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0n the convergence sequence of intuitionistic fuzzy measurable functions on intuitionistic fuzzy measure space

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Abstract : In this paper , the concepts of order-continuity , pseudometric generating property , autocontinuity and null-subtractive of an intuitionistic fuzzy measure on an intuitionistic fuzzy $\sigma -$ algebra of intuitionistic fuzzy sets will be introduced , and we proved Egoroff's Theorem and three forms of Egoroff's Theorem for a sequence of measurable functions on an intuitionistic fuzzy σ −algebra .

1-Introduction :

The concept of fuzzy measure defined on a classical σ −algebra , were first proposed by Sugeno in [5] . Some structural characteristics of fuzzy measure were introduced and discussed by Wang [7] . A generalization of fuzzy measure were established on fuzzy sets by Qiao[9] , and the Lebesgu's theorem and Riesz's theorem for a sequence of measurable functions had been proved on fuzzy σ-algebra of fuzzy set . In 1996 , L.Jun and M. Yasuda[4] show that the Egoroff's theorem for a sequence of fuzzy measurable functions also holds on fuzzy σ-algebra by using the concepts of order continuity and the pseudometric generating property of fuzzy measures .

Many authors defined new types of measures , Adrain I. Ban [2] one of the authors who defined an intuitionistic fuzzy measure on an intuitionistic fuzzy $\sigma -$ algebra $\tilde{\mathcal{A}}$ on an intuitionistic fuzzy sets . The notion of intuitionistic fuzzy sets introduced by Atanassove [1] in

1983 , as a generalization of the notion of fuzzy sets which introduced by Zadeh [8] in 1965 .

In this paper , we will prove Egoroff's theorem and three forms of this theorem for a sequence of intuitionistic fuzzy measurable functions on an intuitionistic fuzzy σ-algebra by using the concepts of order-continuity , pseudometric generating property , autocontinuity and null-subtractive of an intuitionistic fuzzy measure.

2- Intuitionistic fuzzy measure

In this section , we recall some definitions which will be used for this work .

Definition(2.1)[8]:

Let X be a non-empty set and let I be the closed interval [0,1] of the real line . A fuzzy set μ in X is characterized by membership function $\mu: X \longrightarrow I$, which associates with each point $x \in X$ its grade or degree of membership $\mu(x) \in [0,1]$.

Definition(2.2)[1]:

Let X be a non-empty fixed set . An intuitionistic fuzzy set (IFS) A is an object having the form:

A=$\{\langle x, \mu_A(x), v_A(x)\rangle, x \in X\}$,where the functions $\mu_A: X \longrightarrow I$ and $v_A: X \longrightarrow I$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set A ,respectively , and $0 \leq \mu_A(x) + v_A(x) \leq 1$ for each $x \in X$.

Definition(2.3)[3]: $\tilde{0} = \{\langle x, 0,1\rangle, x \in X\}$

$\tilde{1} = \{\langle x, 1,0\rangle, x \in X\}$

are the intuitionistic fuzzy sets corresponding to empty set and the entire universe respectively .

Note : Every fuzzy set A on a non-empty set X is obviously an IFS having the form $\{\langle x, \mu_A(x), 1-\mu_A(x)\rangle, x \in X\}$.

Definition(2.4):

Let A be a subset of a set , we define the intuitionistic characteristic function of A as follows:

$$I\chi_A = \begin{cases} \tilde{1}, & if\ x \in A \\ \tilde{0}, & if\ x \notin A \end{cases}$$

Definition(2.5)[1,6]:

Let X be a non-empty set and let A and B are IFSs in the form $A=\{\langle x, \mu_A(x), v_A(x)\rangle, x \in X\}$, B=$\{\langle x, \mu_B(x), v_B(x)\rangle, x \in X\}$.

Then:

1) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $v_A(x) \geq v_B(x)$ for all $x \in X$.

2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

3) $A^c = \{\langle x, v_A(x), \mu_A(x)\rangle, x \in X\}$.

4) $A \cap B = \{\langle x, min\{\mu_A(x), \mu_B(x)\}, max\{v_A(x), v_B(x)\}\rangle, x \in X\}$.

5) A $\cup B = \{\langle x, max\{\mu_A(x), \mu_B(x)\}, min\{v_A(x), v_B(x)\}, x \in X\rangle\}$.

6) $A/B = A \cap B^c$.

Definition(2.6)[3]:

Let $\{A_i, i \in J\}$ be an arbitrary family of IFSs in X, then

1) $\bigcap_i A_i = \{\langle x, \bigwedge_i \mu_{A_i}(x), \bigvee_i v_{A_i}(x)\rangle, x \in X\}$.

2) $\bigcup_i A_i = \{\langle x, \bigvee_i \mu_{A_i}(x), \bigwedge_i v_{A_i}(x)\rangle, x \in X\}$.

Definition(2.7)[2]:

An intuitionistic fuzzy σ-algebra (σ-field) on $X \neq \emptyset$ is a family $\tilde{\mathcal{A}}$ of IFSs in X satisfying the properties :

1) $\tilde{1} \in \tilde{\mathcal{A}}$;

2) If $A \in \tilde{\mathcal{A}}$ this implies that $A^c \in \tilde{\mathcal{A}}$;

3) If $(A_n)_{n \in N} \in \tilde{\mathcal{A}}$, then $\bigcup_{n \in N} A_n \in \tilde{\mathcal{A}}$.

The pair $(X, \tilde{\mathcal{A}})$ is called an intuitionistic fuzzy measurable space.

Example(2.8)[2]:

Let A=$\{<x, \mu_A(x), v_A(x)>, x \in X\} \in$ IFSs . Let us denote $\Omega_A = \{x \in X;\ \mu_A(x) > 0\}$, $\Lambda_A = \{x \in X;\ v_A(x) > 0\}$ and $N = \{A \in IFS(X)\ ; \Omega_A\ or\ \Lambda_A\ \ is\ a\ finite\ or\ countable\}$, then the family N of IFSs is an intuitionistic fuzzy σ-algebra .

Definition(2.9)[2]:

Let $\tilde{\mathcal{A}}$ be an IF σ-algebra in X . A function $\tilde{m}\colon \tilde{\mathcal{A}} \longrightarrow [0, \infty]$ is said to be an intuitionistic fuzzy measure if it satisfies the following conditions :

1) $\tilde{m}(\ \tilde{0}) = 0$;

2) For any $A, B \in \tilde{\mathcal{A}}$ and $A \subseteq B$ this implies that $\tilde{m}(A) \leq \tilde{m}(B)$.

The intuitionistic fuzzy measure $\tilde{m}$ is called σ-additive if $\tilde{m}(\bigcup_{n \in N} A_n) = \sum_{n \in N} \tilde{m}(A_n)$ for every sequence $(A_n)_{n \in N}$ of pairwise disjoint IFSs in $\tilde{\mathcal{A}}$.

The triple $(X , \tilde{\mathcal{A}}, \tilde{m}\ )$ is called intuitionistic fuzzy measure space .

Definition(2.10):

The intuitionistic fuzzy measure $\tilde{m}$ is called :

1) Finite if $\tilde{m}(\tilde{1})<\infty$, and inini te if $\tilde{m}(\tilde{1})=\infty$.

2) Finitely additive if $\tilde{m}(A \cup B) = \tilde{m}(A) + \tilde{m}(B)$.

Example(2.11)[2]:

The function $\tilde{m}: \tilde{\mathcal{A}} \to [0, \infty]$ defined by

$\tilde{m}(A) = \frac{1}{2}\sum_{x \in X}(\mu_A(x) + 1 - v_A(x))$

for $A = \{<x, \mu_A(x), v_A(x)>, x \in X\} \in \tilde{\mathcal{A}}$, is a σ-additive intuitionistic fuzzy measure .

Definition(2.12):

Let (X, $\tilde{\mathcal{A}}$) be an intuitionistic fuzzy measurable space . An intuitionistic fuzzy measure $\tilde{m}: \tilde{\mathcal{A}} \to [0, \infty]$ is said to be :

1) Order-continuous if $\tilde{m}(A_n) \longrightarrow 0$,whenever $A_n \in \tilde{\mathcal{A}}$ and $A_n \downarrow \tilde{0}$.

2) Auto-continuous from above if $\lim_{n \in N} \tilde{m}(A \cup B_n) = \tilde{m}(A)$, whenever A, $B_n \in \tilde{\mathcal{A}}$, $A \cap B_n = \tilde{0}$ for every $n \in N$ and $\lim_{n \in N} \tilde{m}(B_n) = 0$.

3) Auto-continuous from below if $\lim_{n \in N} \tilde{m}(A / B_n) = \tilde{m}(A)$, whenever A,$B_n \in \tilde{\mathcal{A}}$,$B_n \subseteq A$, for every $n \in N$ and $\lim_{n \in N} \tilde{m}(B_n) = 0$.

4) Auto-continuous if it is auto-continuous from above and auto-continuous from below.

5) Null-subtractive if $\tilde{m}(A \cap B^c) = \tilde{m}(A)$, whenever $A, B \in \tilde{\mathcal{A}}$ and $\tilde{m}(B) = 0$.

6) Have pseudometric generating property if for any $\epsilon > 0$, there exists δ>0 such that $\tilde{m}(E) \vee \tilde{m}(F) < \delta$ this implies that $\tilde{m}(E \cup F) < \epsilon$.

7) Weakly-null-countable additive if $\tilde{m}(\bigcup_{i=1}^{\infty} A_i) = 0$, whenever $A_i \in \tilde{\mathcal{A}}$ with $\tilde{m}(A_i) = 0$.

3-The convergence sequence of an intuitionistic fuzzy measurable functions

In this section , we introduced the definitions of the convergence almost everywhere and the convergence almost uniformly and we proved some relationships between them .

Definition(3.1):

Let $(X, \tilde{\mathcal{A}}, \tilde{m})$ be an intuitionistic fuzzy measure space and f □ $X \to [0, \infty]$ be a function , we say that f is an intuitionistic fuzzy real-valued measurable function on an IF σ-algebra $\tilde{\mathcal{A}}$ if $I\chi_{F_\alpha}$ □ $\tilde{\mathcal{A}}$, where $F_\alpha = \{x: f(x) \geq \alpha\}$ and

$$I\chi_{F_\alpha} = \begin{cases} \tilde{1}, & if\ x \ \square\ F_\alpha \\ \tilde{0}, & if\ x \ \square\ F_\alpha \end{cases}$$

Let $\mathcal{M}$ denoted the collection of all intuitionistic fuzzy real-valued measurable functions on $(X, \tilde{\mathcal{A}}, \tilde{m})$.

Definition(3.2):

Let f □ $\mathcal{M}$, A □ $\tilde{\mathcal{A}}$ and $\{f_n, n \geq 1\}$ □ $\mathcal{M}$ we say that :

1) $\{f_n\}$ converges to f everywhere on A and denote it by $f_n \xrightarrow{e.} f$ on A if there exists a subset D □ X with $I\chi_D$ □ $\tilde{\mathcal{A}}$ such that $\{f_n\}$ converges to f on D and A □ $I\chi_D$.

2) $\{f_n\}$ converges to f almost everywhere on A and denote it by $f_n \xrightarrow{a.e.} f$ on A if there exists a subset D □ X with $I\chi_D$ □ $\tilde{\mathcal{A}}$ and $\tilde{m}(I\chi_D) = 0$ such that $\{f_n\}$ converges to f everywhere on $A \cap I\chi_D^c$.

3) $\{f_n\}$ converges to f pseudo-almost everywhere on A and denote it by $f_n \xrightarrow{p.a.e.} f$ on A if there exists D □ X with $I\chi_D$ □ $\tilde{\mathcal{A}}$ and $\tilde{m}(A \cap I\chi_D^c) = \tilde{m}(A)$ and $\{f_n\}$ converges to f everywhere on $A \cap I\chi_D^c$.

Definition(3.3):

Let $f \,\square\, \mathcal{M}$, $A \,\square\, \tilde{\mathcal{A}}$ and $\{f_n, n \geq 1\} \,\square\, \mathcal{M}$ we say that :

1) $\{f_n\}$ converges to f uniformly on A and denote it by $f_n \overset{u.}{\rightarrow} f$ on A if there exists a subset $D \,\square\, X$ with $I\chi_D \,\square\, \tilde{\mathcal{A}}$ such that $\{f_n\}$ converges to f uniformly on D and $A \,\square\, I\chi_D$.

2) $\{f_n\}$ converges to f almost uniformly on A and denote it by $f_n \overset{a.u.}{\longrightarrow} f$ on A if for any $\delta > 0$ there exists $E \,\square\, \tilde{\mathcal{A}}$ with $\widetilde{m}(E) < \delta$, such that $\{f_n\}$ converges to f uniformly on $A \cap E^c$.

3) $\{f_n\}$ converges to f pseudo almost uniformly on A and denote it by $f_n \overset{p.a.u.}{\longrightarrow} f$ on A if there exists a sequence $\{E_n\}$ of IFSs in $\tilde{\mathcal{A}}$ such that $\lim_{n\to\infty} \widetilde{m}(A \cap E_n^c) = \widetilde{m}(A)$ and $\{f_n\}$ converges to f uniformly on $A \cap E_n^c$.

Theorem(3.4):

Let $\{f_n, f, n \geq 1\} \,\square\, \mathcal{M}, A \,\square\, \tilde{\mathcal{A}}$ and $\widetilde{m}$ is weakly-null-countable additive ,

1- If $f_n \overset{a.e.}{\longrightarrow} f$ and $f_n \overset{a.e.}{\longrightarrow} g$ on A , then $f = g$ $a.e.$ on A .

2- If $f_n \overset{a.e.}{\longrightarrow} f$ on A and g is an intuitionistic fuzzy real-valued measurable function such that $f = g$ $a.e.$ on A , then $f_n \overset{a.e.}{\longrightarrow} g$ on A.

3- If $f_n \overset{a.e.}{\longrightarrow} f$ on A and $\{g_n\}$ is a sequence of an intuitionistic fuzzy real-valued measurable functions such that $f_n = g_n$ $a.e.$ on A , then $g_n \overset{a.e.}{\longrightarrow} g$ on A .

Proof:

1- Since $f_n \overset{a.e.}{\longrightarrow} f$ on A , there exists a subset $D \,\square\, X$ with $I\chi_D \,\square\, \tilde{\mathcal{A}}$ and $\widetilde{m}(I\chi_D) = 0$ such that $f_n \overset{e.}{\rightarrow} f$ on $A \cap I\chi_D^c$.

$\Longrightarrow$ there exists $H \ \square X$ with $I\chi_H \ \square \ \tilde{\mathcal{A}}$ such that f_n converges to f on H and $A \cap I\chi_D^c \ \ \square \ I\chi_H$.

Since $f_n \xrightarrow{a.e.} g$ on A , then there exists a subset $N \ \square \ X$ with $I\chi_N \ \square \ \tilde{\mathcal{A}}$ and $\tilde{m}(I\chi_N) = 0$ such that $f_n \overset{e.}{\rightarrow} g$ on $A \cap I\chi_N^c$.

$\Longrightarrow$ there exists $M \ \square X$ with $I\chi_M \ \square \ \tilde{\mathcal{A}}$ such that f_n converges to g on M and $A \cap I\chi_N^c \ \ \square \ I\chi_M$.

Let $E = I\chi_D \ \square \ I\chi_N = I\chi_{D \square N} \Longrightarrow E \ \square \ \tilde{\mathcal{A}}$.

Since $\tilde{m}$ is weakly-null-countable additive $\Longrightarrow \tilde{m}(E) = 0$.

Since f_n converges to f on $H \Longrightarrow f_n(x) \longrightarrow f(x)$,$\square x \ \square \ H$

and f_n converges to g on $M \Longrightarrow f_n(x) \longrightarrow g(x)$, ,$\square x \ \square \ M$

$\Longrightarrow \ \square x \ \square \ H \cap M$, $f(x) = g(x) \Longrightarrow f = g$ on $H \cap M$

Since $A \cap I\chi_D^c \ \ \square \ I\chi_H$ and $A \cap I\chi_N^c \ \ \square \ I\chi_M$

$\Longrightarrow \ (A \cap I\chi_D^c) \cap (A \cap I\chi_N^c) \ \ \square \ I\chi_H \cap I\chi_M$

$\Longrightarrow \ A \cap (I\chi_D \ \square \ I\chi_N)^c = A \cap E^c \ \square \ I\chi_{H \cap M}$

Therefore, $f = g \ e.$ on $A \cap E^c$.

Since $D \ \square \ N \ \square \ X$, $\tilde{m}(I\chi_{D \square N}) = 0$.

So, $= g \ a.e. \ on \ A$.

2- Since $f_n \xrightarrow{a.e.} f$ on A , there exists a subset $D \ \square \ X$ with $I\chi_D \ \square \ \tilde{\mathcal{A}}$ and $\tilde{m}(I\chi_D) = 0$ such that $f_n \overset{e.}{\rightarrow} f$ on $A \cap I\chi_D^c$.

$\Longrightarrow$ there exists $H \ \square X$ with $I\chi_H \ \square \ \tilde{\mathcal{A}}$ such that f_n converges to f on H and $A \cap I\chi_D^c \ \ \square \ I\chi_H$.

Since $f = g \ a.e.$ on A , then there exists a subset $N \ \square \ X$ with $I\chi_N \ \square \ \tilde{\mathcal{A}}$ and $\tilde{m}(I\chi_N) = 0$ such that $f = g \ e.$ on $A \cap I\chi_N^c$.

$\Longrightarrow$ there exists $M \ \square X$ with $I\chi_M \ \square \ \tilde{\mathcal{A}}$ such that $f = g$ on M and $A \cap I\chi_N^c \ \ \square \ I\chi_M$.

Let $E = I\chi_D \ \square \ I\chi_N = I\chi_{D \square N} \Longrightarrow E \ \square \ \tilde{\mathcal{A}}$.

Since $\tilde{m}$ is weakly-null-countable additive $\Longrightarrow \tilde{m}(E) = 0$.

Since f_n converges to f on $H \Longrightarrow f_n(x) \longrightarrow f(x)$, $\square x \square H$

and $f = g$ on $M \Longrightarrow f(x) = g(x)$, , $\square x \square M$

$\Longrightarrow \square x \square H \cap M$, $f_n(x) \longrightarrow g(x) \Longrightarrow f_n \longrightarrow g$ on $H \cap M$.

Since $A \cap I\chi_D^c \;\square\; I\chi_H$ and $A \cap I\chi_N^c \;\square\; I\chi_M$

$\Longrightarrow (A \cap I\chi_D^c) \cap (A \cap I\chi_N^c) \;\square\; I\chi_H \cap I\chi_M$

$\Longrightarrow A \cap (I\chi_D \square I\chi_N)^c = A \cap E^c \square I\chi_{H \cap M}$

Therefore , $f_n \overset{e.}{\rightarrow} g$ on $A \cap E^c$.

Since $\square N \square X$, $\widetilde{m}(I\chi_{D \square N}) = 0$.

So , $f_n \overset{a.e.}{\longrightarrow} g$ on A .

3- Since $f_n \overset{a.e.}{\longrightarrow} f$ on A , there exists a subset $D \square X$ with $I\chi_D \square \tilde{\mathcal{A}}$ and $\widetilde{m}(I\chi_D) = 0$ such that $f_n \overset{e.}{\rightarrow} f$ on $A \cap I\chi_D^c$.

$\Longrightarrow$ there exists $H \square X$ with $I\chi_H \square \tilde{\mathcal{A}}$ such that f_n converges to f on H and $A \cap I\chi_D^c \;\square\; I\chi_H$.

Since $f_n = g_n\ a.e.$ on , then there exists a sequence $\{E_n\} \square X$ with $I\chi_{E_n} \square \tilde{\mathcal{A}}$ and $\widetilde{m}(I\chi_{E_n}) = 0$ for all $n \geq 1$ such that $f_n = g_n\ e.$ on $A \cap I\chi^c_{\square E_n}$.

$\Longrightarrow$ there exist $M_n \square X$ with $I\chi_{M_n} \square \tilde{\mathcal{A}}$ for all $n \geq 1$ such that $f_n = g_n$ on $\bigcap_{n=1}^{\infty} M_n$ and $A \cap I\chi^c_{\bigcup_{n=1}^{\infty} E_n} \square I\chi_{\bigcap_{n=1}^{\infty} M_n}$.

Let $C = I\chi_D \square I\chi_{\bigcup_{n=1}^{\infty} E_n} = I\chi_{\bigcup_{n=1}^{\infty}(D \square E_n)} \Longrightarrow C \square \tilde{\mathcal{A}}$, and since $\widetilde{m}$ is weakly-null-countable additive $\Longrightarrow \widetilde{m}(C) = 0$.

Since f_n converges to f on $H \Longrightarrow f_n(x) \longrightarrow f(x) \square x \square H$,

and $f_n = g_n$ on $\bigcap_{n=1}^{\infty} M_n \Longrightarrow f_n(x) = g_n(x) \quad \square x \square \bigcap_{n=1}^{\infty} M_n$

$\Longrightarrow g_n(x) \longrightarrow f(x) \;\; \square x \square H \cap (\bigcap_{n=1}^{\infty} M_n) = \bigcap_{n=1}^{\infty}(H \cap M_n)$

Thus , $g_n \longrightarrow f$ on $\bigcap_{n=1}^{\infty}(H \cap M_n)$.

Since $A \cap I\chi_D^c \;\square\; I\chi_H$ and $A \cap I\chi^c_{\bigcup_{n=1}^{\infty} E_n} \square I\chi_{\bigcap_{n=1}^{\infty} M_n}$

$\Longrightarrow (A \cap I\chi_D^c) \cap \left(A \cap I\chi^c_{\bigcup_{n=1}^{\infty} E_n}\right) \subset I\chi_H \cap I\chi_{\bigcap_{n=1}^{\infty} M_n}$

$\Longrightarrow A \cap \left(I\chi_D^c \ \cap I\chi^c_{\bigcup_{n=1}^{\infty} E_n} \right) = A \cap C^c \ \square \ I\chi_{\bigcap_{n=1}^{\infty}(H \cap M_n)}$.

Therefore , $g_n \overset{e.}{\to} g$ on $A \cap C^c$.

Since $\bigcup_{n=1}^{\infty}(D \ \square \ E_n) \ \square \ X$, $\tilde{m}\left(I\chi_{\bigcup_{n=1}^{\infty}(D \square E_n)}\right) = 0$ and $g_n \overset{e.}{\to} g$ on $A \cap C^c$.

So $g_n \overset{a.e.}{\longrightarrow} g$ on A .

Theorem(3.5):

Let $\{f_n, f , n \geq 1\} \ \square \ \mathcal{M}$, $A \ \square \ \tilde{\mathcal{A}}$ and $\tilde{m}$ is weakly-null-countable additive , $f_n \overset{a.e.}{\longrightarrow} f$ on A and $g_n \overset{a.e.}{\longrightarrow} g$ on A , $c \ \square \ R$,then

1) $c \cdot f_n \overset{a.e.}{\longrightarrow} c \cdot f$.

2) $f_n + g_n \overset{a.e.}{\longrightarrow} f + g$.

3) $|f_n| \overset{a.e.}{\longrightarrow} |f|$.

Proof:

1- Since $f_n \overset{a.e.}{\longrightarrow} f$ on A , there exists a subset $D \ \square \ X$ with $I\chi_D \ \square \ \tilde{\mathcal{A}}$ and $\tilde{m}(I\chi_D) = 0$ such that $f_n \overset{e.}{\to} f$ on $A \cap I\chi_D^c$.

$\Longrightarrow$ there exists $H \ \square \ X$ with $I\chi_H \ \square \ \tilde{\mathcal{A}}$ such that f_n converges to f on H and $A \cap I\chi_D^c \ \square \ I\chi_H$.

Since f_n converges to f on $H \Longrightarrow f_n(x) \longrightarrow f(x)$,$\square x \ \square \ H$

$\Longrightarrow \cdot f_n(x) \longrightarrow c \cdot f(x)$,$\square x \ \square \ H \Longrightarrow c \cdot f_n \longrightarrow c \cdot f$ on H .

Therefore, $\cdot f_n \overset{a.e.}{\longrightarrow} c \cdot f$.

2- Since $f_n \overset{a.e.}{\longrightarrow} f$ on A , there exists a subset $D \ \square \ X$ with $I\chi_D \ \square \ \tilde{\mathcal{A}}$ and $\tilde{m}(I\chi_D) = 0$ such that $f_n \overset{e.}{\to} f$ on $A \cap I\chi_D^c$.

$\Longrightarrow$ there exists $H \ \square \ X$ with $I\chi_H \ \square \ \tilde{\mathcal{A}}$ such that f_n converges to f on H and $A \cap I\chi_D^c \ \square \ I\chi_H$.

Since $g_n \overset{a.e.}{\longrightarrow} g$ on A , there exists a subset $N \ \square \ X$ with $I\chi_N \ \square \ \tilde{\mathcal{A}}$ and $\tilde{m}(I\chi_N) = 0$ such that $g_n \overset{e.}{\to} g$ on $A \cap I\chi_N^c$.

$\Longrightarrow$ there exists $M \ \square\ X$ with $I\chi_M \ \square\ \tilde{\mathcal{A}}$ such that g_n converges to g on M and $A \cap I\chi_N^c \ \square\ I\chi_M$.

Let $E = I\chi_D \ \square\ I\chi_N$, since $\tilde{m}$ is weakly-null-countable additive $\Longrightarrow$ $\tilde{m}(E) = 0$.

Since f_n converges to f on $H \Longrightarrow \square x \ \square\ H$ and $f_n(x) \longrightarrow f(x)$ and g_n converges to g on $M \Longrightarrow \square x \ \square\ M$, $g_n(x) \longrightarrow g(x)$

$\Longrightarrow \square x \ \square\ H \cap M$, $f_n(x) + g_n(x) \longrightarrow f(x) + g(x)$.

Thus , $f_n + g_n$ converges to $f + g$ on $H \cap M$.

Since $A \cap I\chi_D^c \ \square\ I\chi_H$ and $A \cap I\chi_N^c \ \square\ I\chi_M$

$\Longrightarrow (A \cap I\chi_D^c) \cap (A \cap I\chi_N^c) \ \square\ I\chi_H \cap I\chi_M = I\chi_{D\cap M}$

$\Longrightarrow A \cap (I\chi_D \ \square\ I\chi_N)^c = A \cap E^c \ \square\ I\chi_{H\cap M}$.

Therefore , $f_n + g_n \overset{e.}{\to} f + g$ on $A \cap E^c$.

Since $\square\ N \ \square\ X$, $\tilde{m}(I\chi_{D\square N}) = 0$ and $f_n + g_n \overset{e.}{\to} f + g$ on $A \cap E^c$, so $f_n + g_n \overset{a.e.}{\longrightarrow} f + g$ on A.

3- Since $f_n \overset{a.e.}{\longrightarrow} f$ on A , there exists a subset $D \ \square\ X$ with $I\chi_D \ \square\ \tilde{\mathcal{A}}$ and $\tilde{m}(I\chi_D) = 0$ such that $f_n \overset{e.}{\to} f$ on $A \cap I\chi_D^c$.

$\Longrightarrow$ there exists $H \ \square\ X$ with $I\chi_H \ \square\ \tilde{\mathcal{A}}$ such that f_n converges to f on H and $A \cap I\chi_D^c \ \square\ I\chi_H$.

Since f_n converges to f on $H \Longrightarrow f_n(x) \longrightarrow f(x)$,$\square x \ \square\ H$

$\Longrightarrow |f_n(x)| \longrightarrow |f(x)|$,$\square x \ \square\ H \Longrightarrow |f_n(x)| \longrightarrow |f(x)|$ on H .

Therefore , $|f_n| \overset{a.e.}{\longrightarrow} |f|$.

Theorem(3.6):

Let $\{f_n, f , n \geq 1\} \ \square\ \mathcal{M}$, $A \ \square\ \tilde{\mathcal{A}}$, if $\tilde{m}$ is p.g.p. and weakly-null-countable additive, then:

1) If $f_n \overset{a.u.}{\longrightarrow} f$ and $f_n \overset{a.u.}{\longrightarrow} g$ on A ,then $f = g$ $a.e.$ on A .

2) If $f_n \overset{a.u.}{\longrightarrow} f$ on A and $f = g$ $a.e.$ on A ,then $f_n \overset{a.u.}{\longrightarrow} g$ on A .

3) If $f_n \overset{a.u.}{\longrightarrow} f$ and $f_n = g_n \ a.e.$ on A ,then $g_n \overset{a.u.}{\longrightarrow} f$ on A.

Proof:

1) since $f_n \overset{a.u.}{\longrightarrow} f$ on A , then for any $\delta > 0$, there exists E □ $\tilde{\mathcal{A}}$ with $\tilde{m}(E) < \delta$ such that f_n converges to f uniformly on $A \cap E^c$

$\Longrightarrow$ there exists a subset D □ X with $I\chi_D$ □ $\tilde{\mathcal{A}}$ and f_n converges to f uniformly on D and $\cap E^c$ □ $I\chi_D$.

Since $f_n \overset{a.u.}{\longrightarrow} g$ on A then for any > 0 , there exists F □ $\tilde{\mathcal{A}}$ with $\tilde{m}(F) < \delta$ such that f_n converges to g uniformly on $A \cap F^c$

$\Longrightarrow$ there exists a subset H □ X with $I\chi_H$ □ $\tilde{\mathcal{A}}$ and f_n converges to g uniformly on H and $\cap F^c$ □ $I\chi_H$.

Since $\tilde{m}$ is p.g.p. $\Longrightarrow \tilde{m}(E$ □ $F) < \epsilon$, since ϵ arbitrary so $\tilde{m}(E$ □ $F) = 0$

Since f_n converges to f uniformly on $D \Longrightarrow$ □x □ D , $f_n(x) \to f(x)$ and f_n converges to g uniformly on $H \Longrightarrow$ □x □ H , $f_n(x) \to g(x)$

$\Longrightarrow$ □x □ $D \cap H$, $f(x) = g(x)$, so $f = g$ on $D \cap H$.

Since $A \cap E^c$ □ $I\chi_D$ and $A \cap F^c$ □ $I\chi_H$

$\Longrightarrow A \cap (E$ □ $F)^c$ □ $I\chi_{D\cap H}$.

Since E □ F □ $\tilde{\mathcal{A}}$ and $\tilde{m}(E$ □ $F) = 0$.

Therefore , $f = g \ a.e.$.

2) since $f_n \overset{a.u.}{\longrightarrow} f$ on A , then for any $\delta > 0$, there exists E □ $\tilde{\mathcal{A}}$ with $\tilde{m}(E) < \delta$ such that f_n converges to f uniformly on $A \cap E^c$

$\Longrightarrow$ there exists a subset D □ X with $I\chi_D$ □ $\tilde{\mathcal{A}}$ and f_n converges to f uniformly on D and $\cap E^c$ □ $I\chi_D$.

Since $f = g \ a.e.$ on A , then there exists F □ X with $I\chi_F$ □ $\tilde{\mathcal{A}}$ and $\tilde{m}(I\chi_F) = 0$ such that $f = g$ everywhere on $A \cap I\chi_F^c$

$\Longrightarrow$there exists a subset H □ X with $I\chi_H$ □ $\tilde{\mathcal{A}}$ and $f = g$ on H and $\cap I\chi_F^c$ □ $I\chi_H$.

Since $\tilde{m}(I\chi_F) = 0 < \delta$ and $\tilde{m}$ is p.g.p. , then $\tilde{m}(E$ □ $I\chi_F) < \epsilon$.

Since $f_n(x) \to f(x)$ uniformly $\square x \square D$ and $f(x) = g(x)$ $\square x \square H$

$\Longrightarrow f_n(x) \to g(x)$ uniformly $\square x \square D \cap H$, so

$f_n \to g$ uniformly on $D \cap H$.

Since $A \cap E^c \square I\chi_D$ and $A \cap I\chi_F^c \square I\chi_H$ this implies that

$A \cap (E \square I\chi_F)^c \square I\chi_{D\cap H}$.

Since $E \square I\chi_F \square \tilde{\mathcal{A}}$ and $\tilde{m}(E \square I\chi_F) < \epsilon$.

So $f_n \overset{a.u.}{\longrightarrow} g$ on A .

3) since $f_n \overset{a.u.}{\longrightarrow} f$ on A , then for any $\delta > 0$, there exists $E \square \tilde{\mathcal{A}}$ with $\tilde{m}(E) < \delta$ such that f_n converges to f uniformly on $A \cap E^c$

$\Longrightarrow$ there exists a subset $D \square X$ with $I\chi_D \square \tilde{\mathcal{A}}$ and f_n converges to f uniformly on D and $\cap E^c \square I\chi_D$.

Since $f_n = g_n\ a.e.$ on A , then there exists a sequence $\{F_n\}$ □X with $I\chi_{F_n} \square \tilde{\mathcal{A}}$ and $\tilde{m}(I\chi_{F_n}) = 0$ for all $n \geq 1$ such that $f_n = g_n\ e.$ on $A \cap I\chi^c_{\square F_n}$.

$\Longrightarrow$ there exist $M_n \square X$ with $I\chi_{M_n} \square \tilde{\mathcal{A}}$ for all $n \geq 1$ such that

$f_n = g_n$ on $\bigcap_{n=1}^{\infty} M_n$ and $A \cap I\chi^c_{\bigcup_{n=1}^{\infty} F_n} \square I\chi_{\bigcap_{n=1}^{\infty} M_n}$.

Since $\tilde{m}(I\chi_{F_n}) = 0$ $\forall n \geq 1$ and $\tilde{m}$ is weakly-null-countable additive

$\Longrightarrow \tilde{m}(\cup I\chi_{F_n}) = \tilde{m}(I\chi_{\cup F_n}) = 0 < \delta$ and since $\tilde{m}$ is p.g.p.

$\Longrightarrow$ $\tilde{m}(E \cup I\chi_{\square F_n}) < \epsilon$.

Since f_n converges to f uniformly on $D \Longrightarrow f_n(x) \to f(x)$ uniformly $\square x \square D$ and $f_n = g_n$ on $\bigcap_{n=1}^{\infty} M_n \Longrightarrow f_n(x) = g_n(x)$ $\square x \square \bigcap_{n=1}^{\infty} M_n$.

This implies that $g_n(x) \to f(x)$ uniformly $\square x \square D \cap (\bigcap_{n=1}^{\infty} M_n)$, so

$g_n \to f$ on $\cap (\bigcap_{n=1}^{\infty} M_n)$.

$A \cap E^c \square I\chi_D$ and $A \cap I\chi^c_{\bigcup_{n=1}^{\infty} F_n} \square I\chi_{\bigcap_{n=1}^{\infty} M_n}$

$\Longrightarrow A \cap (E \cup I\chi_{\square F_n})^c \subset I\chi_{D\cap(\bigcap_{n=1}^{\infty} M_n)}$ and Since $E \square I\chi_{\square F_n} \square \tilde{\mathcal{A}}$

and $\tilde{m}(E \cup I\chi_{\square F_n}) < \epsilon \Longrightarrow$ Therefore , $g_n \overset{a.u.}{\longrightarrow} f$ on A.

Theorem(3.7):

Let $\{f_n, f, n \geq 1\} \Box \mathcal{M}$, $A \Box \tilde{\mathcal{A}}$ and $\tilde{m}$ is p.g.p. , $c \Box R$, $f_n \xrightarrow{a.u.} f$,
$g_n \xrightarrow{a.u.} g$, on A, then:

1) $c \cdot f_n \xrightarrow{a.u.} c \cdot f$.

2) $f_n + g_n \xrightarrow{a.u.} f + g$.

3) $|f_n| \xrightarrow{a.u.} |f|$.

Proof:

We only prove (2) and the proofs of (1) and (3) are similarly .

2) since $f_n \xrightarrow{a.u.} f$ on A , then for any $\delta > 0$, there exists $E \Box \tilde{\mathcal{A}}$ with $\tilde{m}(E) < \delta$ such that f_n converges to f uniformly on $A \cap E^c$.

$\Longrightarrow$ there exists a subset $D \Box X$ with $I\chi_D \Box \tilde{\mathcal{A}}$ and f_n converges to f uniformly on D and $\cap E^c \Box I\chi_D$.

since $g_n \xrightarrow{a.u.} g$ on A , then for any $\delta > 0$, there exists $F \Box \tilde{\mathcal{A}}$ with $\tilde{m}(F) < \delta$ such that g_n converges to g uniformly on $A \cap F^c$.

$\Longrightarrow$ there exists a subset $M \Box X$ with $I\chi_M \Box \tilde{\mathcal{A}}$ and g_n converges to g uniformly on M and $A \cap F^c \Box I\chi_M$.

Since $\tilde{m}$ is p.g.p ,then $\tilde{m}(E \Box F) < \epsilon$, for $\epsilon > 0$.

Since f_n converges to f uniformly on D

$\Longrightarrow \Box x \Box D$, $f_n(x) \longrightarrow f(x)$, and g_n converges to g uniformly on M

$\Longrightarrow \Box x \Box M$, $g_n(x) \longrightarrow g(x)$

$\Longrightarrow \Box x \Box D \cap M$, $f_n(x) + g_n(x) \longrightarrow f(x) + g(x)$, then $f_n + g_n$ converges to $f + g$ uniformly on $D \cap M$.

Since $A \cap E^c \Box I\chi_D$ and $A \cap F^c \Box I\chi_M \Longrightarrow \cap (E \Box F)^c \Box I\chi_{D \cap M}$.

Since $E \Box F \Box \tilde{\mathcal{A}}$ and $\tilde{m}(E \Box F) < \epsilon$.

So , $f_n + g_n \xrightarrow{a.u.} f + g$ on A.

Lemma (3.8):

Let $\tilde{m}$ have pseudometric generating property . If $\lim_{n\to\infty} \tilde{m}(E_n) = 0$, then there exists a sequence $\{\delta_r\}_r$ of positive real numbers and a subsequence $\{E_{n(i)}\}_i$ □ $\{E_n\}_n$ with δ_r □ 0 such that $\tilde{m}\left(\bigcup_{i=r+1}^{\infty} E_{n(i)}\right) < \delta_r$,□$r \geq 1$.

Theorem(3.9):

Let $\{f_n\}_n$□$\mathcal{M}$,f □ $\mathcal{M}$,A □ $\tilde{\mathcal{A}}$ and $A \cap A^c = \tilde{0}$ for every A □ $\tilde{\mathcal{A}}$, then :

1) If $\tilde{m}$ is order-continuous and has p.g.p., then if $f_n \xrightarrow{a.e.} f$ on A $\Longrightarrow f_n \xrightarrow{a.u.} f$ on A .

2) If $\tilde{m}$ is order-continuous , p.g.p and autocontinuous from below , then if $f_n \xrightarrow{a.e.} f$ on A $\Longrightarrow f_n \xrightarrow{p.a.u.} f$ on A .

3) If $\tilde{m}$ is order-continuous, p.g.p and null-subtractive ,then if $f_n \xrightarrow{p.a.e.} f$ on A $\Longrightarrow f_n \xrightarrow{a.u.} f$ on A.

4) If $\tilde{m}$ is order-continuous , p.g.p , null-subtractive and autocontinuous from below , then if $f_n \xrightarrow{p.a.e.} f$ on A $\Longrightarrow f_n \xrightarrow{p.a.u.} f$ on A.

Proof:

1) Since $f_n \xrightarrow{a.e.} f$ on A ,then there exists a subset D □ X with $I\chi_D$ □ $\tilde{\mathcal{A}}$ and $\tilde{m}(I\chi_D) = 0$ such that $f_n \overset{e.}{\rightarrow} f$ on $A \cap I\chi_D^c$.

Let $B = A \cap I\chi_D^c$.

$\Longrightarrow$there exists H □ X with $I\chi_H$ □ $\tilde{\mathcal{A}}$ such that f_n converges to f on H and □ $I\chi_H$.

Let $E_n^{(m)} = \bigcap_{i=1}^{\infty}\{x \in X, |f_i(x) - f(x)| < \frac{1}{m}\}, \forall\, m \geq 1$

Then, for each fixed $m \geq 1$, $E_n^{(m)}$ is increasing on n and , therefore, $I\chi_{E_n^{(m)}}$ is increasing on n .

Since f_n converges everywhere to f on $B \Longrightarrow B$ □ $\lim_{n\to\infty} I\chi_{E_n^{(m)}}$

$\Longrightarrow B$ □ $\bigcup_{n=1}^{\infty} I\chi_{E_n^{(m)}}$.

Therefore we have $B \cap \left(\bigcup_{n=1}^{\infty} I\chi_{E_n^{(m)}}\right)^c = \tilde{0}$

$\Longrightarrow B \cap \left(\bigcup_{n=1}^{\infty} I\chi_{E_n^{(m)}}\right)^c = \bigcap_{n=1}^{\infty}(B \cap I\chi^c_{E_n^{(m)}}) = \tilde{0}$.

Thus , we get $\lim_{n\to\infty} B \cap I\chi^c_{E_n^{(m)}} = \tilde{0} \Longrightarrow B \cap I\chi^c_{E_n^{(m)}}$ □ $\tilde{0}$.

From the order-continuity of $\widetilde{m}$, we have $\lim_{n\to\infty} \widetilde{m}(B \cap I\chi^c_{E_n^{(m)}})$ =0,

and hence , there exists a subsequence $\}B \cap I\chi^c_{E_{n_m}^{(m)}}\{_{m=1}^{\infty}$ of

$\}B \cap I\chi^c_{E_n^{(m)}}\{_{n,m=1}^{\infty}$ such that $\widetilde{m}\left(B \cap I\chi^c_{E_{n_m}^{(m)}}\right) < \frac{1}{m}$ for any $m \geq 1$.

Thus, $\lim_{n\to\infty} \widetilde{m}\left(B \cap I\chi^c_{E_{n_m}^{(m)}}\right) = 0$.

Therefore , by lemma(3.8) , there exists a sequence $\{\delta_r\}_{r=1}^{\infty}$ of

positive real numbers and a subsequence $\}B \cap I\chi^c_{E_{n_{m_i}}^{(m_i)}}\{_i$ of

$\}B \cap I\chi^c_{E_{n_m}^{(m)}}\{_m$ such that δ_r □ 0 and

$\widetilde{m}\left|\bigcup_{i=r+1}^{\infty}\left(B \cap I\chi^c_{E_{n_{m_i}}^{(m_i)}}\right)\right| < \delta_r$, $r \geq 1$.

For any > 0 , since $\widetilde{m}$ has p.g.p. there exists $\sigma > 0$ such that

$\widetilde{m}(E)$ □ $\widetilde{m}(F) < \sigma \Longrightarrow \widetilde{m}(E$ □ $F) < \delta$.

For $\sigma > 0$ above , we can find $r_0 \geq 1$ such that $\delta_{r_0} < \sigma$.

If we take $E = \bigcup_{i=r_0+1}^{\infty}(B \cap I\chi^c_{E_{n_{m_i}}^{(m_i)}})$, then E □ $\tilde{\mathcal{A}}$ and $\widetilde{m}(E) < \sigma$.

Since $\widetilde{m}(I\chi_D) = 0 < \sigma$, therefore $\widetilde{m}(I\chi_D$□$E) < \delta$.

To prove that $\{f_n\}$ converges to f almost uniformly on , we will prove that $\{f_n\}$ converges to f uniformly on $A \cap (I\chi_D$ □ $E)^c$.

Since $A \cap A^c = \tilde{0}$ for every $\square\ \tilde{\mathcal{A}}$, we have $B \cap B^c = \tilde{0}$

$A \cap (I\chi_D \square E)^c = A \cap I\chi_D^c \cap E^c \ \square\ I\chi_{\bigcap_{r_0+1}^{\infty} E_{n_{m_i}}^{(m_i)}}$.

for any $\epsilon > 0$, we take $i_0 > r_0$ such that $m_{i_0+1} > \frac{1}{\epsilon}$.

Then , $x \square \bigcap_{i=r_0+1}^{\infty} E_{n_{m_i}}^{(m_i)} \Longrightarrow x \square E_{n_{m_i}}^{(m_i)}$ as $i > r_0 + 1$.

Thus , $x \square \bigcap_{i=n_{m_{i_0+1}}}^{\infty} \}x \square X, |f_i(x) - f(x)| < \frac{1}{m_{i_0+1}}\{$.

This means that $|f_i(x) - f(x)| < \frac{1}{m_{i_0+1}} < \epsilon$ as $i \geq n_{m_{i_0+1}}$ so that

$f_n$converges to f uniformly on $\bigcap_{i=r_0+1}^{\infty} E_{n_{m_i}}^{(m_i)}$ and since

$A \cap (I\chi_D \square E)^c \square I\chi_{\bigcap_{r_0+1}^{\infty} E_{n_{m_i}}^{(m_i)}}$ and $\tilde{m}(I\chi_D \square E) < \delta$.

Therefore , $f_n$converges to f almost uniformly on A .

2) Since $f_n \xrightarrow{a.e.} f$ on A , there exists a subset $D \square X$ with $I\chi_D \square \tilde{\mathcal{A}}$

and $\tilde{m}(I\chi_D) = 0$ such that $f_n \overset{e.}{\rightarrow} f$ on $A \cap I\chi_D^c$.

Let $B = A \cap I\chi_D^c$.

$\Longrightarrow$ there exists $H \square X$ with $I\chi_H \square \tilde{\mathcal{A}}$ such that f_n converges to f on H and $\square\ I\chi_H$.

Let $E_n^{(m)} = \bigcap_{i=1}^{\infty} \{x \in X, |f_i(x) - f(x)| < \frac{1}{m}\}, \forall m \geq 1$

Then for each fixed $m \geq 1$, $E_n^{(m)}$ is increasing on n and , therefore,

$I\chi_{E_n^{(m)}}$ is increasing on n and as in proof (1) , we have

$\tilde{m}(I\chi_D \square E) < \delta$.

Therefore , $\lim_{n\to\infty} \tilde{m}(I\chi_D \square E) = 0$.

Let $B_n = I\chi_D \square E \Longrightarrow \lim_{n\to\infty} \tilde{m}(B_n) = 0$.

By using the autocontinuity from below of $\tilde{m}$, we have

$\lim_{n\to\infty} \tilde{m}(A \cap B_n^c) = \tilde{m}(A)$.

Now, we will prove that f_n converges to f uniformly on $\cap\ B_n^c$.

Since $B \cap B^c = \tilde{0}$,

$A \cap B_n^c = A \cap I\chi_D^c \cap E^c$ □ $I\chi_{\cap_{r_0+1}^{\infty} E_{n_{m_i}}^{(m_i)}}$,

and as in proof (1) we get f_n converges to f uniformly on $\bigcap_{r_0+1}^{\infty} E_{n_{m_i}}^{(m_i)}$

and therefore , $f_n \xrightarrow{p.a.u.} f$ on A .

3) Since $f_n \xrightarrow{p.a.e.} f$ on , there exists a subset D □ X with $I\chi_D$ □ $\tilde{\mathcal{A}}$

and $\tilde{m}(A \cap I\chi_D^c) = \tilde{m}(A)$ such that $f_n \overset{e.}{\rightarrow} f$ on $A \cap I\chi_D^c$.

Let $B = A \cap I\chi_D^c$.

$\Longrightarrow$ there exists H □X with $I\chi_H$ □ $\tilde{\mathcal{A}}$ such that f_n converges to f on H

and □ $I\chi_H$.

Since $\tilde{m}$ is null-subtractive and $\tilde{m}(A \cap I\chi_D^c) = \tilde{m}(A) \Longrightarrow \tilde{m}(I\chi_D) = 0$.

Let $E_n^{(m)} = \bigcap_{i=1}^{\infty} \{x \in X, |f_i(x) - f(x)| < \frac{1}{m}\}, \forall\, m \geq 1$

Then, for each fixed $m \geq 1$, $E_n^{(m)}$ is increasing on n and , therefore,

$I\chi_{E_n^{(m)}}$ is increasing on n and as in proof (1) , we have

$\tilde{m}(I\chi_D$□$E) < \delta$.

To prove that $\{f_n\}$ converges to f almost uniformly on , we will prove that $\{f_n\}$ converges to f uniformly on $A \cap (I\chi_D$ □ $E)^c$.

Since $\cap B^c = \tilde{0}$,

$A \cap (I\chi_D$ □ $E)^c = A \cap I\chi_D^c \cap E^c$ □ $I\chi_{\cap_{r_0+1}^{\infty} E_{n_{m_i}}^{(m_i)}}$,

and as in proof (1) we get f_n converges to f uniformly on $\bigcap_{r_0+1}^{\infty} E_{n_{m_i}}^{(m_i)}$

and therefore , $f_n \xrightarrow{p.a.u.} f$ on A .

4) Since $f_n \xrightarrow{p.a.e.} f$ on A , there exists a subset D □ X with $I\chi_D$ □ $\tilde{\mathcal{A}}$

and $\tilde{m}(A \cap I\chi_D^c) = \tilde{m}(A)$ such that $f_n \overset{e.}{\rightarrow} f$ on $A \cap I\chi_D^c$.

Let $B = A \cap \chi_D^c$.

$\Longrightarrow$ there exists H □X with $I\chi_H$ □ $\tilde{\mathcal{A}}$ such that f_n converges to f on H and □ $I\chi_H$.

Since $\tilde{m}$ is null-subtractive and $\tilde{m}(A \cap I\chi_D^c) = \tilde{m}(A) \Longrightarrow \tilde{m}(I\chi_D) = 0$.

Let $E_n^{(m)} = \bigcap_{i=1}^{\infty}\{x \in X, |f_i(x) - f(x)| < \frac{1}{m}\}, \forall m \geq 1$

Then, for each fixed $m \geq 1$, $E_n^{(m)}$ is increasing on n and , therefore, $I\chi_{E_n^{(m)}}$ is increasing on n and as in proof (1) , we have

$\tilde{m}(I\chi_D \square E) < \delta$.

Therefore , $\lim_{n\to\infty} \tilde{m}(I\chi_D \square E) = 0$.

Let $B_n = I\chi_D \square E \square \lim_{n\to\infty} \tilde{m}(B_n) = 0$.

By using the autocontinuity from below of $\tilde{m}$, we have

$\lim_{n\to\infty} \tilde{m}(A \cap B_n^c) = \tilde{m}(A)$

Now, we will prove that f_n converges to f uniformly on $\cap B_n^c$.

Since $\cap B^c = \tilde{0}$,

$A \cap B_n^c = A \cap I\chi_D^c \cap E^c \square I\chi_{\bigcap_{r_0+1}^{\infty} E_{n_{m_i}}^{(m_i)}}$,

and as in proof (1) we get f_n converges to f uniformly on $\bigcap_{r_0+1}^{\infty} E_{n_{m_i}}^{(m_i)}$

and therefore , $f_n \xrightarrow{p.a.u.} f$ on A .

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