On Generalized Simple P-injective Rings

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الملخص

في هذا البحث,درسنا الحلقات البسيطة الغامرة من النمط-P المعممة.وحصلنا على نتائج وصفات جديدة لمثل هذه الحلقات. وأخيرا" أعطينا تميزا" للحلقات المقسومة والحلقات المنتظمـــة القوية بدلالة الحلقات ضمن النمط GSP .

ABSTRACT

In this work, we study generalized simple P-injective rings. New properties of such rings are given, and a characterization of division rings and strongly regular rings in terms of GSP rings is obtained.

1. Introduction:

Throughout this paper, R denotes an associative ring with identity, and all modules are unitary right R-modules.

A right R-module M is said to be P-injective if, for any principal right ideal P of R, any right R-homomorphism $f: P \to M$, there exists y in M such that f(b)=yb for all b$\in$P. This concept was introduced by Ming [3].

Recall that: (1) R is called strongly regular if for every a$\in$R there exists b $\in$R such that $a=a^2b$, (2) a ring R is said to be zero commutative (briefly ZC) if for a, b $\in$R, ab=0 implies that ba=0 [2]; (3) For any element a in R, r (a) and $\mathbf{l}(a)$ denote the right and left annihilator of a respectively; (4) A ring R is called reduced if, R contains no non-zero nilpotent element, (5) R is said to be right uniform, if every right ideal of R is essential [4], (6) J(R) will state for the Jacobson radical.

2. GSP-Rings:

<u>Definition 2.1:</u> A ring R is called a GSP-ring (generalized simple P-injective) if, for any maximal right ideal *M* of R, any b $\in$M, bR/bM is p-injective.

Recall the following lemma which are due to Skin [5].

<u>Lemma 2.2:</u> For any ring R, the following statements are equivalent:

(1) R is a ZC ring.

(2) For each a $\in$R, r(a) (equivalently $\mathbf{l}(a)$) is a two-sided ideal of R

We shall begin this section with the following results.

<u>Theorem 2.3:</u> Let R be a ZC, GSP-ring. Then:

(1) J(R)={0}

(2) R is a reduced ring.

<u>Proof (1).</u> Let a $\in$J (R), if aR+r(a) $\neq$R. Then there exists a maximal right ideal M containing aR+r(a). Suppose that aR=aM, then a=ac for some c in M and this implies that a(1-c)=0, so 1-c$\in$r(a)$\subseteq$M, whence 1$\in$M contradicting M$\neq$R. If aR$\neq$aM, the right R-homomorphism g:R/M$\to$aR/aM defined by g(b+M)=ab+aM, for all b in R implies that R/M$\cong$aR/aM. Define $f: aR \to R/M$ as a right R- homomorphism by f(ax)=x+M, for all x in R, then f is a well-defined right R-homomorphism. Indeed; let x_1,x_2 be any two elements in R with $ax_1=ax_2$, implies that (x_1-x_2)

$\in$ r(a) $\subseteq$M, thus x_1+M=x_2+M. Hence, f(ax_1)=x_1+M=x_2+M=f(ax_2). Since R/M is P-injective, then there exists an element c in R such that f(ac)=(c+M) ax =cax+M, yields 1+M=f(a)=da+M, for some d in R, whence 1$\in$ M, again there is a contradiction.

Therefore aR+r(a)=R. In particular ay+d=1 fore some y$\in$ R, d$\in$ r(a), thus we have a^2y=a. Now, since a$\in$ J (R), then there exists an invertible element u in R such that (1-ay)u=1 and this implies that (a-a^2y)u=a, whence a=0. Therefore J (R)=0.

Next, we consider the connection between GSP-ring and division rings.

<u>Proof (2).</u> Let a be a non-zero element of R such that a^2=0. Then there exists a maximal right ideal M of R containing r(a). If aR=aM, then a=ac for some c in M, which implies that 1-c $\in$ r(a)$\subseteq$M, whence 1$\in$ M, contradicting M $\neq$ R. If aR $\neq$aM the right R-homomorphism g: R/M$\rightarrow$aR/aM defined by g(r+M)=ar+aM for all r$\in$ R implies that R/M$\cong$ aR/aM. Then R/M is P-injective. Consider the canonical mapping $f : aR \rightarrow R/M$, then there exists an element b$\in$ R such that 1+M=f(a)=ba+M. Now r(a) is a two sided ideal of R, so ba$\in$ r(a) $\subseteq$M, whence 1$\in$ M, a contradiction. Therefore a=0, whence R is reduced ring.

<u>Definition 2.4:</u> Let R be a ring such that every maximal right ideal is a two sided ideal. Then R is called a quasi-duo ring.

<u>Theorem 2.5:</u> Let R be a quasi-duo, GSP-ring. Then any non-zero divisor of R is a right and left invertible.

<u>Proof.</u> Let y be a non-zero element of R. If yR$\neq$R, let M be a maximal right ideal of R containing yR. Suppose that yR=yM. Then y=yc for some c in M. which implies that 1-c$\in$ r(y), whence 1=c$\in$ M, contradicting M$\neq$R. now, if yR$\neq$yM the right R-homomorphism g:R/M$\rightarrow$yR/yM defined by g(r+M)=yr+yM, for all r in R implies that R/M$\cong$ yR/yM. Since yR/yM is P-injective then R/M is P-injective. Consider the canonical mapping f:yR$\rightarrow$R/M, then there exists an element b in R such that f(y)=1+M=by+M. Hence 1-by$\in$ M, since y$\in$ Mand R is a quasi-duo ring, implies that by$\in$ M. thus 1$\in$ M, again there is contradiction. Therefore yR=R, in particular yr=1, for some r$\in$ R, and then we have yry=y, implies that ry$\in$ r(y)=0, thus ry=1. This proves that y is a right and left invertible.

Corollary 2.6: Let R be a quasi-duo, GSP-ring without zero divisors. Then R is a division ring.

The next result considers other conditions for GSP-ring to be a division ring.

Theorem 2.7: Let R be a ZC, right uniform GSP-ring. Then R is a division ring.

Proof. Let $0 \neq y \in R$, if r(1-y)=0, then $l(1-y)=0$ (R is ZI). Then by Theorem (2.5) 1-y is an invertible element and hence y$\in$ J (R), so by Theorem (2.3) y=0, a contradiction. Therefore r(1-y)$\neq$0, let $0 \neq x \in r(1-y)$, then x=yx. We claim that $xR \cap r(y)=0$, if not, let $z \in xR \cap r(y)$, then z=xr for some $r \in R$ and yz=0, this implies yxr=0, yields xr=0, whence z=0. Therefore $xR \cap r(y)=0$. Since R is a right uniform ring, and xR$\neq$0 then r(y) must be zero. Since R is ZC, then $l(y)=0$. Then by Corollary (2.6) R is a division ring.

Lemma 2.8: If R is a reduced ring, then R/r(a) is a reduced ring.

Proof. See [1]

Theorem 2.9: Let R be a reduced ring, such that for every a$\in$R, R/r(a) is GSP-ring. Then R is strongly regular.

Proof. Let a be a non-zero element in R, and let d=a+ r(a) $\in$ R/r(a). Clearly, $d \neq 0$ because otherwise if d=0, then a+r(a)=r(a), and this yields a^2=0, gives a=0 (since R is reduced). Let $d\bar{x}=0$, we shall prove that $\bar{x}=0$. Observe that (a+r(a))(x+r(a))=r(a), implies that ax+r(a)=r(a) and hence ax$\in$ r(a), so a^2x=0. Thus x$\in$ r(a^2) (since R is reduced). Therefore, x$\in$r(a) implies that ax=0. Hence $\bar{x}=0$, this means that d is a right non-zero divisor. In a similar way we can prove that d is a left non-zero divisor. Since R/r(a) is GSP-ring, then by Theorem (2.5) d is an invertible element. then there exists

$0 \neq \bar{y} = y + r(a) \in R/r(a)$ such that $d\bar{y}=1$, then (a+r(a))(y+r(a))=1+r(a), so ay-1$\in$ r(a) and a(ay-1)=0. Thus a^2y=a. this proves that R is a strongly regular ring.

Theorem 2.10: Let R be a GSP-ring, and let I be a reduced right ideal of R. Then I is a strongly regular ring.

Proof. Since I is reduced, for any $b\in I$, $L(b)\subseteq r(b)$. If $bR+r(b)\neq R$, let M be a maximal right ideal containing bR+r(b). If bR=bM, then b=bc for some $c\in M$ which implies that $1-c\in r(b)\subseteq M$, whence $1\in M$, contradicting $M\neq R$. If $bM\neq bR$, the right R- homomorphism $g:R/M\rightarrow bR/bM$ defined by g(a+M)=ba+bM for all $a\in R$ implies that $R/M\cong bR/bM$, since bR/bM is P-injective, then R/M is P-injective. Consider the canonical mapping $f:bR\rightarrow R/M$, then there exists an element a in R such that f(b)=1+M=ab+M, whence $1\in M$, again there is a contradiction. Thus bR+r(b)=R for any $b\in I$ from which $b=b^2u$, $u\in R$.

Now $b=bbu=b(b^2u)u=b^2v$, where $v=bu^2\in I$, and $(b-bvb)^2=0$ implies that b=bvb which proves that I is a strongly regular ring.

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