□_□^* = 0.1$ .Figure (1) shows the substrate concentration in an ideal bioreactor (S).The physically meaningful solution ($□^* \leq □_0{}^* = 1$)decreases with increasing residence time (tau). This sketch contains one bifurcation point which is limit point.

By superimposing the washout steady-state solution ($□^* = □_0^*$)upon the solution curve for $□ = 0$ we find that there three generic steady-state diagrams that we need to consider.

The figure (1) contains two branches. The first branches corresponds to positive value and the second branches correspond to negative value in the quadratic formula.

We have $\square^{\wedge *} = \frac{-\square \pm \sqrt{\square^{2} - 4\square\square}}{2\square}$ where a = $\square(\square_{\square}{}^{*}\square^{*} + 1) > 0,$

b = $-[(1 - \square_{\square}{}^{*})\ \square^{*} - 1]<0,$

c = $(\square_{\square}{}^{*}\square^{*} + 1)>0.$

We must factorise $\sqrt{\square^{2} - 4\square\square}$ to find the sign of root,

$$\sqrt{\square^{2} - 4\square\square} = \sqrt{(-[(1 - \square_{\square}{}^{*})\ \square^{*} - 1])^{2} - 4\square(\square_{\square}{}^{*}\square^{*} + 1)^{2}} =$$

$$\sqrt{([((1 - \square_{\square}{}^{*})\ \square^{*} - 1) - 2\square^{\frac{1}{2}}(\square_{\square}{}^{*}\square^{*} + 1)][((1 - \square_{\square}{}^{*})\ \square^{*} - 1) + 2\square^{\frac{1}{2}}(\square_{\square}{}^{*}\square^{*} + 1)]}$$

The first factor $((1 - \square_{\square}{}^{*})\ \square^{*} - 1) + 2\square^{\frac{1}{2}}(\square_{\square}{}^{*}\square^{*} + 1)$ is a positive when $((1 - \square_{\square}{}^{*})\ \square^{*} - 1) > 0 \Rightarrow \square^{*} > \frac{1}{(1 - \square_{\square}{}^{*})}$.

The second factor $(((1 - \square_{\square}{}^{*})\ \square^{*} - 1) - 2\square^{\frac{1}{2}}(\square_{\square}{}^{*}\square^{*} + 1))$ will be that

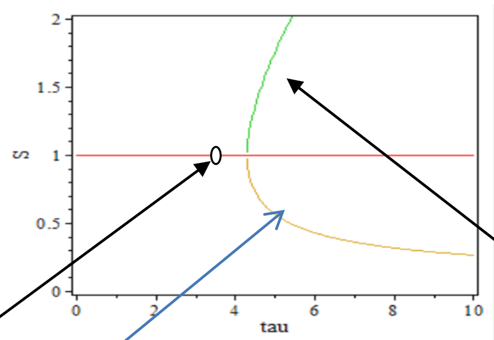
if $(((1 - \square_{\square}{}^{*})\ \square^{*} - 1) - 2\square^{\frac{1}{2}}(\square_{\square}{}^{*}\square^{*} + 1))<0 \Rightarrow$ don't have S.S

if $([((1 - \square_{\square}{}^{*})\ \square^{*} - 1) - 2\square^{\frac{1}{2}}(\square_{\square}{}^{*}\square^{*} + 1)]> 0 \Rightarrow$ have S.S

Hence $|\square| > \sqrt{\square^{2} - 4\square\square}$,

Then, if b >0 $\Rightarrow$ we have two positive solution, with $\left[|\square| - 2\sqrt{\square\square}\right] > 0$;

if b <0⇨ we have two negative solution ,with $\left[|\square| - 2\sqrt{\square}\,\square\right] > 0$.



Limit point $\square^{\wedge *} = \frac{-\square+\sqrt{\square^{2}-4\square\square}}{2\square}$,+ve, UNSTABLE $\square^{\wedge *} = \frac{-\square-\sqrt{\square^{2}-4\square\square}}{2\square}$,-ve,STABLE

Along the branch of the no-washout solution which is a negative sign, we have

$\frac{\square\square^{*}}{\square\square^{*}} < 0$,

Since $\square^{*}=\square^{\wedge *} = \frac{-\square-\sqrt{\square^{2}-4\square\square}}{2\square}$ where a = $\square(\square_{\square}{}^{*}\square^{*} + 1) > 0$

b = $-[(1 - \square_{\square}{}^{*})\ \square^{*} - 1]$<0

c =$(\square_{\square}{}^{*}\square^{*} + 1)$>0

to derivative$\square^{*}$ for $\square^{*}$,we obtain with using MAPLE

$$\frac{\square\square^*}{\square\square^*} = \frac{1}{2}\frac{1}{g\left(1+k_d\tau\right)}\Bigg(1-k_d - \frac{1}{2}\left(2\tau-4k_d\tau-2+2k_d^2\tau+2k_d-8gk_d-8gk_d^2\tau\right)\Big/ \left(\tau^2-2k_d\tau^2-2\tau+k_d^2\tau^2+2k_d\tau+1-4g-8gk_d\tau - 4gk_d^2\tau^2\right)^{1/2}\Bigg) - \frac{1}{2}\frac{1}{g\left(1+k_d\tau\right)^2}\Bigg(\Big(\left(1-k_d\right)\tau-1 - \left(\tau^2-2k_d\tau^2-2\tau+k_d^2\tau^2+2k_d\tau+1-4g-8gk_d\tau - 4gk_d^2\tau^2\right)^{1/2}\Big)k_d\Bigg)$$

<0 ,

Note: this result need more time to prove it’s.

Where $\square_{\square} = \square_{\square}{}^*$, $\square = \square^*$, $\square = \square$,

Thus, the substrate concentration is a decreasing function of the residence time. It follows from the relationship $(\square^*, \square^*) = \left(\square^{*^\wedge}, \frac{\square_0^* - \square^{*^\wedge}}{\square + \square^*\square\square^*}\right)$ that the microorganism’s concentration is an increasing function of the residence time.

4. <u>Physically meaningful solutions</u>

As noted earlier, the quantities $\square^*, \square^*$ are related to the concentrations of the substrate and microorganisms within the reactor. These quantities cannot be negative. Consequently, we are only interested in “physically meaningful” steady-state solutions. That is to say, we are only interested in steady-state with

$\square^* > 0, \square^* > 0.$

There is a critical value of the residence time, $\square^*_{\square\square}$, such that $\square^* > \square^*_{\square\square}$ that “no-washout solution” becomes physically meaningful,

Since $\square^*_{\square\square} = \frac{-\square_1+4\sqrt{\gamma}}{2\square_1}$ = 4.285714286,

Since,the no-washout solution has been the physical unmeaningful with $0 < \square^* < \square^*_{\square\square} \Rightarrow \square^* < 4.285714286$

So, the no-washout solution has been the physical meaningful when $\square^* > \square^*_{\square\square}$.

5. Stability of the steady state solutions

The stability of the steady state solutions for the system composing equations (5) and (6) are determined by the eigenvalues of Jacobin matrix evaluated at the steady state solution$(\square^*_{\square}, \square^*_{\square})$.The Jacobin matrix is defined by

$$\square(\square^*, \square^*) = \begin{bmatrix} \square_{\square^*}(\square^*, \square^*) & \square_{\square^*}(\square^*, \square^*) \\ \square_{\square^*}(\square^*, \square^*) & \square_{\square^*}(\square^*, \square^*) \end{bmatrix}.$$

The eigenvalues,of the Jacobian matrix are defined by

$$|\square - \square\square| = 0 \Rightarrow \square^2 - (\square\square\square)\square + \det \square = 0,$$

where $(\square\square\square)$ is the trace of the Jacobinmatrix $[\square\square\square = \square_{\square} + \square_{\square}]$ and $\det \square$ is the determinant of the Jacobin matrix $[\det \square = \square_{\square} \times \square_{\square} - \square_{\square} \times \square_{\square}]$.The steady state is stable if the real part of all eigenvalues is less than zero. This is true provided that$(\square\square\square) < 0$ $\square\square\square$ $\det \square > 0$.

It is unstable if either $(\square\square\square) > 0$ $\square\square\square/\square\square$ $(\det \square) < 0$.

The Jacobian matrix for the system (1.5) and (2.5) is given by

$$\square(\square^*,\square^*) = \begin{bmatrix} -\frac{1}{\square^*} - \frac{\square^*(1-2\square\square^{*^2})}{(1+\square^*+\square\square^{*^2})^2} & \frac{-\square^*}{1+\square^*+\square\square^{*^2}} \\ \frac{\square^*(1-2\square\square^{*^2})}{(1+\square^*+\square\square^{*^2})^2} & -\frac{\square}{\square^*}+\frac{\square^*}{1+\square^*+\square\square^{*^2}} - \square\square^* \end{bmatrix}$$

5.1 <u>Stability along the washout branch</u>

Evaluating, the Jacobian matrix along the washout branch we obtain

$$\square(\square_0^*,0) = \begin{bmatrix} -\frac{1}{\square^*} & \frac{-\square_0^*}{1+\square_0^*+\square\square_0^{*^2}} \\ 0 & -\frac{\square}{\square^*}+\frac{\square_0^*}{1+\square_0^*+\square\square_0^{*^2}} - \square\square^* \end{bmatrix}.$$

The characteristic equation for the determinate is given by :

$$\left(-\frac{1}{\square^*}-\square\right)\left(-\frac{\square}{\square^*}+\frac{\square_0^*}{1+\square_0^*+\square\square_0^{*^2}} - \square\square^* - \square\right) = 0.$$

Hence, the eigenvalues are:

$$\square_1 = -\frac{1}{\square^*} \quad \text{and} \quad \square_2 = -\frac{\square}{\square^*}+\frac{\square_0^*}{1+\square_0^*+\square\square_0^{*^2}} - \square\square^*.$$

It is clearly that $\square_1 < 0$, what about $\square_2$? When is $\square_2$ negative?

$$\square_2 = -\frac{\square}{\square^*}+\frac{\square_0^*}{1+\square_0^*+\square\square_0^{*^2}} - \square\square^* < 0$$

Note that $\frac{\square_0^*}{1+\square_0^*+\square\square_0^{*^2}} < 1$

It follows that the washout branch is always stable if:

1. $\square\square^{*} \geq \frac{\square_0^{*}}{1+\square_0^{*}+\square\square_0^{*2}}$, $0 < \square \leq 1$
2. $\square\square^{*} > \frac{\square_0^{*}}{1+\square_0^{*}+\square\square_0^{*2}}$ with $\square = 0$.
3. $\square\square^{*} < \frac{\square_0^{*}}{1+\square_0^{*}+\square\square_0^{*2}}$ the washout steady-state is stable provided :

$$\frac{\square}{\square^{*}} > \frac{\square_0^{*}}{1+\square_0^{*}+\square\square_0^{*2}} - \square\square^{*} \quad \square\square\square\ 0 < \square \leq 1.$$

$$\frac{\square}{\square^{*}} > \frac{\square_0^{*}-\left(1+\square_0^{*}+\square\square_0^{*2}\right)\square\square^{*}}{1+\square_0^{*}+\square\square_0^{*2}},$$

$$\square^{*} < \frac{\square\left(1+\square_0^{*}+\square\square_0^{*2}\right)}{\square_0^{*}-\left(1+\square_0^{*}+\square\square_0^{*2}\right)\square\square^{*}} \ .$$

<u>5.2 Stability along the no- washout branch</u>: The Jacobian matrix along the No-washout branch is given by

$$\square(\square^{*},\square^{*}) = \begin{bmatrix} -\frac{1}{\square^{*}} - \frac{(\square_0^{*} - \square^{*\wedge})}{(1+\square^{*}\square_{\square}^{*})}\frac{(1-2\square\,\square^{*\wedge 2})}{(1+\square^{*}+\square\square^{*2})^{2}} & \frac{-\square^{*\wedge}}{(1+\square^{*\wedge}+\square\,\square^{*\wedge 2})} \\ \frac{(\square_0^{*} - \square^{*\wedge})}{(1+\square^{*}\square_{\square}^{*})}.\frac{(1-2\square\square^{*2})}{(1+\square^{*}+\gamma\square^{*2})^{2}} & 0 \end{bmatrix}$$

We have the element $\square_{22} = 0$, $\square h\square\square\ \square\square$ because

$\square_{22} = \square_{\square}$ and $\square_{\square} = -\frac{\square}{\square^{*}} + \frac{\square_0^{*}}{1+\square_0^{*}+\square\square_0^{*2}} - \square\square^{*}$,

Since “No-washout solution” $= -\frac{\square}{\square^{*}} + \frac{\square_0^{*}}{1+\square_0^{*}+\square\square_0^{*2}} - \square\square^{*}$ equal to zero,

So, $\square_{\square} = -\frac{\square}{\square^{*}} + \frac{\square_0^{*}}{1+\square_0^{*}+\square\square_0^{*2}} - \square\square^{*} = 0$

$\Rightarrow \square_{\square} = \square_{22} = 0.$

Since, $\square\square\square = [\square_{\square} + \square_{\square}]$

$$= -\frac{1}{\square^{*}} - \frac{(\square_{0}^{*} - \square^{*^{\wedge}})}{(1+\square^{*}\square_{\square}{}^{*})}\frac{(1-2\square\ \square^{*^{\wedge}2})}{(1+\square^{*}+\square\square^{*2})^{2}}$$

$$= -[\frac{1}{\square^{*}} + \frac{(\square_{0}^{*} - \square^{*^{\wedge}})}{(1+\square^{*}\square_{\square}{}^{*})}\frac{(1-2\square\ \square^{*^{\wedge}2})}{(1+\square^{*}+\square\square^{*2})^{2}}]$$

Since $\square^{*} > 0$, $\square_{0}^{*} > 0$, $\square > 0$ $\square\square\square$ $\square_{0}^{*} > \square^{*^{\wedge}} > 0$,

so, $(\square_{0}^{*} - \square^{*^{\wedge}}) > 0$,

since $(1+\square^{*}\square_{\square}{}^{*}) > 0$,

So, we have $\frac{1}{\square^{*}}$, $\frac{(\square_{0}^{*} - \square^{*^{\wedge}})}{(1+\square^{*}\square_{\square}{}^{*})}$ are positive.

Since $(1 + \square^{*} + \square\square^{*2})^{2} > 0$,

We need have $(1 - 2\square\ \square^{*^{\wedge}2}) > 0 \Longrightarrow \square^{*^{\wedge}} < \frac{1}{\sqrt{2\square}}$

so, $\frac{(1-2\square\ \square^{*^{\wedge}2})}{(1+\square^{*}+\square\square^{*2})^{2}} > 0$,

Then, we have $-[\frac{1}{\square^{*}} + \frac{(\square_{0}^{*} - \square^{*^{\wedge}})}{(1+\square^{*}\square_{\square}{}^{*})}\frac{(1-2\square\ \square^{*^{\wedge}2})}{(1+\square^{*}+\square\square^{*2})^{2}}] < 0 \Rightarrow \square\square\square < 0.$

Since $\square\square tJ = \frac{\square^{*^{\wedge}}(\square_{0}^{*} - \square^{*^{\wedge}})}{(1+\square^{*}\square_{\square}{}^{*})}\frac{(1-2\square\square^{*2})}{(1+\square^{*}+\square\square^{*2})^{3}}$, since $\square_0^* > 0, \square_0^* > \square^{*^\wedge} > 0, (\square_0^* - \square^{*^\wedge}) > 0$ and $(1+\square^*\square_\square^*) > 0,$

So, $\frac{\square^{*^\wedge}(\square_0^* - \square^{*^\wedge})}{(1+\square^*\square_\square^*)} > 0.$

Since $(1 + \square^* + \square\square^{*2})^3 > 0 \;\; \square\square\square \; \left(1 - 2\square\square^{*2}\right) > 0 \;\; \square h\square\square \;\; \square^{*^\wedge} < \frac{1}{\sqrt{2\square}}$,

So $\frac{(1-2\square\square^{*2})}{(1+\square^*+\square\square^{*2})^3} > 0,$

Then, we have $\frac{\square^{*^\wedge}(\square_0^* - \square^{*^\wedge})}{(1+\square^*\square_\square^*)}\frac{(1-2\square\square^{*2})}{(1+\square^*+\square\square^{*2})^3} > 0 \Rightarrow \square\square tJ > 0$.

Thus, the no-washout solution is stable when $\square^{*^\wedge} < \frac{1}{\sqrt{2\square}}$.

<u>6. Conclusion</u> : In this project we have investigated an ideal bioreactor model for the interaction between a substrate and a microorganism. We started with dimensional model equations and from these we derived the non-dimensional model equations. The steady-state solutions of this model were found and their stability determined as a function of the residence time. We have found the importance of the critical value of the residence time. If the residence time is between zero and the critical value the process fails and the reactor stops working.

<u>7. References</u>

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