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Glance on Delay Partial Differential Problems of Order One

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Abstract

Hither the delay partial differential problems of order one are introduced. Here is swamped in proving some related theorems with its conditions which is supporting by remarks, and implementation. Also, some counter examples are given to show that the converse of some the presented theorems that have been proved may not be true. Finally, conclusion is given.

Keywords: Eventually positive, Eventually negative, Oscillatory, Linear differential equation, Partial differential equation, Partial differential problems.

نظرة على تأخير المسائل التفاضلية الجزئية من الرتبة الأولى

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الخلاصة

تم تقديم المشـكلات التفاضـلية الجزئية التاخيرية من الرتبة الأولى، وتم اثبات بعض النظريات ذات الصـلة التي تدعم الملاحظات والتنفيذ، وأيضـا، تم إعطاء بعض الأمثلة المضـادة لاظهار ان عكس بعض النظريات المقدمة التي تثبت انها قد لا تكون صـحيحة. واخيرا تم الاسـتنتاج، ان المعادلة التفاضـلية الجزئية التاخيرية من الرتبة الأولى يمكن ان يكون لها حلول إيجابية او حلول سـلبية او حلول متذبذبة في نهاية المطاف، وتم مناقشة بعض الأمثلة التي تثبت ان العكس لا يكون صحيحا.

1. Introduction

The delay partial differential equations simply is described by at least two independent variables, an unknown function of the independent variables, the behaviour of the unknown function at some prior τ value of the independent variables, and partial derivatives of the unknown function. Really, many authors have studied the concept of delay differential problems of order two as in [1-3]. While the delay differential equations ordinary and partial are interested in [4-5]. Authors in [6-7] focus on oscillating. Moreover [8] is submitted basic concepts in this field of mathematics.

Herein catch up in introducing new problems in delay partial differential equations, which are denoted in this paper by (DPDPs). Where are proved theorems supplied by conditions with applications on it which is represented by examples to discuss the converse of theorems. Inasmuch are getting at remarks and implementation to clarify the theorems and its conditions.

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2. Main Results

The purpose of this paper is to introduce a new formula of DPDPs of order one. Additionally, examples, theories, and mathematical results are used to explain all the possible and problematic properties of DPDPs. Furthermore, some counterexamples are given to clear the converse of some of the presented theorems.

3. Definition

The DPDPs are a set of an equation and two inequalities in modern mathematics of the first degree, which is shown below:

$$t\psi_t(t) + \alpha(t)\psi(t) + 9p(t)\psi(t-\tau) \leq 0 \quad (1)$$

$$t\psi_t(t) + \alpha(t)\psi(t) + 9p(t)\psi(t-\tau) \geq 0 \quad (2)$$

$$t\psi_t(t) + \alpha(t)\psi(t) + 9p(t)\psi(t-\tau) = 0 \quad (3)$$

Where $\alpha(t) \geq 0, p(t) > 0$ are continuous function on some interval a ≤ t ≤ b and, τ is a positive constant, with the following sufficient conditions are satisfied under theorems (1), (2), and (3) in this work.

- Inequalities (1) do not have eventually positive solutions.
- Inequalities (2) do not have eventually negative solutions.
- Equation (3) has just oscillatory solution.

4. Theorem Consider the delay partial differential inequality

$$t\psi_t(t) + \alpha(t)\psi(t) + 9p(t)\psi(t-\tau) \leq 0 \quad (1)$$

Where τ is a positive constant and $\alpha(t) \geq 0$,$p(t) > 0$ are continuous function, $t \in \mathbb{R}^+$. Assume that

$$\lim_{t\to\infty} inf\ 9\int_{t-\tau}^{t} p(s)ds \geq -\lim_{t\to\infty} \inf \int_{t-\tau}^{t} \alpha(s)ds \quad (2)$$

And,

$$\lim_{t\to\infty} inf\ 9\int_{t-\frac{\tau}{2}}^{t} p(s)ds > 0 \quad (3)$$

Then (2.4) has eventually negative solutions only.

Proof: Starting the proof by supposing that $\psi(t)$ is an eventually positive solution, which leads to a contradiction

$$\psi(t) \geq 0 \quad \text{for} \quad t > t_o$$

Then $\psi(t-\tau) > 0$ for $t > t_o + \tau$ and from 2.4

$$\psi_t(t) < 0 \quad \text{for } t > t_o + \tau$$

So $\psi(t)$ is decreasing

Hence, $\psi(t) < \psi(t-\tau)$ for $t > t_o + 2\tau$

Set that $$Q(t) = \frac{\psi(t-\tau)}{\psi(t)} \text{ for } t > t_o + 2\tau \quad (4)$$

Then, $Q(t) > 1$.

Dividing both sides of (2.4) by $\psi(t)$, for $t > t_o + 2\tau$, to obtain that

$$\frac{t\psi_t(t)}{\psi(t)} + \frac{\alpha(t)\psi(t)}{\psi(t)} + \frac{9p(t)\psi(t-\tau)}{\psi(t)} \leq 0 \ , t > t_o + 2\tau \quad (5)$$

Integrating both sides of (2.8) from $t-\tau$ to t, for $t > t_o + 3\tau$, to get:

$$\int_{t-\tau}^{t} s\ \frac{\psi_t(s)}{\psi(s)}ds + \int_{t-\tau}^{t}\alpha(s)ds + \int_{t-\tau}^{t} 9p(s)Q(s)ds \le 0\ ,\ t > to + 3\tau \qquad (6)$$

So the integral of the first term from the left of Eq. (2.9) is

$$\int_{t-\tau}^{t} s\ \frac{\psi_t(s)}{\psi(s)}ds = s\ .ln\,\psi(s)ds\Big|_{t-\tau}^{t} - \int_{t-\tau}^{t} ln\,\psi(s)\ ds \qquad (7)$$

$$= t.\,ln\,\psi(t) - (\,t-\tau).\,ln\,\psi(\,t-\tau) - [\ \psi(s).\,ln\,\psi(s) - \psi(s)]\Big|_{t-\tau}^{t}$$

$$= t.\,ln\,\psi(t) - (\,t-\tau).\,ln\,\psi(\,t-\tau) - [\psi(t).\,ln\,\psi(t) - \psi(t) - (\,\psi\big((\,t-\tau)ln\,\psi((\,t-\ \tau) - \psi(\,t-\tau)\big)] \qquad (8)$$

$$= t.\,ln\,\psi(t) - (\,t-\tau).\,ln\,\psi(\,t-\tau) - \psi(t).\,ln\,\psi(t) + \psi(t) + (\,\psi\big((\,t-\tau)ln\,\psi((\,t-\tau) + \psi(\,t-\tau)\big)$$

$$= ln\,\psi(t)[t - \psi(t)] - ln\,\psi(\,t-\tau)[\,(\,t-\tau) + \psi(\,t-\tau)] + [\,\psi(t)\ +\ \psi(\,t-\tau)] \qquad (9)$$

So, one can write (9) as in the following:

$$ln\,\psi\,(t)[t - \psi(t)] - ln\,\psi(\,t-\tau)[\,(\,t-\tau) + \psi(\,t-\tau)] + [\,\psi(t)\ -\ \psi(\,t-\tau)] + \int_{t-\tau}^{t}\alpha(s)ds - \int_{t-\tau}^{t} 9p(s)Q(s)ds \le 0 \qquad t > t_0 + 3\,\tau$$

$$ln\,\psi(t)[t - \psi(t)] - ln\,\psi(\,t-\tau)[\,(\,t-\tau) + \psi(\,t-\tau)] + [\,\psi(t)\ -\ \psi(\,t-\tau)] \le -\int_{t-\tau}^{t}\alpha(s)ds - \int_{t-\tau}^{t} 9p(s)Q(s)ds \qquad t > t_0 + 3\,\tau \qquad (10)$$

Now, integrating (4) from $t-\frac{\tau}{2}$ to t and using the fact that $\psi(t)$ is decreasing, one gets that:

$$\int_{t-\frac{\tau}{2}}^{t} s\psi_t(s)ds + \psi(t)\int_{t-\frac{\tau}{2}}^{t}\alpha(s)ds + \psi(t-\tau)\int_{t-\frac{\tau}{2}}^{t} 9p(s)ds \le 0, \text{ for } t > t_0 + \frac{\tau}{2} \qquad (11)$$

Then the integral of the first tern from the left of (11) becomes

$$\int_{t-\frac{\tau}{2}}^{t} s\psi_t(s)ds = t.\psi(t) - (\,t-\frac{\tau}{2})\,\psi(\,t-\frac{\tau}{2}) - \psi(t)\int_{t-\frac{\tau}{2}}^{t}\alpha(s)\ ds$$

So, here can (8) as in the following:

$$t.\psi(t) - \left(t-\frac{\tau}{2}\right)\psi\left(t-\frac{\tau}{2}\right) - \psi(t).\frac{\tau}{2} + \psi(t)\int_{t-\frac{\tau}{2}}^{t}\alpha(s)ds + \psi(t-\tau)\int_{t-\frac{\tau}{2}}^{t} 9p(s)ds \le 0$$

Dividing the last inequality first by $\psi(t)$ and then by $\psi\left(t-\frac{\tau}{2}\right)$, to obtain respectively:

$$\frac{t.\psi(t)}{\psi(t)} - \frac{\psi\left(t-\frac{\tau}{2}\right)\left(t-\frac{\tau}{2}\right)}{\psi(t)} - \frac{\psi(t)}{\psi(t)}.\frac{\tau}{2} + \frac{\psi(t)}{\psi(t)}\int_{t-\frac{\tau}{2}}^{t}\alpha(s)ds + \frac{\psi(t-\tau)}{\psi(t)}\int_{t-\frac{\tau}{2}}^{t} 9p(s)ds \le 0$$

$$t - \frac{\psi\left(t-\frac{\tau}{2}\right)\left(t-\frac{\tau}{2}\right)}{\psi(t)} + \frac{\tau}{2} + \int_{t-\frac{\tau}{2}}^{t}\alpha(s)ds + \frac{\psi(t-\tau)}{\psi(t)}\,9\int_{t-\frac{\tau}{2}}^{t} p(s)ds \le 0 \qquad (12)$$

$$\frac{t.\psi(t)}{\psi(\,t-\frac{\tau}{2})} - \left(t-\frac{\tau}{2}\right) + \frac{\tau}{2}.\frac{\psi(t)}{\psi\left(t-\frac{\tau}{2}\right)} + \frac{\psi(t)}{\psi(\,t-\frac{\tau}{2})}\int_{t-\frac{\tau}{2}}^{t}\alpha(s)ds + \frac{\psi(t-\tau)}{\psi(\,t-\frac{\tau}{2})}\int_{t-\frac{\tau}{2}}^{t} 9p(s)ds \le 0 \qquad (13)$$

Set that $Q(t) = \frac{\psi(t)}{\psi\left(t-\frac{\tau}{2}\right)}$ for $t > t_o + 3\tau$

Since $Q \le 1.$ & $\psi(t)$ is decreasing, so when t→ ∞

Then Q(t)→ 0

That is why $\lim_{t\to\infty} inf\ 9\int_{t-\frac{\tau}{2}}^{t} p(s)ds \le (t-\frac{\tau}{2}).\frac{\psi(\,t-\frac{\tau}{2})}{\psi(t-\tau)}$

Inasmuch $\lim_{t\to\infty} inf\ 9\int_{t-\frac{\tau}{2}}^{t} p(s)ds \le 0$

This result contradicts (3)

Now:

Use $\lim_{t\to\infty}\inf$ on the Eq.(10) to be

$$\lim_{t\to\infty}\inf \ln \frac{\psi(t)}{\psi(t-\tau)}.[\frac{[t-\psi(t)]}{\psi(t-\tau)[(t-\tau)+\psi(t-\tau}\Big)]+\lim_{t\to\infty}\inf[\psi(t)+\psi(t-\tau)] \le -\lim_{t\to\infty}\inf\int_{t-\tau}^{t}\alpha(s)ds - \lim_{t\to\infty}\inf\int_{t-\tau}^{t}9p(s)Q(s)ds \quad (14)$$

Since $\lim_{t\to\infty}\frac{\psi(t)}{\psi(t-\tau)}$=0 and $\psi(t)$>0 , $\psi(t-\tau)$>0

Then

$$0 \le -\lim_{t\to\infty}\inf\int_{t-\tau}^{t}\alpha(s)ds - \lim_{t\to\infty}\inf\int_{t-\tau}^{t}9p(s)Q(s)ds$$

$$\lim_{t\to\infty}\inf 9\int_{t-\tau}^{t}p(s)ds] \le -\lim_{t\to\infty}\inf\int_{t-\tau}^{t}\alpha(s)ds$$

which contradicts condition (2)

Finally, the proof is complete.

5. Remark The converse of theorem (1) may not be true as in the following

6. Example Consider the delay partial differential inequality

$$t\psi_t(t) + 9\,\psi\left(t-\frac{3\pi}{12}\right) \le 0 \quad (15)$$

So the solution $\psi(t) = -\sin t$ is eventually negative.

But the condition (3) is not satisfied.

7. Theorem Consider the delay partial differential inequality.

$$t\psi_t(t) + \alpha(t)\psi(t) + 9p(t)\psi(t-\tau) \ge 0 \quad (16)$$

Where τ is a positive constant and $\alpha(t) \ge 0$,$p(t) > 0$ are continuous function, $t \in \mathbb{R}^+$. Assume that

$$\lim_{t\to\infty} inf\ 9\int_{t-\tau}^{t}p(s)ds > -\lim_{t\to\infty}\inf\int_{t-\tau}^{t}\alpha(s)ds$$

And,

$$\lim_{t\to\infty} inf\ 9\int_{t-\frac{\tau}{2}}^{t}p(s)ds > 0$$

Then (16) has eventually positive solutions only.

Proof. Starting the proof by supposing that (t) is an eventually negative solution.

$$\psi(t) \le 0, \quad \text{for } t<t_o$$

Then $\psi(t-\tau) < 0$, for $t < t_o + \tau$ and from Eq.(16)

$$\psi_t(t) > 0 \quad \text{for } t < t_o + \tau$$

So $\psi(t)$ is decreasing

Hence, $\psi(t)$<$\psi(t-\tau)$ for $t < t_o + 2\tau$ Set that

$$Q(t) = \frac{\psi(t-\tau)}{\psi(t)} \text{ for } t < t_o + 2\tau \quad (17)$$

Then, $Q(t) < 1$.

Dividing both sides of Eq.(16) by $\psi(t)$, for $t < t_o + 2\tau$, to obtain that

$$\frac{t\psi_t(t)}{\psi(t)}+\frac{\alpha(t)\psi(t)}{\psi(t)}+\frac{9p(t)\psi(t-\tau)}{\psi(t)}\le 0 \quad t<t_o+2\tau \tag{18}$$

Which is the same inequality (8) & continue the proof to get a contradiction as the same steps in the proof of theorem (1)

8. Remark The converse of theorem (2) may not be true as show in the following example

9.Example Consider the delay partial differential inequality

$$t\psi_t(t)+9\psi(t-2\pi)\ge 0 \tag{19}$$

Then the solution $\psi(\mathrm{t})=\cos t$ when t satisfies the inequality is eventually positive but the inequality is not satisfied the condition (3).

10.Theorem Consider the delay partial differential inequality.

$$t\psi_t(t)+\alpha(t)\psi(t)-9p(t)\psi(t-\tau)=0 \tag{20}$$

Where τ is a positive constant and $\alpha(t)\ge 0$,$p(t)>0$ are continuous function, $t\in\mathbb{R}^+$. Assume that

$$\lim_{t\to\infty} inf\ 9\int_{t-\tau}^{t}p(s)ds > -\lim_{t\to\infty}\inf\int_{t-\tau}^{t}\alpha(s)ds$$

And,

$$\lim_{t\to\infty} inf\ 9\int_{t-\frac{\tau}{2}}^{t}p(s)ds>0$$

Then (20) has oscillatory solution only.

Proof: Let $\psi(t)$ be a non-oscillatory solution, then $\psi(t)$ either eventually positive i.e. $\psi(t)>0$.

So, it satisfies one of the following cases:

1.$t\psi_t(t)+\alpha(t)\psi(t)+9p(t)\psi(t-\tau)<0$ and leads to a contradiction by Theorem (1)

2. $t\psi_t(t)+\alpha(t)\psi(t)+9p(t)\psi(t-\tau)>0$ the solution is eventually positive by (2)

Or $\psi(t)$ is eventually negative i.e. $\psi(t)<0$.

So, it satisfies one of the following cases:

1. $t\psi_t(t)+\alpha(t)\psi(t)+9p(t)\psi(t-\tau)>0$ which contradiction is a by (2)
2. $t\psi_t(t)+\alpha(t)\psi(t)+9p(t)\psi(t-\tau)<0$ is eventually negative. By (1)

From the above discussions, the solutions are only oscillating

11. Remark: the converse of theorem (3) may not be true as show in the following example

12. Example Consider the delay partial differential inequality

$$t\psi_t(t)+9\psi\left(t-\frac{3\pi}{2}\right)=0 \tag{21}$$

Then $\psi(t)=\sin t\sin x$ is oscillatory

But the condition (2) is not satisfied.

Implementations

Now some illustrative examples are mentioned to solve DPDPs, which is represented by examples below as an application on the above theorems.

13. Example Consider the delay partial differential inequality

$$t\psi_t(t) + e^{-t}\psi(t) + 9e^{-2t}\psi(t-\tau) \leq 0 \quad (22)$$

Then inequality is satisfied the conditions (2) and (3) where
$\alpha(t) = e^{-t}, p(t) = e^{-2t}$ and the solution $\psi(t) = -e^{-2(t+x)}$
is an eventually negative where $\psi_t(t) = 2e^{-2(t+x)}$.

14. Example Consider the delay partial differential inequality

$$t\psi_t(t) + 9\psi(t-\tau) \geq 0 \quad (23)$$

So the conditions (2) and (3) where $\alpha(t) = 0,\ p(t) = 1$ are satisfied
Also $\psi(t) = e^{-t}$ is an eventually positive solution
Where $\psi_t(t) = e^{-t}$.

15.Example Consider the delay partial differential inequality

$$t\psi_t(t) - 9\psi(t - \frac{\pi}{2}) = 0 \quad (24)$$

The conditions (2) and (3) where $\alpha(t) = 0$, $p(t) = 1$ are satisfied and the solution $\psi(t) = sinxsint$ is an oscillatory solution
Where $\psi_t(t) = sinxcost$.

Conclusion

This study found that DPDPs, ultimately has a variety of solutions (eventually positive, eventually negative, oscillatory). In addition, the above results are explained through examples, theorems. Moreover, some remarks are presented which show the converse of some of the presented theorems.

Future Work

Our work in the near future shows in the solutions demeanour of the delay partial differential problems of order two

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