Stability Analysis for Steady State Solutions of Huxley Equation

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الملخص

تمت دراسة استقرارية الحلول اللازمنية لمعادلة Huxley باستخدام تحليل الأستقرارية من النمط Fourier في حالتين الأولى في حالة السعة الثابتة والثانية في حالة السعة المتغيرة. إذ تم الاستنتاج بان الحلين $u_1 = 0$ و $u_1 = 1$ مستقران دوماً بينما الحلين $u_1 = a$ و $u_1 = u_1(X)$ مستقرين على نحو مشروط. وفي الحالة الثانية تمت المقارنة بين النتائج التي تم الحصول عليها من الحل التحليلي مع نتائج طريقة Galerkin العددية وقد تبين إن النتائج هي نفسها في كلتا الحالتين.

ABSTRACT

Stability analysis of steady state solutions of Huxley equation using Fourier mode stability analysis in two cases is investigated. Firstly when the amplitude is constant and secondly when the amplitude is variable and the results were found to be: The solutions $u_1 = 0$ and $u_1 = 1$ are always stable while the solutions $u_1 = a$ and $u_1 = u_1(X)$ are conditionally stable. In the second case, a comparison between the analytical solution and the numerical solution of Galerkin method is done and the results are the same.

1. Introduction

Consider a system of any nature whatsoever that exists in a state S. We say that S is stable, in one sense or another, if small perturbations or changes in the system do not drastically affect the state S. For example, the solar system currently exists in a time-dependent state in which the planets move about the sun in an orderly fashion. It is known that if a small additional celestial body is introduced into the system, then the original state is not disturbed to any significant degree. We say that the original state is stable to small perturbations. Similar questions of stability arise in every physical problem (Logan (1987)). McKean (1970) investigated the steady state solutions of Huxley equation for some values of the wave velocity c. Maginu (1978) studied by the use of Liapunove's second method the stability of such stationary solutions. Fife (1979) analyzed the possible asymptotic behavior of stationary solutions of bistable equation. Conley and Smoller (1980) used some topological concepts in the study of stationary solutions. Smoller and Wasserman (1981) obtained the exact number of steady state solutions subject to Dirichlet boundary conditions. Manoranjan et al (1984) obtained the estimates for the critical lengths of the domain at which bifurcation occurs in the cases $b = 0, a\ (0 < a \leq 1/2),\ and\ 1$. Manoranjan (1984) studied in detail the solutions bifurcating from the equilibrium state $u = a$.

Eilbeck and Manoranjan (1986) considered different types functions for the pseudo-spectral method applied to the nonlinear reaction-diffusion equation in 1- and 2- space dimensions. Eilbeck (1986) extended the pseudo-spectral method to follow steady state solutions as a function of the problem parameter, using path-following techniques.

Fath and Domanski (1999) studied the cellular differentiation in a developing organism via a discrete bistable reaction-diffusion model and they investigated some properties of the bifurcation of steady state solutions. Lewis and Keener (2000) studied the propagation failure using the one-dimensional scalar bistable equation by a passive gap and they reduced the problem of finding conditions for block to the problem of finding the existence of steady state solutions.

In this paper, the stability of steady state solutions of Huxley equation is analyzed.

2. The Mathematical Model

One of the famous nonlinear reaction-diffusion equations is the generalized Burgers-Huxley (gBH) equation:

$$\frac{\partial u}{\partial t} + au^{d}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = bu\left(1-u^{d}\right)\left(u^{d}-a\right) \qquad (1)$$

$a \geq 0$, $b \geq 0$, $d > 0$ & $a \in (0,1)$

If we take $d = 1$, $a \neq 0$, *and* $b \neq 0$, equation (1) becomes the following Burgers-Huxley (BH) equation:

$$\frac{\partial u}{\partial t} + au\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = bu(1-u)(u-a) \qquad (2)$$

Equation (2) shows a prototype model for describing the interaction mechanism, convection transport. When $b = 0, d = 1$, equation (1) is reduced to Burgers equation which describes the far field of wave propagation in nonlinear dissipative systems

$$\frac{\partial u}{\partial t} + au\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad (3)$$

When $a = 0, d = 1$, equation (1) is reduced to the Huxley equation which describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = bu(1-u)(u-a) \qquad (4)$$

It is known that non-linear diffusion equations (3) and (4) play important roles in nonlinear physics. They are of special significance for studying nonlinear phenomena (Wang et al, 1990). Zeldovich and Frank-Kamenetsky formulated the equation (4) in 1938 as a model for flame front propagation and for this reason this equation is sometimes named Zeldovich-Frank-Kamenetsky (ZF) equation, which has been extensively studied as a simple nerve model (Binczak et al, 2001). In 1952, Hodgkin and Huxley proposed their famous Hodgkin-Huxley model for nerve propagation.

Because of the mathematical complexity of this model, it led to the introduction of the simpler Fitzhugh-Nagumo system. The simplified model of the Fitzhugh-Nagumo system is Huxley equation (Wang,1985). Because Huxley equation is a special case of Fitzhugh-Nagumo system, it is sometimes named Fitzhugh-Nagumo (FN) equation (Estevez and Gordoa, 1990) or the reduced Nagumo equation or Nagumo equation (Pesin and Yurchenko, 2004). In sixties, Fitzhugh used equation (4) as an approximate equation for the description of dynamics of the giant axon. This equation was among the first models of excited media (Landa,1996).

In this paper, we shall take the Huxley equation as a model problem (Manoranjan et al ,1984):

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=bu(1-u)(u-a)$$

$$x\in[-L,L],\ t\geq 0$$

$$u(x,0)=(b-H)x^2+H,\ b\geq 0,\ H>0$$

$$u(-L,t)=u(L,t)=b \qquad (5)$$

For a dimensionless form, we introduce the following dimensionless quantities:

$X = x/L\ ,\ T = t/L^2$

Substitute these non-dimensional quantities in equation (4), it follows that:

$$\frac{\partial u}{\partial T}=\frac{\partial^2 u}{\partial X^2}+L^2 bu(1-u)(u-a) \qquad (6)$$

$$u(-1,t)=u(1,t)=b \quad ,\quad b\geq 0\ ,\ -1\leq X\leq 1 \qquad (7)$$

The equations (6) and (7) represent the non-dimensional form of Huxley equation in x and t.

3. Fourier Mode Stability Analysis

Assume that the solution of equation (6) can be written in the form (Logan ,1987):

$$u(X,T)=u_1(X)+u_2(X,T) \qquad (8)$$

where $u_1(X)$ is the steady state solution and $u_2(X,T)$ is the disturbance. Substitute (8) in (6), with its boundary conditions, we get the following equation:

$$\frac{\partial u_2}{\partial T}=\frac{d^2u_1}{dX^2}+\frac{\partial^2 u_2}{\partial X^2}-bL^2u_1^3-bL^2u_2^3-3bL^2u_1^2u_2-3bL^2u_1u_2^2$$
$$+(1+a)bL^2u_1^2+(1+a)bL^2u_2^2+2(1+a)bL^2u_1u_2-abL^2u_1-abL^2u_2$$

If we separate the two cases, we obtain

$$\frac{\partial u_2}{\partial T}=\frac{\partial^2 u_2}{\partial X^2}-bL^2u_2^3-3bL^2u_1^2u_2-3bL^2u_1u_2^2+$$
$$(1+a)bL^2u_2^2+2(1+a)bL^2u_1u_2-abL^2u_2 \qquad (9)$$

$$\frac{d^2u_1}{dX^2}-bL^2u_1^3+(1+a)bL^2u_1^2-abL^2u_1=0$$

which can be written as

$$\frac{d^2u_1}{dX^2}+L^2bu_1(1-u_1)(u_1-a)=0 \qquad (10)$$

$$u_1(-1)=u_1(1)=b,\quad b\geq 0,\quad -1\leq X\leq 1 \qquad (11)$$

The solution of steady state case (10) and (11) is (Manoranjan et al ,1984):

$$u_1(X)=\begin{bmatrix} 3a/\sqrt{(2-a)(1/2-a)}\cosh\left(\sqrt{a}LX\right)+1+a,\ 0<a<1/2, b=1 \\ 1/2+a\,\mathrm{sn}\left(\frac{LX}{\sqrt{2}}\left(1/2-a^2\right)^{1/2}, a/\left(1/2-a^2\right)^{1/2}\right), a=1/2, b=1 \end{bmatrix} \qquad (12)$$

$$0<a<1/2$$

where $sn(v,d)$ represents the Jacobi elliptic function of argument v and modulus d .The Taylor expansion of $sn(v,d)$ is (Clarke (2000)):

$$sn(v,d)=v-\left(1+d^2\right)\frac{v^3}{3!}+\left(1+14d^2+d^4\right)\frac{v^5}{5!}-\left(1+135d^2+135d^4+d^6\right)\frac{v^7}{7!}+\ldots \qquad (13)$$

Where $sn(v,0)=\sin(v),\ sn(v,1)=\tanh(v)$

Note that the case $a\in(1/2,1)$ can be reduced to the case $a\in(0,1/2)$ by replacing u_1 with $1-u_1$ (Pesin and Yurchenko ,2004).

3.1. Stability Analysis in the Case of the Constant Amplitude

We assume that the disturbance has the following Fourier mode form (Logan ,1987):

$$u_2(X,T)=Ae^{ik(X-cT)} \qquad (14)$$

$$A>0,\ k>0, c=c_1+ic_2, i=\sqrt{-1}$$

Where A is the wave amplitude, k is the wave number and c is the wave velocity, the solution is stable if $c_2<0$, unstable if $c_2>0$ and $c_2=0$, gives the neutral stability curve which separate between the stable region and the unstable region (when $c_2 < 0$ the disturbance decays as $t \to \infty$ and the stationary solution will be stable, while in the case $c_2 > 0$ the disturbance (perturbation) grows as $t \to \infty$ and the equilibrium state (rest point) solution will be unstable, the quantity c_2 is called the stability indicator).

Neglecting the non-linear terms in equation (9), we have:

$$\frac{\partial u_2}{\partial T} = \frac{\partial^2 u_2}{\partial X^2} - 3bL^2u_1^2u_2 + 2(1+a)bL^2u_1u_2 - abL^2u_2 \qquad (15)$$

Substitute (14) in (15), we get

$$-ic_1 + c_2 = -k - \frac{3bL^2u_1^2}{k} + \frac{2(1+a)bL^2u_1}{k} - \frac{abL^2}{k}$$

Equating the real and imaginary parts, we get

$$c_1 = 0$$

$$c_2 = -\left[\frac{k^2 + bL^2(3u_1^2 + a) - 2(1+a)bL^2u_1}{k}\right] \qquad (16)$$

According to the sign of c_2, we have the following three cases:

, then $c_2 > 0$, and the solution $k^2 + bL^2(3u_1^2 + a) < 2(1+a)bL^2u_1$ (i) If is unstable.

, then $c_2 < 0$, and the solution $k^2 + bL^2(3u_1^2 + a) > 2(1+a)bL^2u_1$ (ii) If is stable.

then $c_2 = 0$, which gives $k^2 + bL^2(3u_1^2 + a) = 2(1+a)bL^2u_1$, (iii) If the neutral stability curve

$$k = \sqrt{2(1+a)bL^2u_1 - bL^2(3u_1^2 + a)} \qquad (17)$$

If $\quad 2(1+a)bL^2u_1 > bL^2(3u_1^2 + a)$

Now, we shall apply the results above on the following four cases:

(a) When $u_1 = 0$, substitute in (16), we have

$$c_2 = -\left[\frac{k^2 + abL^2}{k}\right] < 0$$

i.e. the steady state case $u_1 = 0$ is always stable.

When $u_1 = 1$, substitute in (16), we have (b)

$$c_2 = -\left[\frac{k^2 + bL^2(1-a)}{k}\right] < 0$$

i.e. the steady state case $u_1 = 1$ is always stable.

(c) When $u_1 = a$, substitute in (16), we get

$$c_2 = -\left[\frac{k^2 + a^2 bL^2 - abL^2}{k}\right] \qquad (18)$$

From (18), we have

(i) If $k^2 + a^2 bL^2 < abL^2$, and the solution is unstable. $c_2 > 0$, then

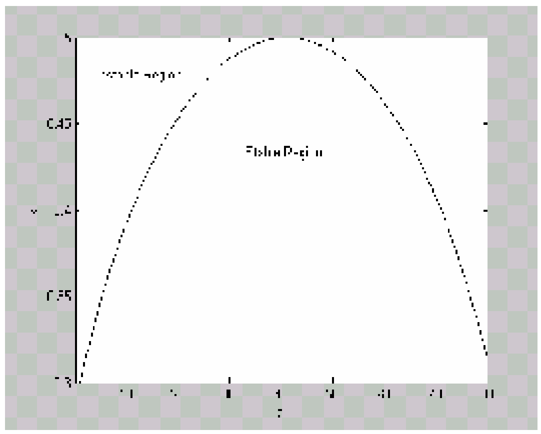
(ii) If $k^2 + a^2 bL^2 > abL^2$ and the solution is stable. $c_2 < 0$, , then

(iii) If $k^2 + a^2 bL^2 = abL^2$ in Eq.(18), we get $c_2 = 0$, substitute $c_2 = 0$, then

$k^2 + a^2 bL^2 - abL^2 = 0 \Rightarrow k^2 = abL^2 - a^2 bL^2 = abL^2(1-a) \Rightarrow$

$$k = \sqrt{abL^2(1-a)} \qquad (19)$$

which is the neutral stability curve as in figure (1).



for the steady $k = \sqrt{abL^2(1-a)}$ Figure (1) shows the neutral stability curve $b = 1, L = 1$, *and* $0.1 \le a \le 0.9$ when $u_1 = a$ state solution

(d) When $u_1 = u_1(X)$ (function of X) as in the equation (12), substitute in (16), we get

$$c_2 = -\left[(k^2 + bL^2(3u_1^2(X) + a) - 2(1+a)bL^2 u_1(X)) / k\right] \qquad (20)$$

From (20), we have

(i) If $k^2 + bL^2(3u_1^2(X) + a) < 2(1+a)bL^2 u_1(X)$, then $c_2 > 0$, and the solution is unstable.

(ii)If $k^2 + bL^2(3u_1^2(X) + a) > 2(1+a)bL^2 u_1(X)$, then $c_2 < 0$, and the solution is stable.

(iii) If $k^2 + bL^2(3u_1^2(X) + a) = 2(1+a)bL^2 u_1(X)$ which gives $c_2 = 0$, , then the neutral stability curve

$$k=\sqrt{2(1+a)bL^2u_1(X)-bL^2(3u_1^2(X)+a)} \qquad (21)$$

If $2(1+a)bL^2u_1(X)>bL^2(3u_1^2(X)+a)$

3.2. Stability Analysis in the Case of the Variable Amplitude

We shall compare the analytical solution with the solution of Galerkin numerical technique.

3.2.1. The Analytical Solution

We assume that the disturbance case has the following Fourier mode form (Logan ,1987):

$$u_2(X,T)=A(X)e^{ik(X-CT)} \qquad (22)$$

Substitute (22) in (15) and neglecting the imaginary terms in the resulting equation, we have

$$A''(X)+\left(2(1+a)bL^2u_1-3bL^2u_1^2-abL^2-k^2-kc_2\right)A(X)=0$$

The equation above can be rewritten in the following form:

$$A''(X)+DA(X)=0 \qquad (23)$$

$$A(-1)=A(1)=b \quad , \quad -1\le X\le 1 \quad , \quad b\ge 0 \qquad (24)$$

$$D=2(1+a)bL^2u_1-3bL^2u_1^2-abL^2-k^2-kc_2$$

The boundary conditions of the amplitude function are the same as the boundary conditions of the original problem.

The solution of (23) and (24) has the following three cases (Logan ,1987):

(i) If $D<0$, then the general solution of (23) and (24) is:

$$A(X)=a_1e^{\sqrt{-D}X}+a_2e^{-\sqrt{-D}X}$$

Since $\cosh x=(e^x+e^{-x})/2$ and $\sinh x=(e^x-e^{-x})/2$

The solution in above can be written as:

$$A(X)=B\cosh(\sqrt{-D})X+C\sinh(\sqrt{-D})X \qquad (25)$$

Where B, C are arbitrary constants.

Calling the constant $-D$ does not mean that
it is negative, we use the negative sign only for convenience.

From the boundary conditions, we have

$$B\cosh\sqrt{-D}-C\sinh\sqrt{-D}=b \qquad (26)$$

$$B\cosh\sqrt{-D}+C\sinh\sqrt{-D}=b$$

By adding the two equations in above, we get

$$B=b/\cosh\sqrt{-D}$$

Since $\sinh\sqrt{-D}\neq 0$

substitute B in one of the two equations above, we have
$C\sinh\sqrt{-D}\neq 0$. Because $D\neq 0$ which implies that $C=0$
Substitute B and C in the general solution (25), we get

$$A(X)=\left(\frac{b}{\cosh\sqrt{-D}}\right)\cosh\sqrt{-D}X \tag{27}$$

(ii) If $D=0$, then the general solution of (23) and (24) is:

$$A(X)=BX+C \tag{28}$$

From the boundary conditions, we have

$$\left.\begin{aligned} C-B&=b\\ C+B&=b\end{aligned}\right\} \tag{29}$$

By solving the algebraic system (29), we get

$B=0$, $C=b$

Substitute B and C in the general solution (28), we get

$$A(X)=b \tag{30}$$

(iii) If $D>0$, then the general solution of (23) and (24) is:

$$A(X)=B\cos\sqrt{D}X+C\sin\sqrt{D}X \tag{31}$$

From the boundary conditions, we have

$$\left.\begin{aligned} B\cos\sqrt{D}-C\sin\sqrt{D}&=b\\ \mathrm{B}\cos\sqrt{\mathrm{D}}+C\sin\sqrt{D}&=b\end{aligned}\right\} \tag{32}$$

By solving the algebraic system (32), we get

$$B=b$$
$$D=(np)^2,\ n=1,2,3,... \tag{33}$$

Substitute B *and* D in the general solution (31), we get

$$A(X)=b\cos(np)X+C\sin(np)X \tag{34}$$

From (33), we have

$$\mathrm{c}_2=-\left[(k^2+(np)^2+bL^2(3u_1^2+a)-2(1+a)bL^2u_1)/k\right] \tag{35}$$

Equation (35), has the following three cases:
(i) If $k^2+(np)^2+bL^2(3u_1^2+a)<2(1+a)bL^2u_1$, then $c_2>0$, and the solution is unstable.
(ii) If $k^2+(np)^2+bL^2(3u_1^2+a)>2(1+a)bL^2u_1$, then $c_2<0$, and the solution is stable.
(iii) If $k^2+(np)^2+bL^2(3u_1^2+a)=2(1+a)bL^2u_1$, then $c_2=0$, to get the neutral stability curve, we take the smallest eigenvalue when $n=1$

$$k=\sqrt{2(1+a)bL^2u_1-bL^2\left(3u_1^2+a\right)-9.869604401} \qquad (36)$$

If $2(1+a)bL^2u_1 > bL^2\left(3u_1^2+a\right)+9.869604401$

The values for which the problem has a nontrivial solution are called the eigenvalues and the corresponding solutions are called the eigenfunctions.

According to these results, we have the following three cases:

(a) When $u_1=0$, substitute in (35), we have

$$c_2=-\left[(k^2+(np)^2+abL^2)/k\right]<0$$

i.e. the steady state case $u_1=0$ is always stable.

(b) When $u_1=1$, substitute in (35), we have

$$c_2=-\left[(k^2+(np)^2+bL^2(1-a))/k\right]<0$$

i.e. the steady state case $u_1=1$ is always stable.

(c) When $u_1=a$, substitute in (35), we get

$$c_2=-\left[(k^2+(np)^2+a^2bL^2-abL^2)/k\right] \qquad (37)$$

From (37), we have

(i) If $k^2+(np)^2+a^2bL^2<abL^2$, and the solution is unstable. $c_2>0$, then

(ii) If $k^2+(np)^2+a^2bL^2>abL^2$, and the solution is stable. $c_2<0$, then

(iii) If $k^2+(np)^2+a^2bL^2>abL^2$, to get the neutral stability $c_2=0$, then curve , we take the smallest eigenvalue when $n=1$

$$k=\sqrt{abL^2(1-a)-9.869604401}$$

If $abL^2(1-a)>9.869604401$

(d) When $u_1=u_1(X)$ (function of X) as in the equation (12), substitute in (35), we get

$$c_2=-\left[(k^2+(np)^2+bL^2\left(3u_1^2(X)+a\right)-2(1+a)bL^2u_1(X))/k\right] \qquad (38)$$

From (38), we have

(i) If $k^2+(np)^2+bL^2\left(3u_1^2(X)+a\right)<2(1+a)bL^2u_1(X)$, then $c_2>0$, and the solution is unstable.

(ii) If $k^2+(np)^2+bL^2\left(3u_1^2(X)+a\right)>2(1+a)bL^2u_1(X)$, then $c_2<0$, and the solution is stable.

(iii) If $k^2+(np)^2+bL^2\left(3u_1^2(X)+a\right)=2(1+a)bL^2u_1(X)$, then $c_2=0$, to get the neutral stability curve , we take the smallest eigenvalue when $n=1$

$$k=\sqrt{2(1+a)bL^2u_1(X)-bL^2\left(3u_1^2(X)+a\right)-9.869604401} \qquad (39)$$

If $2(1+a)bL^2u_1(X) > bL^2(3u_1^2(X)+a)+9.869604401$

3.2.2. Stability Analysis Using Galerkin Method

The residual methods as Galerkin method usually starts with a governing boundary value problem. The differential equation is written so that zero occurs on one side of the equal sign. If the exact solution T could be substituted into the equation, the result would be zero . The exact solution is not known, so some approximation of the exact solution $T^* \cong T$ is employed instead. Substitution of the approximate solution into the differential equation results in an erroneous value r, rather than zero, the error r is then multiplied by weighting function w, and the product is integrated over the solution region. The result is called the residual R and is set equal to zero (Allaire, 1985).

Let the solution of the equations (23) and (24) be in the form:

$$A(X) = \sum_{n=1}^{\infty} (B_n \cos l_n X + C_n \sin l_n X) \tag{40}$$

To find l_n ,we can take any typical term of the summation above as follows:

$$A(X) = B_n \cos l_n X + C_n \sin l_n X$$

By using the boundary conditions, we have

$$\left.\begin{aligned} B_n \cos l_n - C_n \sin l_n &= \mathrm{b} \\ \mathrm{B_n} \cos l_n + C_n \sin l_n &= b \end{aligned}\right\} \tag{41}$$

By solving the algebraic system (41), we get

$$l_n = np \quad , \qquad n = 1,2,3,... \tag{42}$$

$$B_n = b \tag{43}$$

Substitute (42) and (43) in (40), we get

$$A(X) = \sum_{n=1}^{\infty} (b\cos(np)X + C_n \sin(np)X) \tag{44}$$

Substitute (44) in (23), we get

$$\sum_{n=1}^{\infty} D(b\cos(np)X + C_n \sin(np)X) - \sum_{n=1}^{\infty} ((np)^2 b\cos(np)X + (np)^2 C_n \sin(np)X) = 0$$

The residual is:

$$\sum_{n=1}^{p} \left[D - (np)^2\right](b\cos(np)X + C_n \sin(np)X) = R \tag{45}$$

Using Galerkin assumption (Al-Obaidi and Ibrahim ,2001), we have

$$\int_{-1}^{1}\left[\sum_{n=1}^{p}\left[D-(np)^2\right](b\cos(np)X+C_n\sin(np)X)\right]f_m(X)\,dX=0$$

$f_m(X)=\cos\ l_m X\ ,\ l_m=mp\ ,\ m=1,2,3,....$ (46)

Any element of the matrix will have the form:

$$X(n,m)=\int_{-1}^{1}\sum_{n=1}^{p}\sum_{m=1}^{p}\left[D-(np)^2\right](b\cos(np)X+C_n\sin(np)X)\cos(mp)X\,dX \qquad (47)$$

The following analytical integrals will be useful:

(1) $\int_{-1}^{1}\cos(np)X\cos(mp)XdX=\begin{bmatrix}1 & if & n=m\\ 0 & if & n\neq m\end{bmatrix}$

(2) $\int_{-1}^{1}\sin(np)X\cos(mp)XdX=0 \quad for\ any \quad n,m$

To determine the value of c_2, from (47), we get

$D=(np)^2=0\ \Rightarrow$

$2(1+a)bL^2u_1-3bL^2u_1^2-abL^2-k^2-kc_2=(np)^2$

After some mathematical manipulation, one may obtain the algebraic equation:

$$c_2=-\left[k^2+(np)^2+bL^2(3u_1^2+a)-2(1+a)bL^2u_1/k\right] \qquad (49)$$

We observe that equation (49) which is obtained by using Galerkin method is the same as equation (35) which is obtained from the analytical solution i.e. the equation (49) will give the same results which are obtained by using the equation (35) in the cases: $u_1=0, u_1=1, u_1=a$, and $u_1=u_1(X)$, if we use it to compute those results , this indicates the efficiency and accuracy of the Galerkin method.

The solution of the eigenvalue problem in 2x2 matrix is:

$l_1=p=3.141592654 \quad , \quad l_2=2p=6.283185307$

The solution in the 4x4 matrix is:

$l_1=p=3.141592654 \quad , \quad l_2=2p=6.283185307$

$l_3=3p=9.424777961 \quad , \quad l_4=4p=12.56637061$

To get the neutral stability curve, we put $c_2=0$ in equation (49) and we take the smallest eigenvalue when $n=1$, we have

$$k=\sqrt{2(1+a)bL^2u_1-bL^2\left(3u_1^2+a\right)-9.869604401} \qquad (50)$$

If $\ 2(1+a)bL^2u_1>bL^2\left(3u_1^2+a\right)+9.869604401$

We note that the equation (50), which is obtained by using Galerkin method, is the same as equation (36), which is obtained by the use of analytical solution.

4. Conclusions

We concluded in the constant amplitude case that:

(i) The steady state solution $u_1 = 0$ is always stable.

(ii) The steady state solution $u_1 = 1$ is always stable.

(iii) The steady state solution $u_1 = a$ is stable if:

$$k^2 + a^2 b\ L^2 > ab\ L^2$$

The neutral stability curve in this case is:

$$k = \sqrt{ab\ L^2 (1-a)}$$

(iv) The steady state solution $u_1 = u_1(X)$ as in the equation (12), is stable if:

$$k^2 + b\ L^2 \left(3u_1^2(X) + a\right) > 2(1+a) b\ L^2 u_1(X)$$

The neutral stability curve in this case is:

$$k = \sqrt{2(1+a) b L^2 u_1(X) - \left(3u_1^2(X) + a\right)}$$

If $\quad 2(1+a) b\ \mathrm{L}^2 u_1(X) > \left(3u_1^2(X) + a\right)$

In the case of variable amplitude the comparison between the analytical solution and the numerical solution of Galerkin method has been done and the results are the same in the analytical and numerical solution and the results were found to be:

(i) The steady state solution $u_1 = 0$ is always stable.

(ii) The steady state solution $u_1 = 1$ is always stable.

(iii) The steady state solution $u_1 = a$ is stable if:

$$k + (np)^2 + a^2 b\ \mathrm{L}^2 > ab\ L^2$$

The neutral stability curve in this case is:

$$k = \sqrt{ab\ L^2 (1-a) - 9.869604401}$$

If $\quad ab\ \mathrm{L}^2 (1-a) > 9.869604401$

(iv) The steady state solution $u_1 = u_1(X)$ as in the equation (12), is stable if:

$$k^2 + (np)^2 + b\ L^2 \left(3u_1^2(X) + a\right) > 2(1+a) b\ L^2 u_1(X)$$

The neutral stability curve in this case is:

$$k = \sqrt{2(1+a)bL^2u_1(X) - b\ L^2\left(3u_1^2(X) + a\right) - 9.869604401}$$

If $\quad 2(1+a)b\ \mathrm{L}^2u_1(X) > b\ \mathrm{L}^2\left(3u_1^2(X) + a\right) + 9.869604401$

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REFERENCES

[1] Allaire, P.E., (1985), **Basics of the Finite Element Method**, *Wm. C. Brown Publishers.*

[2] Al-Obaidi, M.F. and Ibrahim, B.M., (2001), **Stability Analysis and Chaos in a Bend Duct**, *Raf. J. Sci., Vol. 12, No. 1,* pp. 91-99.

[3] Binczak, S., Eilbeck, J.C. and Scott, A.C., (2001), **Ephaptic Coupling of Myelinated Nerve Fibers**, *Physica D 148* pp. 159-174.

[4] Clarke, F., (2000), **The Taylor Series Coefficients of the Jacobi Elliptic Functions**, *http://www-maths.swan.ac.uk/staff/fwc*.

[5] Conley, C. and Smoller, J., (1980), **Topological Techniques in Reaction-Diffusion Equations**, In: Jager, W., Rost, H. and Tautu, P., Eds., Biological Growth and Spread, *Lecture Notes in Biological Mathematics (Springer, Berlin) 38*, pp. 473-483.

[6] Eilbeck, J.C. and Manoranjan, V.S., (1986), **A Comparison of Basis Functions for the Pseudo-Spectral Method for a Model Reaction-Diffusion Problem**, *J. Comput. Appl. Math., 15*, pp. 371-378.

[7] Eilbeck, J.C., (1986), **The Pseudo-Spectral Method and Path Following in Reaction-Diffusion Bifurcation Studies**, *SIAM J. Sci. Stat.Comput.,Vol. 7, No. 2*, pp. 599-610.

[8] Estevez, P.G. and Gordoa, P.R., (1990), **Painleve Analysis of the Generalized Burgers-Huxley Equation**, *J. Phys. A: Math. Gen.,23*, pp. 4831-4837.

[9] Fath, G. and Domanski, Z, (1999), **Avalanche of Bifurcations and Hysteresis in a Model of Cellular Differentiation**, *Phys. Rev. E., Vol. 60, No. 4*, pp. 4604-4609.

[10] Fife, P.C., (1979), **Long Time Behavior of Solutions of Bistable Nonlinear Diffusion Equation, Arch. Rat. Mech**. *Anal., Vol. 70, No. 1*, pp. 31-46.

[11] Landa, P.S., (1996), **Nonlinear Oscillations and Waves in Dynamical Systems**, *Kluwer Academic Publishers.*

[12] Lewis, T. J. and Keener, J.P., (2000), **Wave-Block in Excitable Media Due to Regions of Depressed Excitability**, *SIAM J. Appl. Math., Vol. 61, No.1*, pp. 293-316.

[13] Logan, J.D., (1987), **Applied Mathematics**, *John Wiley & Sons.*

[14] Maginu, K., (1978), **Stability of Stationary Solutions of a Semilinear Parabolic Partial Differential Equation**, *J.Math. Anal. Appl. 63*, pp. 224-243.

[15] Manoranjan, V.S., Mitchell, A.R., Sleeman, B.D. and Yu, K. P., (1984), **Bifurcation Studies in Reaction-Diffusion**, J. *Comput. Appl. Math.*, 11, pp. 27-37.

[16] Manoranjan, V.S., (1984), **Bifurcation Studies in Reaction-Diffusion II**, *J. Comput. Appl. Math., 11*, pp. 307-314.

[17] McKean, H.P., (1970), **Nagumo's Equation,** *Adv. In Math., 4*, pp. 209-223.

[18] Pesin, Y. and Yurchenko, A., (2004), **Some Physical Models of the Reaction-Diffusion Equation and Coupled Map Lattices**, *Russian Math. Surveys, Vol. 59, No. 3,* to appear.

[19] Smoller, J. and Wasserman, A., (1981), **Global Bifurcation of Steady State Solutions**, *J. Differential Equations*, *39*, pp. 269-290.

[20] Wang, X.Y., (1985), **Nerve Propagation and Wall in Liquid Crystals**,*Phys. Lett., Vol. 112 A*, No. 8, pp. 402-406.

[21] Wang, X.Y., Zhu, Z.S. and Lu, Y.K., (1990), **Solitary Wave Solutions of the Generalized Burgers-Huxley Equation**, *J. Phys. A: Math.Gen.*, 23, pp. 271-274.