Three-mode bifurcation of extremals in the analysis of bifurcation solutions of sixth order differential equation

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Abstract:

In this paper we are interested in the study of bifurcation solutions of nonlinear wave equation of sixth order of elastic beams on elastic foundations by using local method of Lyapunov –Schmidt. The normal form of the key function corresponding to the functional related to this equation has been found. Also, it was found a new geometrical description of caustic with bifurcation spreading of the critical points.

Key words: Bifurcation solutions, bifurcation spreading , caustic.

الملخص:

قدمنا في هذا البحث دراسة حلول التفرع لمعادلة الموجة للأنابيب المرنة الغير خطية من الدرجة السادسة بالاعتماد على أساس المرونة باستخدام طريقة ليابيونوف - شمدت المحلية في الحالة المتغايرة. تم أيجاد الصيغة القياسية للدالة الأساسية المقابلة للدالي .كذلك تم أيجاد وصف هندسي جديد للمجموعة المميزة(مخطط التفرع) مع توزيع كامل للنقاط الحرجة في المناطق المتممة للمجموعة المميزة.

1. Introduction

Many of the nonlinear problems in mathematics and physics can be written in the form of operator equation,

$$F(x,\lambda) = b, x \in O \subset X, b \in Y, \lambda \in R^n \ldots(1)$$

in which F is a smooth Fredholm map of index zero, X, Y Banach spaces and O open subset of X. For these problems, the method of reduction to finite dimensional equation Sapronov(1973),

$$\Phi(\xi,\lambda) = \beta, \xi \in \widehat{M}, \beta \in \widehat{N} \ldots(2)$$

can be used, where $\hat{M}$ and $\hat{N}$ are smooth finite dimensional manifolds.

Passage from equation (1) into equation (2) (variant local scheme of Lyapunov – Schmidt) with the conditions, that equation (2) has all the topological and analytical properties of equation (1) (multiplicity, bifurcation diagram, *etc*) dealing withArnold(1989),Sapronov(1973),Vainberg andTrenogin(1969) and Zachepa andSapronov (2002).

Definition 1.1 Suppose that *E* and *M*are Banach spaces and $A: E \to M$ be a linear continuous operator. The operator A is called Fredholm operator, if

1- The kernel of A, Ker(A), is finite dimensional,

2- The range of A, Im(A), is closed in M,

3- The Cokernel of A, Coker(A), is finite dimensional.

The number

dim(*Ker A*)- *dim*(*Coker A*)

is called Fredholm index of the operator A .

Suppose that $F:\Omega \to M$ is a nonlinear Fredholm map of index zero. A smooth map $F:\Omega \to M$ has variational property, if there exist a functional $V:\Omega \to R$ such that $F = grad_H V$ or equivalently,

$$\frac{\partial V}{\partial x}(x,\lambda)h = \langle F(x,\lambda),h\rangle_H, \quad \forall\ x \in \Omega,\ h \in E.$$

where ($\langle \cdot,\cdot \rangle_H$ is the scalar product in Hilbert space H). In this case the solutions of equation $F(x,\lambda)=0$ are the critical points of functional $V(x, \lambda)$. By using the method of finite dimensional reduction (Local method of Lyapunov-Schmidt) the problem,

$$V(x,\lambda) \to extr, \quad x \in E,\ \lambda \in R^n.$$

can be reduced into an equivalent problem,

$$W(\xi,\lambda) \to extr,\ \xi \in R^n.$$

the function $W(\xi,\lambda)$ is called Key function.

If $N = span\{e_1,...,e_n\}$ is a subspace of E, where $e_1, \dots, e_n$ are orthonormal basis, then the key function $W(\xi,\lambda)$ can be defined in the form,

$$W(\xi,\lambda) = \inf_{x:\langle x,e_i\rangle = \xi_i\ \forall i} V(x,\lambda), \quad \xi = (\xi_1,...,\xi_n).$$

The function W has all the topological and analytical properties of the functional V (multiplicity, bifurcation diagram, *etc*) Thompson and Stewart(1986).The study of bifurcation solutions of functional V is equivalent to the study of bifurcation solutions of key function. If F has variational property, then it is easy to check that,

$$\theta(\xi,\lambda) = grad\, W(\xi,\lambda).$$

Equation $\theta(\xi,\lambda)=0$ is called bifurcation equation.

Definition 1.2 The caustic is defined to be the set of all λ in which the functional $V(\cdot,\lambda), \quad \lambda\in\Lambda\subset R^n$ have in $\Omega\subset R^n$ degenerate critical points**.**

The oscillations and motion of waves of the elastic beams on elastic foundations can be described by means of the following ODE,

$$\frac{d^6w}{dx^6}+\delta\frac{d^4w}{dx^4}+\alpha\frac{d^2w}{dx^2}+\beta w+w^3=\psi \qquad \dots (3)$$

where, z is the deflection of beam, $\delta=1+\rho$, ρ is small

parameter and ψ is a continuous function. In this work equation (3) has been studied with the following boundary conditions,

$$\frac{d^4w}{dx^4}(0)=\frac{d^2w}{dx^2}(0)=w(0)=\frac{d^4w}{dx^4}(\pi)=\frac{d^2w}{dx^2}(\pi)=w(\pi)=0$$

Equation (3) has been studied by Thompson and Stewart (1986) they showed numerically the existence of periodic solutions of equation (3) for some values of parameters. Bardin and Furta (2001) used the local method of Lyapunov-Schmidt and found the sufficient conditions of existence of periodic waves of equation (3), also they are introduced the solutions of equation (3) in the form of power series. Sapronov (1973,1996) and Zachepa andSapronov(2002) applied the local method of Lyapunov –Schmidt and found the bifurcation solutions of equation (3). Abdul Hussain (2005,2009) studied equation (3) with small perturbation when the nonlinear part has quadratic term and Mohammed (2007) studied equation (3) in the variational case when the nonlinear part has quadratic term. In this paper we studied the bifurcation solutions of equation (3) near the critical point when the dimension of the null space is equal to three.

2. Bifurcation of Extremals from the cusp point.

In the bifurcation analysis of extremals the following two questions need to be answered (1) Describe geometrical structure of caustic (bifurcation diagram of the function) and (2) Determine the bifurcation spreading of the critical points in the complements of caustic. In this section we studied the bifurcation of extremals of the following function

$$\widetilde{W}(\xi,\lambda) = \xi_1^4 + \xi_2^4 + \xi_3^4 + 4\left(\xi_1^2\xi_2^2 + \xi_1^2\xi_3^2 + \xi_2^2\xi_3^2 + \xi_1\xi_2^2\xi_3 - \tfrac{1}{3}\xi_1^3\xi_3\right) + \frac{n_1}{6}\xi_1^2 + \frac{32n_1}{3}\xi_2^2 + \frac{243n_1}{2}\xi_3^2 + k_1\xi_1^2 + k_2\xi_2^2 + k_3\xi_3^2 - q_1\xi_1 - q_3\xi_3 \quad \dots (4)$$

where, $n_1 = \sqrt{3}\sqrt{8}\pi^{\frac{1}{2}}$ and $\xi = (\xi_1, \xi_2, \xi_3)$, $\gamma = (k_1, k_2, k_3, q_1, q_3) \in R^5$

To find the caustic of the function $\widetilde{W}$ we used the same manner in Abdul Hussain (2009). Since, the function $\widetilde{W}(\xi,\gamma)$ is even with respect to the second variable ξ_2 then we can write function (4) as follows:

$$\widetilde{W}(\xi,\gamma) = \xi_2^4 + W_2(\xi_1,\xi_3)\xi_2^2 + W_0(\xi_1,\xi_3) \quad \dots (5)$$

where

$$W_2(\xi_1,\xi_3) = 4(\xi_1^2 + \xi_3^2 + \xi_1\xi_3) + \frac{32n_1}{3} + k_2$$

and

$$W_0(\xi_1,\xi_2) = \xi_1^4 + \xi_3^4 + 4\left(\xi_1^2\xi_3^2 - \tfrac{1}{3}\xi_1^3\xi_3\right) + \frac{n_1}{6}\xi_1^2 + \frac{243n_1}{2}\xi_3^2 + k_1\xi_1^2 + k_3\xi_3^2 - q_1\xi_1 - q_3\xi_3$$

Function (5) has three nondegenerate critical points with respect to the variable ξ_2. The existence of these points depends on the coefficient of the quadratic term $W_2(\xi_1,\xi_3)$. If the condition

$$\inf W_2(\xi_1,\xi_3) > 0 \quad \dots (6)$$

is satisfied then the investigation of critical points of the function $\widetilde{W}$ is reduced to the investigation of critical points of the function,

$$W_0(\xi_1,\xi_3) = \inf_{\eta_2} \widetilde{W}(\xi_1,\xi_2,\xi_3). \dots (7)$$

If condition (6) is not satisfied, then the space of variables ξ_1, ξ_2, and ξ_3 is decomposed into the domains $D_1 = \{(\xi_1,\xi_2,\xi_3) : W_2(\xi_1,\xi_3) > 0\}$ and $D_2 = \{(\xi_1,\xi_2,\xi_3) : W_2(\xi_1,\xi_3) < 0\}$. The investigation of $\widetilde{W}$ is reduced to the investigation of the function W_0 in the former domain and to the investigation of the pair of functions

$$U_1(\xi_1,\xi_3)=\sup_{\xi_2}\tilde{W}(\xi_1,\xi_2,\xi_3)=W_0(\xi_1,\xi_3),$$
$$U_2(\xi_1,\xi_3)=\inf_{\xi_2}\tilde{W}(\xi_1,\xi_2,\xi_3)=W_0(\xi_1,\xi_3)-\frac{1}{4}[W_2(\xi_1,\xi_3)]^2 \quad \ldots(8)$$

$$U_2(\xi_1,\xi_3)=-\frac{127n_1}{6}\xi_1^2+\frac{601n_1}{6}\xi_3^2-\frac{28}{3}\xi_1^3\xi_3-8\,\xi_1^2\xi_3^2-3\xi_1^4-3\xi_3^4+k_1\xi_1^2+k_3\xi_3^2-q_1\xi_1-q_3\xi_3-8\,\xi_1\xi_3^3-2k_2\xi_3^2-2k_2\xi_1^2-\frac{256n_1^2}{9}-\frac{16n_1}{3}k_2-\frac{k_2^2}{4}-\frac{64n_1}{3}\xi_1\xi_3$$

in the latter domain.

Critical points of the function (7) are determined by the system of equations

$$4\xi_1^3+8\,\xi_1\xi_3^2-4\xi_1^2\xi_3+\frac{n_1}{3}\,\xi_1+2k_1\,\xi_1-q_1=0,$$
$$4\xi_3^3+8\,\xi_1^2\xi_3-\frac{4}{3}\xi_1^3+243n_1\,\xi_3+2k_3\,\xi_3-q_3=0.$$

The points are degenerate when $\det(\partial^2 W_0/\partial\xi^2)=0$. By using the following transformation $k_1=d_1+d_2$, $k_3=d_1-d_2$, we have
the Caustic of function (7) in
space of parameters (q_1,q_3,d_2) for $d_1=-3$ in the form,

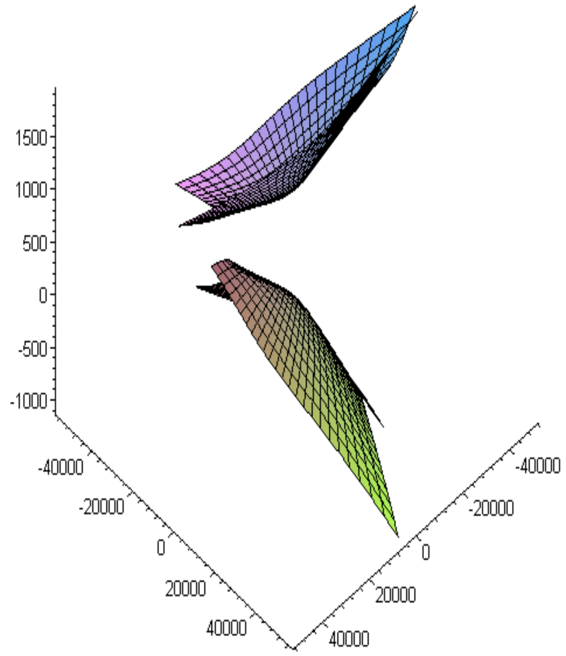

Fig. 1 Describes the Caustic of function (7).

The critical points of function $U_2(\xi_1,\xi_3)$ are the solutions of the following system of equations

$$-\frac{127n_1}{3}\xi_1 - 28\xi_1^2\xi_3 - 16\xi_1\xi_3^2 - 12\xi_1^3 + 2k_1\xi_1 - q_1 - 8\xi_3^3 - 4k_2\xi_1 - \frac{64n_1}{3}\xi_3 - 2k_2\xi_3 = 0,$$

$$-\frac{601n_1}{3}\xi_3 - \frac{28}{3}\xi_1^3 - 16\xi_3\xi_1^2 - 12\xi_3^3 + 2k_3\xi_3 - q_3 - 24\xi_1\xi_3^2 - 4k_2\xi_3 - \frac{64n_1}{3}\xi_1 - 2k_2\xi_1 = 0$$

The points are degenerate when $\det(\partial^2 U_2/\partial\xi^2) = 0$. Also, by using the following transformation $k_1 = d_1 + d_2, \quad k_3 = d_1 - d_2,$

we have the Caustic of function $U_2(\xi_1,\xi_3)$ in space of parameters (q_1,q_3,d_2) for $d_2=k_2=-1.5$ in the form,

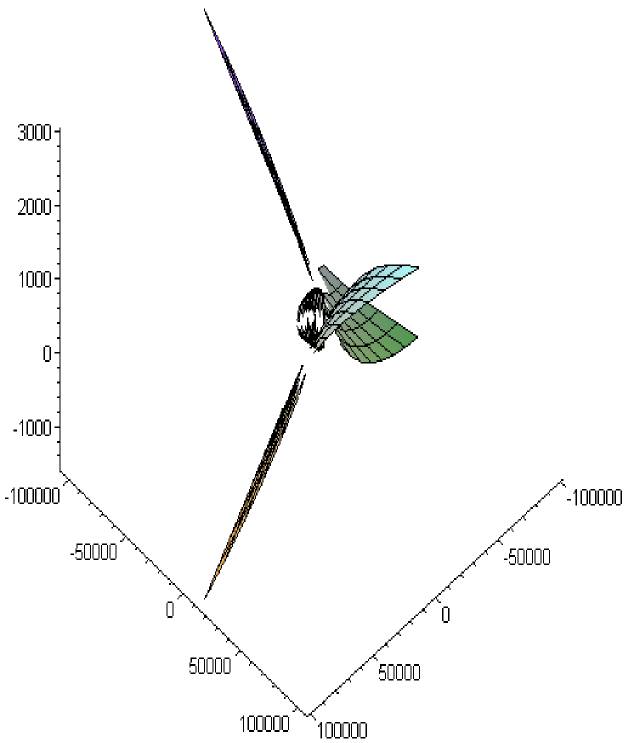

Fig. 2 Describes the Caustic of function U_2.

and hence the Caustic of pair functions (8) for $d_1=k_2=-3$ is given by the following graph,

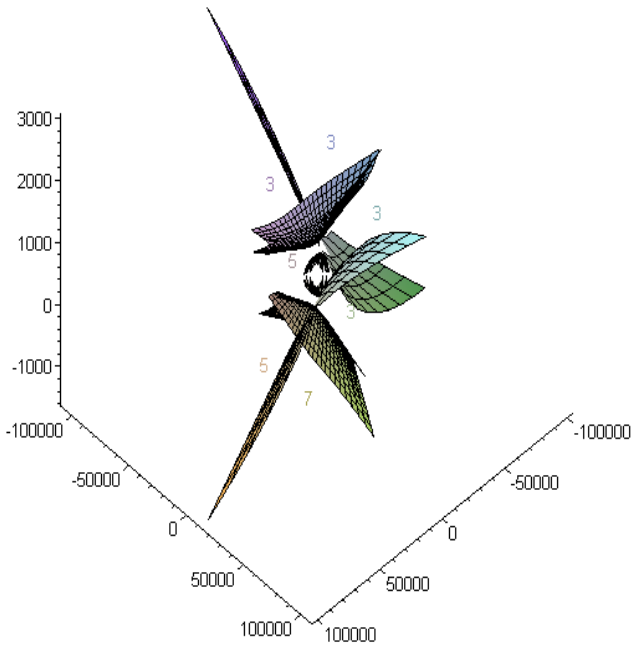

Fig. 3 Describes the Caustic of functions (8).

The bifurcation diagram (Caustic) of pair functions (8) decomposes the space of parameters into 12 regions; every region contains a fixed number of critical points. The following spreading of bifurcating critical points corresponds to the above regions:

If $(q_1, q_3, d_2) = (0, -1000, -500)$ then we have 7 critical points (4-min., 3-saddle), if $(q_1, q_3, d_2) = (3000, -3000, 1500)$ then we have 7 critical points (4-min., 3-saddle) , if $(q_1, q_3, d_2) = (-10000, 20000, 50)$ then we have 3 critical points (2-min., 1-saddle), if $(q_1, q_3, d_2) = (-40000, -30000, 1500)$ then we have 3 critical points (1-min., 2-max.), if $(q_1, q_3, d_2) = (-10000, 50000, 1000)$ then, we have 3 critical points (1-min., 2-saddle), if $(q_1, q_3, d_2) = (-40000, 10000, 1000)$ then we have 3 critical points (1-min., 2-saddle), if $(q_1, q_3, d_2) = (50000, -10000, 0)$ then we have 5 critical points (1-min, 2-saddle, 2-max.),if $(q_1, q_3, d_2) = (0, -20000, 400)$ then we have 5 critical points (3-min., 2-saddle) ,

3. Applications

Consider the boundary value problem,

$$\frac{d^6W}{dx^6}+\delta\frac{d^4W}{dx^4}+\alpha\frac{d^2W}{dx^2}+\beta W+W^3=\psi$$
$$\frac{d^4W}{dx^4}(0)=\frac{d^2W}{dx^2}(0)=W(0)=\frac{d^4W}{dx^4}(\pi)=\frac{d^2W}{dx^2}(\pi)=W(\pi)=0 \qquad \text{... (9)}$$

which described the oscillations and motion of wave on elastic foundations, where, $\delta=1+\rho$, ρ is small parameter and ψ is a symmetric function with respect to the involution

$$I:\psi(x)\mapsto\psi(\pi-x).$$

Suppose that $F:E\to M$ is a nonlinear Fredholm operator of index zero from Banach space *E* to Banach space *M*, where $E=C^6([0,\pi],R)$ is the space of all continuous functions that have derivative of order at most four, $M=C^0([0,\pi],R)$ is the space of all continuous functions and *F* is given in the form of operator equation:

$$F(w,\lambda)=\frac{d^6w}{dx^6}+\delta\frac{d^4w}{dx^4}+\alpha\frac{d^2w}{dx^2}+\beta w+w^3$$

where

$$w=w(x),$$
$$x\in[0,\pi],\quad \lambda=(\delta,\alpha,\beta).$$

Every solution of equation (9) is a solution of operator equation,

$$F(w,\lambda)=\psi,\qquad \psi\in M. \qquad \text{… (10)}$$

Since, operator f has variational property, then there exists functional *V*, such that

$$F(w,\lambda)=grad_H V(w,\lambda,0).$$

and then every solution of equation (10) is a critical point of the functional *V* where,

$$V(w,\lambda,\psi)=\int_0^\pi\left(\frac{(w''')^2}{2}-\delta\frac{(w'')^2}{2}+\alpha\frac{(w')^2}{2}+\beta\frac{w^2}{2}+\frac{w^4}{4}-w\psi\right)dx$$

Thus, the study of equation (10) is equivalent to the study extremely problem,

$$V(w,\lambda,\psi)\to extr,\qquad w\in E.$$

The analysis of bifurcation can be found by using the local method of Lyapunov-Schmidt to reduce it into finite dimensional space and by localized parameters,

$$\delta=\tilde{\delta}+\delta_1,\ \alpha=\tilde{\alpha}+\delta_2,\ \beta=\tilde{\beta}+\delta_3.$$

this reduction leads to the function in three variables,

$$W(\tau,\gamma)=\inf_{\langle w,e_i\rangle=\tau_i,\, i=1,2} V(w,\gamma),$$
$$\tau=(x_1,x_2,x_3),\ \gamma=(\delta_1,\delta_2,\delta_3).$$

It is well known that in the reduction of Lyapunov-Schmidt the function $W(\tau,\gamma)$ is smooth. This function has all the topological and analytical properties of functional *V*Thompson and Stewart(1986)In particular, for small γ there is a one-to-one corresponding between the critical points of functional *V* and the smooth function *W*, preserving the type of critical points (multiplicity, Morse index, *etc*) Thompson and Stewart(1986). We note that the functional *V* is symmetric with respect to the involution $I:\psi(x)\mapsto\psi(\pi-x)$. By using the method of Lyapunov-Schmidt, the linearized equation corresponding to the equation (10) has the form,

$$y'''''' + \delta y'''' + \alpha y'' + \beta y = 0, y\in E$$
$$\frac{d^4y}{dx^4}(0)=\frac{d^2y}{dx^2}(0)=y(0)=\frac{d^4y}{dx^4}(\pi)=\frac{d^2y}{dx^2}(\pi)=y(\pi)=0 \quad \ldots(11)$$

Localized parameters,

$$\delta=\frac{65}{4}+\delta_1,\ \alpha=\frac{241}{4}+\delta_2,\ \beta=45+\delta_3.$$

lead to bifurcation along the modes $e_1(x)=c_1\sin(x)$, $e_2(x)=c_2\sin(2x)$, $e_3(x)=c_3\sin(3x)$ where $\|e_1\|=\|e_2\|=\|e_3\|=1$ and $c_1=c_2=c_3=\sqrt{\frac{2}{\pi}}$.

Let $N=Ker(A)=Span\{e_1,e_2,e_3\}$

where,

$$A=F_z(0,\lambda)=\frac{d^6}{dx^6}+\delta\frac{d^4}{dx^4}+\alpha\frac{d^2}{dx^2}+\beta$$

then, the space E can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$E=N\oplus\hat{E},\ \hat{E}=N^{\perp}\cap E=\{v\in E: v\perp N\}.$$

Similarly, the space M can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$M=N\oplus\hat{N}^{\perp},\ \hat{N}^{\perp}=N^{\perp}\cap M=\{v\in M: v\perp N\}.$$

There exists projections $p:E\to N$ and $I-p:E\to\hat{E}$ such that $pz=w$ and $(I-p)w=v$, (I is the identity operator). Hence every vector $w\in E$ can be written in the unique form,

$$w=z+v,\ w=\sum_{i=1}^{3}\tau_i e_i\in N,\ N\perp v\in\hat{E},\ \tau_i=\langle w,e_i\rangle.$$

Similarly, there exists projections $Q:M\to N$ and $I-Q:M\to\hat{N}^{\perp}$ such that,

$$F(w,\lambda)=QF(w,\lambda)+(I-Q)F(w,\lambda) \qquad \ldots (12)$$

Since $\psi\in M$ implies that

$$\psi=\psi_1+\psi_2,\quad \psi_1=t_1e_1+t_2e_2+t_3e_3\in N,\quad \psi_2\in\hat{N}^{\perp}$$

Accordingly, equation (10) can be written in the form,

$$QF(z+v,\lambda)=\psi_1,$$
$$(I-Q)F(z+v,\lambda)=\psi_2.$$

By the implicit function theorem, there exist a smooth map $\Phi : N \to \hat{E}$, such that

$$W(\tau,\gamma,\psi)=V(\Theta(\tau,\gamma,\psi),\gamma,\psi),$$
$$\gamma=(\delta_1,\delta_2,\delta_3).$$

and then the key function can be written in the form,

$$W(\tau,\gamma)=V(x_1 e_1+x_2 e_2+x_3 e_3+$$
$$\Theta(x_1 e_1+x_2 e_2+x_3 e_3,\gamma),\gamma)$$
$$=\tilde{U}(\tau,\gamma)+o(|\tau|^4)+$$
$$O(|\tau|^4)O(\gamma).$$

The function $\tilde{U}(\tau,\gamma)$ can be found as follows, substitute the value of *w* in the above functional we have that,

$$\int_0^\pi \frac{(w''')^2}{2}\,dx=(\frac{1}{2}x_1^2+32x_2^2+\frac{729}{2}x_3^2),$$
$$\int_0^\pi \frac{(w'')^2}{2}\,dx=(\frac{1}{2}x_1^2+8x_2^2+\frac{81}{2}x_3^2),$$
$$\int_0^\pi \frac{(w')^2}{2}\,dx=(\frac{1}{2}x_1^2+2x_2^2+\frac{9}{2}x_3^2),$$
$$\int_0^\pi \frac{w^2}{2}\,dx=\frac{1}{2}(x_1^2+x_2^2+x_3^2),$$
$$\int_0^\pi \psi_1\,w\,dx=\tilde{q}_1\,x_1+\tilde{q}_2\,x_2+\tilde{q}_3\,x_3.,$$
$$\int_0^\pi \frac{w^4}{4}\,dx=\frac{1}{8\pi}(12\,x_2^2x_1x_3+3x_2^4+3x_3^4-$$
$$4x_1^3x_3$$
$$+12x_3^2x_2^2+12x_3^2x_1^2+3x_1^4+12x_2^2x_1^2),$$

The symmetry of the function $\psi(x)$ with respect to the involution $I:\psi(x)\mapsto\psi(\pi-x)$ implies that $\tilde{q}_2=0$ and then we have,

$$\tilde{U}(\tau,\gamma) = \frac{3}{8\pi}\,(x_1^4 + x_2^4 + x_3^4) + \frac{3}{2\pi}(x_1^2 x_3^2$$
$$+ x_1^2 x_2^2 + x_3^2 x_2^2 + x_2^2 x_1 x_3) - \frac{1}{2\pi} x_1^3 x_3$$
$$+ \frac{1}{2}\, x_1^2 + 32\, x_2^2 + \frac{729}{2}\, x_3^2 + \tilde{k}_1\, x_1^2 + \tilde{k}_2\, x_2^2$$
$$+ \tilde{k}_3\, x_3^2 + \tilde{q}_1\, x_1 + \tilde{q}_3\, x_3.$$

Let

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sqrt[4]{\frac{8\pi}{3}} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \quad \text{and} \quad n_1 = \sqrt{3}\sqrt{8}\pi^{\frac{1}{2}}$$

we have the function $\tilde{U}(\tau,\gamma)$ is equivalent to the following function,

$$U(\xi,\gamma) = \xi_1^4 + \xi_2^4 + \xi_3^4 + 4(\xi_1^2\xi_2^2 + \xi_1^2\xi_3^2 +$$
$$\xi_2^2\xi_3^2 + \xi_1\xi_2^2\xi_3 - \frac{1}{3}\xi_1^3\xi_3)$$
$$+ \frac{n_1}{6}\,\xi_1^2 + \frac{32\,n_1}{3}\,\xi_2^2 + \frac{243\,n_1}{2}\,\xi_3^2 +$$
$$k_1\,\xi_1^2 + k_2\,\xi_2^2 + k_3\,\xi_3^2 - q_1\xi_1 - q_3\,\xi_3\,,$$
$$\xi = (\xi_1,\xi_2,\xi_3),$$
$$\gamma = (k_1,k_2,k_3,q_1,q_3) \in R^5. \qquad \ldots(13)$$

and hence we proved the following result,

Theorem 3.1 The normal form of the key function *W* corresponding to the functional *V* is given by

$$\tilde{W}(\xi,\gamma) = U(\xi,\gamma) + o(|\xi|^4) +$$
$$O(|\xi|^4)O(\gamma)$$
$$= \xi_1^4 + \xi_2^4 + \xi_3^4 + 4(\xi_1^2\xi_2^2$$
$$+ \xi_1^2\xi_3^2 + \xi_2^2\xi_3^2 + \xi_1\xi_2^2\xi_3 - \frac{1}{3}\xi_1^3\xi_3)$$
$$+ \frac{n_1}{6}\,\xi_1^2 + \frac{32\,n_1}{3}\,\xi_2^2 +$$
$$\frac{243\,n_1}{2}\,\xi_3^2 + k_1\,\xi_1^2 + k_2\,\xi_2^2 + k_3\,\xi_3^2$$
$$- q_1\xi_1 - q_3\,\xi_3 + o(|\xi|^4)$$
$$+ O(|\xi|^4)O(\gamma). \qquad \ldots(14)$$

The geometrical form of bifurcations of critical points and the first asymptotic of branches of bifurcating for the function $\tilde{W}$ are completely determined by its principal part $U(\xi,\gamma)$. This mean that the study of Caustic of the function $\tilde{W}$ is equivalent to the study of Caustic of the function $U(\xi,\gamma)$.Function $U(\xi,\gamma)$ has all the topological and analytical properties of functional *V*, so the study of bifurcation analysis of equation (10) is equivalent to the study of bifurcation analysis of the function $U(\xi,\gamma)$. Note that the function $U(\xi,\gamma)$ is similar to the function $\tilde{W}(\xi,\gamma)$ in the previous section, so all results which we need from the Lyapunov-Schmidt reduction have been found.

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