On a Certain Class of Harmonic Multivalent Functions Defined By Generalized Ruscheweyh Derivative Operator and Hadamard Product

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Abstract: We introduce a new class of harmonic multivalent functions define by generalized Ruscheweyh derivative operator and Hadamard product. We obtain coefficient conditions, application of fractional calculus operators and integral operator of this class.

Keywords: Multivalent function, Harmonic function, Generalized Ruscheweyh derivative operator, Hadamard product, Fractional calculus, Integral operator.

الملخص :
في هذا البحث قدمنا نوع جديد من دالة التعدديه التوافقية المعرفه بمؤثر Ruscheweyh التفاضلي وضرب Hadamard. حصلنا على شروط المعاملات وتطبيقها على مؤثر التفاضل الكسري ومؤثر التكامل لهذا الصنف من الدوال .

1. Introduction:

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subseteq \mathbb{C}$, we can write $f = h + \bar{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. Note that $f = h + \bar{g}$ reduces to h if the co-analytic part g is zero. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$. See Clunie and Sheil-Small [5] (see also [3], [8] and [13]).

For $p \geq 1$, denote by $HA(p)$ the set of all multivalent harmonic functions f of the form:

$$f = h + \bar{g} \ , \tag{1}$$

where h and g are of the form:

$$h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=p}^{\infty} b_k z^k, \qquad |b_p| < 1$$

which are analytic multivalent harmonic functions and sense preserving in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.
Let F be a fixed multivalent harmonic functions given by

$$F(z) = H(z) + \overline{G(z)} = z^p + \sum_{k=p+1}^{\infty} |c_k| z^k + \overline{\sum_{k=p}^{\infty} |d_k| z^k}, \ \ |d_p| < 1. \tag{2}$$

The class HA(p) for p=1 was defined and studied by Jahangiri et al. in [10].
For fixed positive integers p, and for $0 \le \alpha < 1$, $\beta \ge 0$ we let $H_\lambda^n A(p,\alpha,\beta)$ denote the class of multivalent harmonic functions of the form (1), then the functions $f \in$HA(p) is said to be in the class $H_\lambda^n A(p,\alpha,\beta)$ if satisfy the condition

$$\mathrm{Re}\left\{\frac{\left[D_\lambda^{n+p-1}(f*F)(z)\right]'}{pz^{p-1}}\right\} > \beta\left|\frac{\left[D_\lambda^{n+p-1}(f*F)(z)\right]'}{pz^{p-1}} - 1\right| + \alpha, \qquad (3)$$

where

$$D_\lambda^{n+p-1}(f*F)(z) = D_\lambda^{n+p-1}(h*H)(z) + D_\lambda^{n+p-1}(g*G)(z) \qquad (4)$$

and $f * F$ is a harmonic convolution of f and F.
The operator D_λ^{n+p-1} denotes the generalized Ruscheweyh derivative operator introduced in [4].
For h and g given by (1), we obtain

$$D_\lambda^{n+p-1}(h)(z) = z^p + \sum_{k=p+1}^{\infty}(1+(k-p)\lambda)C(n,k,p)a_k\, z^k \qquad (5)$$

and

$$D_\lambda^{n+p-1}(g)(z) = \sum_{k=p}^{\infty}(1+(k-p)\lambda)C(n,k,p)b_k\, z^k, \qquad (6)$$

where $\lambda \ge 0$, $p \in \mathbb{N}$, $n > -p$ and $C(n,k,p) = \binom{k+n-1}{n+p-1}$.
Finally, we denote by HT(p) the subclass of functions of the form $f = h + \bar{g}$ of the class HA(p), where

$$h(z) = z^p - \sum_{k=p+1}^{\infty}|a_k|\, z^k, \qquad g(z) = -\sum_{k=p}^{\infty}|b_k|\, z^k, \quad |b_P| < 1. \qquad (7)$$

Let $H_\lambda^n T(p,\alpha,\beta) = HT(p) \cap H_\lambda^n A(p,\alpha,\beta)$.
Silverman [12], proved that the coefficient conditions

$$\sum_{k=2}^{\infty} k\,(|a_k| + |b_k|) \le 1 \quad \text{and} \quad \sum_{k=2}^{\infty} k^2\,(|a_k| + |b_k|) \le 1$$

are necessary if $f = h + \bar{g}$ has negative coefficient, also Jahangiri [8] showed that the coefficient condition

$$\sum_{k=2}^{\infty}(k-\alpha)\,|a_k| + \sum_{k=1}^{\infty}(k+\alpha)\,|b_k| \le 1-\alpha$$

and

$$\sum_{k=2}^{\infty} k(k-\alpha)\,|a_k| + \sum_{k=1}^{\infty} k(k+\alpha)\,|b_k| \le 1-\alpha,$$

are necessary and sufficient conditions for functions $f = h + \bar{g}$ to be harmonic starlike with negative coefficient and harmonic convex with negative coefficient respectively.

In this paper, we obtain the coefficient condition for the function $f = h + \bar{g}$, where h and g given by (1) to be in the class $H_\lambda^n A(p,\alpha,\beta)$ and it is shown that the coefficient condition for the function in the class $H_\lambda^n T(p,\alpha,\beta)$. Furthermore, we determine application of fractional calculus operators and integral operator for the function in the class $H_\lambda^n T(p,\alpha,\beta)$.

2. Main Results:

In the first theorem, we introduce a sufficient coefficient bound for harmonic functions in the class $H_\lambda^n A(p,\alpha,\beta)$.

Theorem 1: Let $f = h + \bar{g}$ with h and g are given by (1). Let

$$\sum_{k=p+1}^{\infty} \frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)|a_k c_k| + \sum_{k=p}^{\infty} \frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)|b_k d_k| \le 1, \quad (8)$$

where $a_p = c_p = 1, \beta \ge 0, \lambda \ge 0, 0 \le \alpha < 1$ *and* $p \in \mathbb{N}$. Then f is harmonic multivalent, sense preserving in U and $f \in H_\lambda^n A(p,\alpha,\beta)$.

Proof: For $|z_1| \le |z_2| < 1$, $z_1 \ne z_2$, we have by inequality (8)

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \ge 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$= 1 - \left|\frac{\sum_{k=p}^{\infty} b_k(z_1^k - z_2^k)}{(z_1^p - z_2^p) + \sum_{k=p+1}^{\infty} a_k(z_1^k - z_2^k)}\right| > 1 - \left|\frac{\sum_{k=p}^{\infty} k|b_k|}{1 - \sum_{k=p+1}^{\infty} k|a_k|}\right|$$

$$\ge 1 - \frac{\sum_{k=p}^{\infty} \frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)|b_k d_k|}{1 - \sum_{k=p+1}^{\infty} \frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)|a_k c_k|} \ge 0.$$

Hence, f is multivalent in U. f is sense preserving in U. This is because

$$|h'(z)| \geq p|z|^{p-1} - \sum_{k=p+1}^{\infty} k|a_k||z|^{k-1} > p - \sum_{k=p+1}^{\infty} k|a_k|$$

$$> 1 - \sum_{k=p+1}^{\infty} \frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)|a_k c_k|$$

$$\geq \sum_{k=p}^{\infty} \frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)|b_k d_k|$$

$$\geq \sum_{k=p}^{\infty} k|b_k| > \sum_{k=p}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|.$$

Now, we show that $f \in H_\lambda^n A(p,\alpha,\beta)$. Using the fact

$$\mathrm{Re}\{w\} > \beta|w-1| + \alpha \Leftrightarrow \mathrm{Re}\left(\left(1+\beta e^{i\theta}\right)w - \beta e^{i\theta}\right) \geq \alpha,$$

it follows that $f \in H_\lambda^n A(p,\alpha,\beta)$ if and only if

$$\mathrm{Re}\left(\frac{\left(1+\beta e^{i\theta}\right)\left[D_\lambda^{n+p-1}(f*F)(z)\right]'}{pz^{p-1}} - \beta e^{i\theta}\right) \geq \alpha, \qquad 0 \leq \alpha < 1$$

We need to prove that $\mathrm{Re}\{w\} > 0$, where

$$w = \frac{\left(1+\beta e^{i\theta}\right)\left[D_\lambda^{n+p-1}(f*F)(z)\right]' - p\left(\alpha+\beta e^{i\theta}\right)}{pz^{p-1}} = \frac{A(z)}{B(z)}.$$

Also, using the fact $\mathrm{Re}\{w\} > 0 \Leftrightarrow |1+w| \geq |1-w|$, it sufficient to show that

$$|A(z)+B(z)| - |A(z)-B(z)| \geq 0.$$

Therefore

$$|A(z)+B(z)| - |A(z)-B(z)|$$

$$= \left| p(2-\alpha)z^{p-1} + \sum_{k=p+1}^{\infty} k\left(1+\beta e^{i\theta}\right)(1+(k-p)\lambda)C(n,k,p)a_k c_k z^{k-1} \right.$$
$$\left. - \sum_{k=p}^{\infty} k\left(1+\beta e^{i\theta}\right)(1+(k-p)\lambda)C(n,k,p)\overline{b_k d_k}\,\overline{z}^{k-1} \right|$$

$$-\left|-\alpha p z^{p-1}+\sum_{k=p+1}^{\infty} k\left(1+\beta e^{i\theta}\right)(1+(k-p)\lambda)C(n,k,p)a_k c_k z^{k-1}\right.$$
$$\left.-\sum_{k=p}^{\infty} k\left(1+\beta e^{i\theta}\right)(1+(k-p)\lambda)C(n,k,p)\overline{b_k d_k}\bar{z}^{k-1}\right|$$
$$\geq p(2-\alpha)|z|^{p-1}-\sum_{k=p+1}^{\infty} k\left(1+\beta e^{i\theta}\right)(1+(k-p)\lambda)C(n,k,p)|a_k c_k||z|^{k-1}$$
$$-\sum_{k=p}^{\infty} k\left(1+\beta e^{i\theta}\right)(1+(k-p)\lambda)C(n,k,p)|b_k d_k||z|^{k-1}$$
$$-\alpha p|z|^{p-1}-\sum_{k=p+1}^{\infty} k\left(1+\beta e^{i\theta}\right)(1+(k-p)\lambda)C(n,k,p)|a_k c_k||z|^{k-1}$$
$$-\sum_{k=p}^{\infty} k\left(1+\beta e^{i\theta}\right)(1+(k-p)\lambda)C(n,k,p)|b_k d_k||z|^{k-1}$$
$$\geq 2p(1-\alpha)\left\{1-\sum_{k=p+1}^{\infty}\frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)|a_k c_k|\right.$$
$$\left.-\sum_{k=p}^{\infty}\frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)|b_k d_k|\right\}.$$

By hypothesis.
The above expression is non negative by (8), and so $f\in H_\lambda^n A(p,\alpha,\beta)$. The coefficient bounds (8) is sharp for the function

$$f(z)=z^p+\sum_{k=p+1}^{\infty}\frac{p(1-\alpha)}{k(1+\beta)(1+(k-p)\lambda)C(n,k,p)}X_k z^k$$
$$+\sum_{k=p}^{\infty}\frac{p(1-\alpha)}{k(1+\beta)(1+(k-p)\lambda)C(n,k,p)}\overline{Y}_k(\bar{z})^k,$$

where

$$\sum_{k=p+1}^{\infty}|X_k|+\sum_{k=p}^{\infty}|Y_k|=1.$$

<u>Theorem 2:</u> Let $f=h+\bar{g}$ be given by (7). Then $f\in H_\lambda^n T(p,\alpha,\beta)$ if and only if

$$\sum_{k=p+1}^{\infty}\frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)|a_k c_k|$$
$$+\sum_{k=p}^{\infty}\frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)|b_k d_k|\leq 1, \qquad (9)$$

where $a_p = c_p = 1, \beta \geq 0, \lambda \geq 0, 0 \leq \alpha < 1$ *and* $p \in \mathbb{N}$.

Proof: Since $H_\lambda^n T(p,\alpha,\beta) \subset H_\lambda^n A(p,\alpha,\beta)$, we need only to prove the 'only if' part of the theorem. For function f of the form (7), we note that the condition

$$\operatorname{Re}\left\{\frac{\left[D_\lambda^{n+p-1}(f*F)(z)\right]'}{pz^{p-1}}\right\} > \beta\left|\frac{\left[D_\lambda^{n+p-1}(f*F)(z)\right]'}{pz^{p-1}} - 1\right| + \alpha,$$

is equivalent to

$$\operatorname{Re}\left(\frac{\left(1+\beta e^{i\theta}\right)\left[D_\lambda^{n+p-1}(f*F)(z)\right]' - p\left(\alpha+\beta e^{i\theta}\right)}{pz^{p-1}}\right) \geq 0. \quad (10)$$

Choosing the values of $z = r$ on positive real axis where $0 \leq z = r < 1$, and using $\operatorname{Re}\left(-e^{i\theta}\right) \geq -\left|e^{i\theta}\right| = -1$, the inequality (10) reduces to

$$= \operatorname{Re}\left\{\frac{\left(1+\beta e^{i\theta}\right)\left(pz^p - \sum_{k=p+1}^{\infty} k(1+(k-p)\lambda)C(n,k,p)|a_k c_k| z^k - \sum_{k=p}^{\infty} k(1+(k-p)\lambda)C(n,k,p)|b_k d_k| z^k\right) - p\left(\alpha+\beta e^{i\theta}\right)z^p}{pz^p}\right\}$$

$$\geq \left\{\frac{(1+\beta)\left(pr^p - \sum_{k=p+1}^{\infty} k(1+(k-p)\lambda)C(n,k,p)|a_k c_k| r^k - \sum_{k=p}^{\infty} k(1+(k-p)\lambda)C(n,k,p)|b_k d_k| r^k\right) - p(\alpha+\beta)r^p}{pr^p}\right\}$$

$$= \left\{\frac{p(1-\alpha)r^p - \sum_{k=p+1}^{\infty} k(1+\beta)(1+(k-p)\lambda)C(n,k,p)|a_k c_k| r^k - \sum_{k=p}^{\infty} k(1+\beta)(1+(k-p)\lambda)C(n,k,p)|b_k d_k| r^k}{pr^p}\right\}.$$

Letting $r \longrightarrow 1^-$, we obtain

$$\frac{p(1-\alpha) - \sum_{k=p+1}^{\infty} k(1+\beta)(1+(k-p)\lambda)C(n,k,p)|a_k c_k| - \sum_{k=p}^{\infty} k(1+\beta)(1+(k-p)\lambda)C(n,k,p)|b_k d_k|}{pr^p} \geq 0. \quad (11)$$

If the condition (8) does not hold, then the numerator in (11) is negative for $|z| = r$ sufficiently close to 1. Hence, there exist $z_0 = r_0$ in (0,1) for which the inequality of (11) is negative. Therefore, it follows that $f \notin H_\lambda^n T(p,\alpha,\beta)$ and so the proof is complete. □

Theorem 3: Let $f \in H_\lambda^n T(p,\alpha,\beta)$, $(\beta \geq 0, \lambda \geq 0, 0 \leq \alpha < 1)$. Then *f* is convex in the disk

$$|z| < \min_k \left[\frac{1-[(1+\beta)/(1-\alpha)]|b_p d_p|}{k}\right]^{1/k-1}, \qquad k = p+1, p+2, \dots \qquad ,$$

$[(1+\beta)/(1-\alpha)]|b_p d_p| < 1.$

Proof: Let $f \in H_\lambda^n T(p,\alpha,\beta)$ and r be fixed such that $0< r<1$, then if $r^{-1}f(rz) \in H_\lambda^n T(p,\alpha,\beta)$ and we have

$$\sum_{k=p+1}^{\infty} k^2 \left(|a_k c_k| + |b_k d_k|\right) r^{k-1} = \sum_{k=p+1}^{\infty} k \left(|a_k c_k| + |b_k d_k|\right)(kr^{k-1})$$

$$\leq \sum_{k=p+1}^{\infty} \frac{k(1+\beta)}{p(1-\alpha)} (1+(k-p)\lambda) C(n,k,p)(|a_k c_k| + |b_k d_k|)(kr^{k-1})$$

$$\leq 1 - [(1+\beta)/(1-\alpha)]|b_p d_p|.$$

Provided $kr^{k-1} \leq 1 - [(1+\beta)/(1-\alpha)]|b_p d_p|$, which is true if

$$|z| < \min_k \left[\frac{1-[(1+\beta)/(1-\alpha)]|b_p d_p|}{k}\right]^{1/k-1}, \qquad k = p+1, p+2, \dots \qquad ,$$

$[(1+\beta)/(1-\alpha)]|b_p d_p| < 1.$

The proof is complete. □

3. Applications of fractional calculus operators:

Various operators of fractional calculus (that is, fractional integral and fractional derivatives) have been studied in the literature rather extensively (cf., e.g., [1, 11]; see also [7] the various references cited therein). For our present investigation, we recall the following definitions.

Definition 1: Let *f* be analytic in a simply connected region of the z-plane containing the origin. The fractional integral of *f* of order ν is defined by

$$D_z^{-\nu} f(z) = \frac{1}{\Gamma(\nu)} \int_0^z \frac{f(t)}{(z-t)^{1-\nu}} dt, \qquad (\nu > 0), \tag{12}$$

where the multiplicity of $(z-t)^{\nu-1}$ is removed by requiring that $\log(z-t)$ is real for $z - t > 0$.

Definition 2: Let f be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order ν is defined by

$$D_z^{\nu} f(z) = \frac{1}{\Gamma(1-\nu)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\nu}} dt, \qquad (0 \le \nu < 1), \tag{13}$$

where the multiplicity of $(z-t)^{-\nu}$ is removed by requiring that log(z − t) is real for z – t > 0.

Definition 3: Under the hypothesis of Definition 2, the fractional derivative of order n + ν is defined, for a function f, by

$$D_z^{n+\nu} f(z) = \frac{d^n}{dz^n}\{D_z^{\nu} f(z)\}, \qquad (0 \le \nu < 1; n \in \mathbb{N}_0).$$

From Definition (1) and (2) by applying a simple calculation for the function f given by (7) and using (2), then by Hadamard product, we get

$$\mathcal{D}_z^{-\nu}\{(f*F)(z)\} = \frac{\Gamma(p+1)}{\Gamma(p+\nu+1)} z^{p+\nu} - \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+\nu+1)} |a_k c_k| z^{k+\nu} - \sum_{k=p}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+\nu+1)} |b_k d_k| \overline{z}^{k+\nu} \tag{14}$$

and

$$\mathcal{D}_z^{\nu}\{(f*F)(z)\} = \frac{\Gamma(p+1)}{\Gamma(p-\nu+1)} z^{p-\nu} - \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\nu+1)} |a_k c_k| z^{k-\nu} - \sum_{k=p}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\nu+1)} |b_k d_k| \overline{z}^{k-\nu}. \tag{15}$$

Theorem 4: Let the function f defined by (7) be in the class $H_{\lambda}^{n}T(p,\alpha,\beta)$. Then

$$|\mathcal{D}_z^{-\nu}\{(f*F)(z)\}| \le \frac{\Gamma(p+1)}{\Gamma(p+\nu+1)} \left(1 + |b_p d_p|\right) r^{p+\nu} + \frac{p\Gamma(p+1)}{(1+\lambda)(p+n)\Gamma(p+\nu+2)} \left(\frac{(1-\alpha)}{(1+\beta)} - |b_p d_p|\right) r^{p+\nu+1} \tag{16}$$

and

$$|\mathcal{D}_z^{-\nu}\{(f*F)(z)\}| \ge \frac{\Gamma(p+1)}{\Gamma(p+\nu+1)} \left(1 - |b_p d_p|\right) r^{p+\nu} - \frac{p\Gamma(p+1)}{(1+\lambda)(p+n)\Gamma(p+\nu+2)} \left(\frac{(1-\alpha)}{(1+\beta)} - |b_p d_p|\right) r^{p+\nu+1}. \tag{17}$$

Proof: From Definition 1, we note that

$$\frac{\Gamma(p+1)}{\Gamma(p+\nu+1)} z^{-\nu}\mathcal{D}_z^{-\nu}\{(f*F)(z)\} = z^{-p} - \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)\Gamma(p+\nu+1)}{\Gamma(p+1)\Gamma(k+\nu+1)} |a_k c_k| z^{k+\nu} - \sum_{k=p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+\nu+1)}{\Gamma(p+1)\Gamma(k+\nu+1)} |b_k d_k| \overline{z}^{k+\nu}$$

$$= z^{-p} - |b_p d_p| \overline{z}^p - \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)\Gamma(p+\nu+1)}{\Gamma(p+1)\Gamma(k+\nu+1)} \left(|a_k c_k| z^k + |b_k d_k| \overline{z}^k \right)$$

$$= z^{-p} - |b_p d_p| \overline{z}^p - \sum_{k=p+1}^{\infty} \varphi(k) \left(|a_k c_k| z^k + |b_k d_k| \overline{z}^k \right),$$

where

$$\varphi(k) = \frac{\Gamma(k+1)\Gamma(p+\nu+1)}{\Gamma(p+1)\Gamma(k+\nu+1)} \; ; \; k \geq p+1.$$

Noting that $\varphi(k)$ is decreasing function of k, we have

$$0 < \varphi(k) \leq \varphi(p+1) = \frac{p+1}{p+\nu+1}.$$

Therefore, we obtain

$$\left| \frac{\Gamma(p+\nu+1)}{\Gamma(p+1)} z^{-\nu}\mathcal{D}_z^{-\nu}\{(f*F)(z)\} \right| \leq \left(1 + |b_p d_p|\right)|z|^p + \varphi(p+1)|z|^{p+1} \sum_{k=p+1}^{\infty} (|a_k c_k| + |b_k d_k|)$$

$$\leq \left(1 + |b_p d_p|\right) r^p + \frac{p}{(p+\nu+1)(1+\lambda)(p+n)} \left(\frac{(1-\alpha)}{(1+\beta)} - |b_p d_p| \right) r^{p+1},$$

which is equivalent (16), we also have

$$\left| \frac{\Gamma(p+\nu+1)}{\Gamma(p+1)} z^{-\nu}\mathcal{D}_z^{-\nu}\{(f*F)(z)\} \right| \geq \left(1 - |b_p d_p|\right)|z|^p - \varphi(p+1)|z|^{p+1} \sum_{k=p+1}^{\infty} (|a_k c_k| + |b_k d_k|)$$

$$\geq \left(1 - |b_p d_p|\right) r^p - \frac{p}{(p+\nu+1)(1+\lambda)(p+n)} \left(\frac{(1-\alpha)}{(1+\beta)} - |b_p d_p| \right) r^{p+1},$$

which is equivalent (17).

Theorem 5: Let the function f defined by (7) be in the class $H_\lambda^n T(p, \alpha, \beta)$. Then

$$|\mathcal{D}_z^{\nu}\{(f*F)(z)\}| \leq \frac{\Gamma(p+1)}{\Gamma(p-\nu+1)} \left(1 + |b_p d_p|\right) r^{p-\nu}$$

$$+\frac{p\Gamma(p+1)}{(1+\lambda)(p+n)\Gamma(p-\nu+2)}\left(\frac{(1-\alpha)}{(1+\beta)}-\left|b_p d_p\right|\right)r^{p-\nu+1} \qquad (18)$$

and

$$|\mathcal{D}_z^{\nu}\{(f*F)(z)\}| \geq \frac{\Gamma(p+1)}{\Gamma(p-\nu+1)}\left(1-\left|b_p d_p\right|\right)r^{p-\nu} - \frac{p\Gamma(p+1)}{(1+\lambda)(p+n)\Gamma(p-\nu+2)}\left(\frac{(1-\alpha)}{(1+\beta)}-\left|b_p d_p\right|\right)r^{p-\nu+1}. \qquad (19)$$

Proof: From Definition 2, we note that

$$\frac{\Gamma(p-\nu+1)}{\Gamma(p+1)}z^{\nu}\mathcal{D}_z^{\nu}\{(f*F)(z)\} = z^p - \sum_{k=p+1}^{\infty}\frac{\Gamma(k+1)\Gamma(p-\nu+1)}{\Gamma(p+1)\Gamma(k-\nu+1)}|a_k c_k|z^k - \sum_{k=p}^{\infty}\frac{\Gamma(k+1)\Gamma(p-\nu+1)}{\Gamma(p+1)\Gamma(k-\nu+1)}|b_k d_k|\overline{z}^k$$

$$= z^p - \left|b_p d_p\right|\overline{z}^p - \sum_{k=p+1}^{\infty}\frac{\Gamma(k+1)\Gamma(p-\nu+1)}{\Gamma(p+1)\Gamma(k-\nu+1)}\left(|a_k c_k|z^k + |b_k d_k|\overline{z}^k\right)$$

$$= z^p - \left|b_p d_p\right|\overline{z}^p - \sum_{k=p+1}^{\infty}\Phi(k)\left(|a_k c_k|z^k + |b_k d_k|\overline{z}^k\right),$$

where

$$\Phi(k) = \frac{\Gamma(k+1)\Gamma(p-\nu+1)}{\Gamma(p+1)\Gamma(k-\nu+1)}\ ;\ \ k \geq p+1.$$

Noting that $\Phi(k)$ is decreasing function of k, we have

$$0 < \Phi(k) \leq \Phi(p+1) = \frac{p+1}{p-\nu+1}.$$

Therefore, we obtain

$$\left|\frac{\Gamma(p-\nu+1)}{\Gamma(p+1)}z^{\nu}\mathcal{D}_z^{\nu}\{(f*F)(z)\}\right| \leq \left(1+\left|b_p d_p\right|\right)|z|^p + \Phi(p+1)|z|^{p+1}\sum_{k=p+1}^{\infty}\left(|a_k c_k| + |b_k d_k|\right)$$

$$\leq \left(1+\left|b_p d_p\right|\right)r^p + \frac{p}{(p-\nu+1)(1+\lambda)(p+n)}\left(\frac{(1-\alpha)}{(1+\beta)}-\left|b_p d_p\right|\right)r^{p+1},$$

which is equivalent (18), we also have

$$\left|\frac{\Gamma(p-\nu+1)}{\Gamma(p+1)}z^{\nu}\mathcal{D}_z^{\nu}\{(f*F)(z)\}\right| \geq \left(1-\left|b_p d_p\right|\right)|z|^p - \Phi(p+1)|z|^{p+1}\sum_{k=p+1}^{\infty}\left(|a_k c_k| + |b_k d_k|\right)$$

$$\geq \left(1-\left|b_p d_p\right|\right) r^p-\frac{p}{(p-\nu+1)(1+\lambda)(p+n)}\left(\frac{(1-\alpha)}{(1+\beta)}-\left|b_p d_p\right|\right) r^{p+1},$$

which is equivalent (19).

Corollary 1: Let the function $f=h+\bar{g}$ defined by (7) be in the class $H_\lambda^n T(p,\alpha,\beta)$, r → 1 in the inequality (17) of Theorem 4. Then

$$\left\{w:|w|<\frac{\big((1+\lambda)(p+n)(p+\nu+1)-p[(1-\alpha)/(1+\beta)]\big)\Gamma(p+1)}{(1+\lambda)(p+n)(p+\nu+2)}-\frac{\big((1+\lambda)(p+n)(p+\nu+1)-p\big)\Gamma(p+1)}{(1+\lambda)(p+n)(p+\nu+2)}\left|b_p d_p\right|\right\}\subset f(U).$$

Corollary 2: Let the function $f=h+\bar{g}$ defined by (7) be in the class $H_\lambda^n T(p,\alpha,\beta)$, r → 1 in the inequality (19) of Theorem 5. Then

$$\left\{w:|w|<\frac{\big((1+\lambda)(p+n)(p-\nu+1)-p[(1-\alpha)/(1+\beta)]\big)\Gamma(p+1)}{(1+\lambda)(p+n)(p-\nu+2)}-\frac{\big((1+\lambda)(p+n)(p-\nu+1)-p\big)\Gamma(p+1)}{(1+\lambda)(p+n)(p-\nu+2)}\left|b_p d_p\right|\right\}\subset f(U).$$

4. Integral operator:

We will examine the closure properties of the class $H_\lambda^n T(p,\alpha,\beta)$ under the generalized Bernardi-Libera-Livingston integral operator $\mathcal{L}_{p,c}(f)$ which is defined by

$$\mathcal{L}_{p,c}(f)=\frac{c+p}{z^c}\int_0^z t^{c-1} f(t)\,dt, \qquad c>-1.$$

Theorem 6: Let $f\in H_\lambda^n T(p,\alpha,\beta)$. Then $\mathcal{L}_{p,c}\big((f*F)(z)\big)\in H_\lambda^n T(p,\alpha,\beta)$.

Proof: From the representation of $\mathcal{L}_{p,c}(f)$, it follows that

$$\mathcal{L}_{p,c}\big((f*F)(z)\big)=\frac{c+p}{z^c}\int_0^z t^{c-1}\left[(h*H)(z)+\overline{(g*G)(z)}\right]dt$$

$$=\frac{c+p}{z^c}\left(\int_0^z t^{c-1}\left(t^p-\sum_{k=p+1}^{\infty}|a_k c_k|\,t^k\right)dt-\overline{\int_0^z t^{c-1}\left(\sum_{k=p}^{\infty}|b_k d_k|\,t^k\right)dt}\right)$$

$$=z^p-\sum_{k=p+1}^{\infty}\left(\frac{c+p}{c+k}\right)|a_k c_k|\,z^k-\sum_{k=p}^{\infty}\left(\frac{c+p}{c+k}\right)|b_k d_k|\,z^k.$$

Using the inequality (9), we get

$$\sum_{k=p+1}^{\infty}\frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)\left(\frac{c+p}{c+k}|a_k c_k|\right)$$
$$+\sum_{k=p}^{\infty}\frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)\left(\frac{c+p}{c+k}|b_k d_k|\right)$$

$$\leq\sum_{k=p+1}^{\infty}\frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)|a_k c_k|$$
$$+\sum_{k=p}^{\infty}\frac{k(1+\beta)}{p(1-\alpha)}(1+(k-p)\lambda)C(n,k,p)|b_k d_k|\leq 1,$$

since $f\in H_{\lambda}^{n}T(p,\alpha,\beta)$.
Hence by Theorem 2, $\mathcal{L}_{p,c}\big((f*F)(z)\big)\in H_{\lambda}^{n}T(p,\alpha,\beta)$.

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