# UNSTEADY ENERGY EQUATION OF ELECTRICALLY CONDUCTING FLUID OF SECOND GRADE OVER A STRETCHING SHEET SUBJECT TO A TRANSVERSE MAGNETIC FIELD

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**Abstract**

In this paper, we obtain an expression for unsteady energy equation for an incompressible viscous electrically conducting second grad fluid over a stretching sheet subject to a transverse magnetic field. Homotopy analysis method (HAM) is needed to solve the governing equations. Also we examines the effects of internal heat parameter, viscoelastic parameter, magnetic parameter, Prandtl number, Eckert number and time which they control the energy equation.

**Keyword**: Homotopy analysis method (HAM); Unsteady; viscoelasticity; magnetic parameter; Prandtl number; Eckert number.

# معادلة الطاقة اللامستقرة لمائع من الرتبة الثانية قابل للتوصيل الكهربائي على صفيحة مستعرضة مطاطية في حقل مغناطيسي

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**الخلاصة**

في هذا البحث نتناول دراسة معادلة الطاقة اللامستقرة لمائع من الرتبة الثانية غير قابل للانضغاط ثابت اللزوجة قابل للتوصيل الكهربائي على مقطع عرضي لصفيحة مطاطية تحت تأثير مجال مغناطيسي. وتم التعبير عن الحل باستعمال طريقة هوموتوبي التحليلية (HAM). كما درست تأثير الإعداد اللابعدية التي تحكم معادلة الطاقة وهي الزمن ($\xi$)، الحرارة الداخلية ($\alpha$)، اللزوجة ($K$)، الحقل المغناطيسي ($M_n$)، عدد براندل ($\Pr$)، وعدد إيكرت ($Ec$).

**كلمات مفتاحية:** طريقة الهوموتوبي التحليلية (HAM), اللامستقر, اللزوجة, المعامل المغناطيسي, عدد براندل، عدد إيكرت.

## Introduction

The study of flow and heat transfer is of considerable interest in many industrial applications such as processes involving continuous pulling of a sheet through a reaction zone, as in metallurgy, textile and paper industries, manufacture of polymeric sheets, sheet glass and crystalline materials. Since the pioneering work of Sakiadis [1, 2], various aspects of the problem have been investigated by many authors. Crane [3], Vleggaar [4] and Gupta and Gupta [5] have analyzed the stretching problem with constant surface temperature while Soundalgekar and Ramana Murty [6] investigated the constant surface velocity case with power-law temperature variation. This flow was examined by Siddappa and Khapate [7] for a special class of non-Newtonian fluids known as second-order fluids which are viscoelastic in nature.

Rajagopal et al. [8] independently examined the same flow as in [7] and obtained similarity solutions of the boundary layer equations numerically for the case of small viscoelastic parameter $K$. It is shown that skin-friction decreases with increase in $K$. Dandapat and Gupta [9] examined the same problem with heat transfer. In [9], an exact analytical solution of the non-linear equation governing this self-similar flow which is consistent with the numerical results in [8] is given and the solutions for the temperature for various values of $K$ are presented . Later, Cortell [10] extended the work of Dandapat and Gupta [9] to study the heat transfer in an incompressible second order fluid caused by a stretching sheet with a view to examining the influence of the viscoelastic parameter on temperature distributions. It is found that temperature distribution depends on $K$, in accordance with the results in [9]. Numerical solutions for the flow of a fluid of grade three past an infinite porous flat plate subject to suction at the plate are to be found in Rajagopal et al. [11] and in Cortell [12] . Hayat et al. [13] studied the flow of a third-grade fluid over a wall with suction or blowing and Gupta et al. [14] investigated the steady flow of a power law fluid past an infinite porous flat plate subject to suction or blowing with heat transfer. Arbitrary injection/suction in a power -law fluid is analyzed in [15]. Flow and heat transfer characteristics were investigated in [16] for a viscoelastic fluid over a stretching sheet with power-law surface temperature and in [17] with a non-linearly stretching sheet. Very recently, Vajravelu and Rollings [18] assumed additional effects such as the flow in an electrically conducting fluid permeated by a trans verse uniform magnetic field with uniform suction at the surface, however, heat transfer in such flow was not studied.

Furthermore, they augmented the missing boundary condition and used a proper sign for the normal stress modulus (i.e. $\alpha \geq 0$).

## Flow analysis

An incompressible homogeneous second grade fluid has a constitutive equation given by [19]:

$$\mathrm{T} = -p\mathrm{I} + \mu \mathrm{A}_1 + \alpha_1 \mathrm{A}_2 + \alpha_2 \mathrm{A}_1^2 \qquad (1)$$

Here $\mathrm{T}$ is the Cauchy stress tensor, $p$ is the indeterminate pressure constrained by the incompressibility, $\mu$ is the coefficient viscosity, $\alpha_1$ and $\alpha_2$ are the moduli of the viscoelastic fluid, and $\mathrm{A}_1$ and $\mathrm{A}_2$ are the first two Rivilin-Ericksen tensors defined as:

$$\mathrm{A}_1 = L + L^{\mathrm{T}}$$

$$\mathrm{A}_2 = \frac{d\mathrm{A}_1}{dt} + \mathrm{A}_1 L + L^{\mathrm{T}} \mathrm{A}_1, \qquad (2)$$

Where $d/dt$ is the material derivative and $L = \nabla V$. If the fluid of second grade is to satisfy the Clausius-Dehum inequality for all motions and the assumption that the specific Helmholtz free energy of the fluid is a minimum when it is locally at rest, then the requirements for the moduli of the second grad fluid are

$$\mu \geq 0,\ \alpha_1 > 0, \text{and}\ \ \alpha_1 + \alpha_2 = 0 \qquad (3)$$

Though the sign of $\alpha_1$ has been a subject of much controversy.

If the second grade fluid is electrically conducting, the Lorentz force $J \times B$ where $J$ is the electrical current and $B$ is the magnetic field, must be included in the momentum equation when a transverse uniform magnetic field $B = (0, B_0, 0)$ is applied to the fluid layer. The terms due to Lorentz force can be simplified if the following assumption are made: (i) all physical quantities are constant; (ii) the magnetic field $B$ is perpendicular to the velocity $V$ and the induced magnetic field is small compared with the applied magnetic field; (iii) the electrical field is assumed to be zero. These assumptions are valid when the magnetic Reynolds number is small and there is no displacement current [20]. Thus, in the boundary layer approximation the Lorentz force is simply the term $-\sigma B_0^2 u$, where $\sigma$ is the electrical conductivity, $B_0$ is the uniform magnetic field in the $y$-direction , and $u$ is the $x$-component of the velocity $V$.

The flow problem of non-Newtonian fluids, characterized by Bingham plastic and the power law models, in a magnetic field has been investigated by Sarpkaya [21]. Sarpkaya also pointed out that some non-Newtonian fluids such as nuclear fuel slurries, liquid metals, mercury amalgams, biological fluids, plastic extrusions, paper coating, lubrication oils and greases, have applications in many areas in the absence as well as in the presence of magnetic field.

In this paper, using homotopy analysis method (HAM), one of the most effective methods [22, 23]. We present a general solution for unsteady energy equation of a laminar boundary layer flow of an electrically conducting second grad fluid subject to a transverse uniform magnetic field over a stretching sheet with prescribed power-law surface temperature and prescribed power-law surface heat flux, the viscoelastic modulus $\alpha_1$ of the second grad fluid is taken to be positive to satisfy thermodynamic restriction Eq. (3).

Consider the unsteady, two dimensional laminar flow of electrically conducting fluid caused by an impulsive stretching flat surface in two lateral direction in an otherwise quiescent fluid in the presence of transverse magnetic field. It is assumed that the contribution due to the normal stress is of the same order of magnetic as that due to shear stress [19].

The basic boundary layer equations for the unsteady flow are:

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = 0 \qquad (4)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho}u$$

$$+\frac{\alpha_1}{\rho}\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial u}{\partial y}\frac{\partial^2 \upsilon}{\partial y^2}+\upsilon\frac{\partial^3 u}{\partial y^3}\right] \quad (5)$$

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+\upsilon\frac{\partial T}{\partial y}\right)=k\frac{\partial^2 T}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

$$\alpha_1\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}+\upsilon\frac{\partial u}{\partial y}\right)$$

$$+q(T-T_\infty)+\sigma B_0^2 u^2 \quad (6)$$

where $u$ and $\upsilon$ are velocity component in the $x$ and $y$ direction, $T$ is the temperature, $T_\infty$ is the temperature of the ambient fluid, $\rho$ is the density, $q$ is the specific heat generation rate, $\nu=\mu/\rho$ is kinematic viscosity, $k$ is the conductivity and $c_p$ is the specific heat at constant pressure. In deriving (5) and (6) it is assumed that the contribution due to the normal stress is of the same order of magnitude as that due to the shear stress. The term $\frac{\sigma B_0^2}{\rho}u$ in (5) is the Lorentz Force and the last three terms in (6) are work done due to deformation, internal heat generation or absorption and the Joule heating. We assumed that the gravity force is neglected and the modified pressure gradient is absent since the flow is driven by the stretching sheet.

The series solution for motion equation (5) can be found in Adel Rashed [24].

The boundary conditions for energy equation (6) are

$$T=T_w \quad \text{at } y=0$$
$$T\to T_\infty \quad \text{as } y\to\infty \quad (7)$$

**Non-Dimensional form of energy Equation**

We can write down the Velocity Equation in non-dimensional form through using the transformations:

$$u=Bxf'(\eta),\ \upsilon=-(B\nu)^{1/2}f(\eta)\xi^{1/2}$$
$$\eta=-(B/\nu)^{1/2}y\xi^{-1/2},\ \tau=Bt$$
$$\xi=1-e^{-\tau} \quad (8)$$

substitution these quantities into energy equation (6), the Dimensionless form of energy equation is

$$\theta''+P_r f\xi\theta'+\frac{1}{2}P_r\eta(1-\xi)\theta'-P_r(1-\xi)\xi\frac{\partial\theta}{\partial\xi}$$
$$-P_r\xi(2f'-\alpha)\theta+P_rE_c\left[(f'')^2+M_n\xi(f')^2\right.$$
$$\left.+Kf''(f'f''-ff''')\right]=0 \quad (9)$$

Where a prime denotes the differentiation with respect to $\eta$, $f=f(\eta,\xi)$, $\theta=\theta(\eta,\xi)$, $K=\alpha_1 B/\mu$ is the viscoelastic parameter, $P_r=\mu c_p/k$ is the Prandtl number, $\alpha=q/B\rho c_p$ is the internal heat parameter, $E_c=B^2x^2/\Delta T c_p$ is the Eckert number and $M_n$ is the magnetic parameter.

The corresponding boundary conditions (7) became:

$\theta=1$ at $\eta=0$,

$$\theta \to 0 \text{ as } \eta \to \infty \quad (10)$$

**Basic ideas of HAM**

Let us consider the following differential equation

$$N[u(\tau)] = 0 \quad (11)$$

where $N$ is a nonlinear operator, $\tau$ denotes independent variable, $u(\tau)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [25] constructs the so-called zero-order deformation equation

$$(1-p)L[\varphi(\tau,p) - u_0(\tau)] = phH(\tau)N[\varphi(\tau,p)] \quad (12)$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $H(\tau) \neq 0$ is an auxiliary function, $L$ is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, $\varphi(\tau,p)$ is unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, it holds $\varphi(\tau,0) = u_0(\tau)$ and $\varphi(\tau,1) = u(\tau)$ respectively. Thus, as p increases from 0 to 1, the solution $\varphi(\tau,p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\varphi(\tau,p)$ in Taylor series with respect to p, we have

$$\varphi(\tau,p) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau) p^m \quad (13)$$

where

$$u_m(\tau) = \frac{1}{m!} \left. \frac{\partial^m \varphi(\tau,p)}{\partial p^m} \right|_{p=0} \quad (14)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (13) converges at $p = 1$, then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{\infty} u_m(\tau) \quad (15)$$

which must be one of solutions of original nonlinear equation, as proved by Liao [25]. As $h = -1$ and $H(\tau) = 1$, Eq. (12) becomes

$$(1-p)L[\varphi(\tau,p) - u_0(\tau)] + pN[\varphi(\tau,p)] = 0 \quad (16)$$

Which is used mostly in the homotopy perturbation method, where as the solution obtained directly, without using Taylor series [26, 27].

According to the definition (14), the governing equation can be deduced from the zero-order deformation equation (12). Define the vector

$$\overrightarrow{u_n} = \{u_0(\tau), u_1(\tau), \cdots, u_n(\tau)\}$$

Differentiating equation (12) m times with respect to the embedding parameter $p$ and then setting $p = 0$ and finally dividing them by m!, we have the so-called mth-order deformation equation

$$L[u_m(\tau) - \chi_m u_{m-1}(\tau)] = hH(\tau) R_m(\overrightarrow{u_{m-1}}) \quad (17)$$

where

$$R_m(\overrightarrow{u_{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(\tau,p)]}{\partial p^{m-1}}\bigg|_{p=0} \quad (18)$$

And

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} \quad (19)$$

It should be emphasized that $u_m(\tau)$ for $m \ge 1$ is governed by the linear equation (17) with the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

**Homotopy Analysis Solution**

In order to solve dimensionless equation we select

$$\theta_0(\eta,\xi) = e^{-\eta} \quad (20)$$

as initial approximation of $\theta$, Besides to choose

$$L[\phi(\xi,\eta,q)] = \frac{\partial^2\phi}{\partial\eta^2} - \phi \quad (21)$$

as the auxiliary linear operator with the property

$$L[C_1 \exp(-\eta) + C_2 \exp(\eta)] = 0 \quad (22)$$

$\theta_0(\eta,\xi)$ Satisfy the linear operator and the corresponding boundary conditions.

**Zero-order deformation equation**

Based on dimensionless equation we are define the nonlinear operator

$$\mathrm{N}[\phi(\eta,\xi,p),\varphi(\eta,\xi,p)] = (\phi'')^2 + P_r\xi\varphi\phi' + \frac{1}{2}P_r\xi(1-\xi)\phi' - P_r\xi(1-\xi)\frac{\partial\phi}{\partial\xi} - P_r\xi(2\varphi' - \alpha)\phi$$
$$+ P_r E_c\left[(\varphi'')^2 + K\varphi''(\varphi'\varphi'' - \varphi\varphi''')\right.$$
$$\left.+ M_n\xi(\varphi')^2\right] \quad (23)$$

Let $h$ denote the non-zero auxiliary parameter. We construct the zero-order deformation equation

$$(1-p)\,L[\phi(\eta,\xi,p) - \theta_0(\eta,\xi)] = p\,h\,H(r,t)\,\mathrm{N}[\phi(\eta,\xi,p)] \quad (24)$$

Subject to the boundary conditions

$$\phi(\eta,\xi)\big|_{\eta=0} = 1, \quad \phi(\eta,\xi)\big|_{\eta\to\infty} \to 0 \quad (25)$$

Where $\phi(\eta,\xi,p)$ is the solution which depends not only upon $\theta_0(\eta,\xi), L, H(\eta,\xi)$ and $h$ but on the embedding parameter $p \in [0,1]$. When $p = 0$ and $p = 1$ the zero-order deformation equation have the solutions $\phi(\eta,\xi,0) = \theta_0(\eta,\xi)$ and $\phi(\eta,\xi,1) = \theta(\eta,\xi)$ respectively. Thus $p$ increases from 0 to 1, $\phi(\eta,\xi,p)$ vary from the initial guesses $\theta_0(\eta,\xi)$ to the solution $\theta(\eta,\xi)$ of the considered unsteady problem. So expanding $\phi(\eta,\xi,p)$ in Taylor's series with respect to the embedding parameter $p$, we have

$$\phi(\eta,\xi,p) = \phi(\eta,\xi,0) + \sum_{m=1}^{\infty}\frac{1}{m!}\left[\frac{\partial^m\phi}{\partial p^m}\bigg|_{p=0}\right]p^m \; \phi(\eta,\xi,p) = \theta_0 + \sum_{m=1}^{\infty}\theta_m(\eta,\xi)\,p^m \quad (26)$$

Assuming that $h$ properly chosen so that the solution sequence of (24) convergent at $p = 1$, we have, using the boundary conditions (25), the solution series

$$\theta(\eta,\xi)=\theta_0(\eta,\xi)+\sum_{m=1}^{\infty}\theta_m(\eta,\xi) \qquad (27)$$

**High-order deformation equation**

For simplicity, we define the vector

$$\overrightarrow{\theta_m}=\{\theta_0,\theta_1,\theta_2,...,\theta_m\} \qquad (28)$$

Differentiating the zero-order deformation equation $m$ times with respect to $p$, then setting $p=0$, and finally divided it by $m!$, we obtain the m-th order deformation equation

$$L[\theta_m(\eta,\xi)-\chi_m\theta_{m-1}(\eta,\xi)]=hR_m(\vec{\theta}_{m-1}) \qquad (29)$$

Subject to the boundary conditions

$$\theta_m(\eta,\xi)\big|_{\eta=0}=\theta_m(\eta,\xi)\big|_{\eta\to\infty}=0 \qquad (30)$$

where

$$R_m(\vec{\theta}_{m-1})=\theta''_{m-1}+\frac{1}{2}P_r\eta(1-\xi)\theta'_{m-1}$$

$$-P_r(1-\xi)\xi\frac{\partial\theta_{m-1}}{\partial\xi}+\alpha P_r\xi\theta_{m-1}+\sum_{i=0}^{m-1}\Big[P_r\xi f_i\theta'_{m-1-i}$$

$$-2P_r\xi f'_i\theta_{m-1-i}+P_rE_c f''_i f''_{m-1-i}+Kf''_{m-1-i}\sum_{j=0}^{i}f''_{i-j}\ f'_j-Kf_{m-1-i}\sum_{j=0}^{i}f''_{i-j}\ f'_j$$

$$+M_n\xi\ f'_i\ f'_{m-1-i}\Big] \qquad (31)$$

And $$\chi_m=\begin{cases}0, & m=1\\ 1, & m>1\end{cases} \qquad (32)$$

In this way, it is easy to solve the linear equation (29) one after the other in the order $\mathrm{m}=1,2,3,...$ by means of the symbolic computation software such as Mathematica, and Maple.

**General solution**

The first step in the HAM is to find a set of base functions to express the sought solution of the problem under investigation. As mentioned by Liao[9a, 12a], a solution may be expressed with different base functions, among which some converge to the exact solution of the problem faster than others. Here, due to many boundary-layer flows decay exponentially at infinity, we assume that $\theta(\eta,\xi)$ can be expressed by a set of functions $\left\{\xi^{r+1-m}\eta^j\exp(-k\eta)\big|r\ge 0,n\ge 0,j\ge 0)\right\}$ in the form

$$\phi_m(\eta,\xi)=\sum_{k=0}^{m+1}\sum_{r=0}^{2m-1}\Psi_{m,r,k}(\eta)\ \ \xi^{r+1-m}\exp(-k\eta)$$

$$m\ge 1 \qquad (33)$$

where

$$\left.\begin{array}{l}\Psi_{m,r,0}=d^{0}_{m,r,0},\ m\geq 1,\ k=0\\ \Psi_{m,0,k}=d^{0}_{m,0,k},\ m\geq 1,\ k\geq 1,\ r=0\\ \Psi_{m,0,0}=d^{0}_{m,0,0},\ m\geq 1,\ k=r=0\\ \Psi_{m,r,k}(\eta)=\sum_{i=0}^{2m} d^{i}_{m,r,k}\ \eta^{i}\\ m\geq 1,\ 1\leq k\leq m+1,\ 1\leq r\leq 2m-1\end{array}\right\} \quad (34)$$

And from the initial guess $\theta_0(\eta,\xi)$ we have

$$d^{0}_{0,0,0}=0, \quad d^{0}_{0,0,1}=1 \quad (35)$$

For simplicity we will define $\Lambda^{i}_{m,r,k}$ as the following

$$\Psi_{m,r,k}(\eta)=\sum_{i=0}^{2m} \Lambda^{i}_{m,r,k}\ d^{i}_{m,r,k}\ \eta^{i} \quad (36)$$

where

$$\Lambda^{i}_{m,r,k}=\left\{\begin{array}{l}0,\ if\ \ m=k=r=0,\ \ i>0\\ 0,\ if\ \ m>0,\ k=0,\ \ i\geq 1\\ 0,\ if\ \ m>0,\ k\geq 1, r=0,\ \ i\geq 1\\ 0,\ if\ \ m>0,\ k=r=0,\ \ i\geq 1\\ 0,\ if\ \ k>m+1\\ 0,\ if\ \ \mathrm{i}>2m\\ 0,\ if\ \ \mathrm{r}>2\mathrm{m}-1\\ 1,\ \ \text{otherwies}\end{array}\right\} \quad (37)$$

From equation (36) and (37) we can get

$$\Psi'_{m,r,k}(\eta)=\sum_{i=1}^{2m} i\,\Lambda^{i}_{m,r,k}\ d^{i}_{m,r,k}\ \eta^{i-1}$$

This can be written (using definition of $\Lambda^{i}_{m,r,k}$) as:

$$\phi'_m(\eta,\xi)=\sum_{k=0}^{m+1}\sum_{r=0}^{2m-1}\sum_{i=0}^{2m} w^{i}_{m,r,k}\eta^{i}\ \ \xi^{r+1-m}\exp(-k\eta) \quad (38)$$

$$\phi''_m(\eta,\xi)=\sum_{k=0}^{m+1}\sum_{r=0}^{2m-1}\sum_{i=0}^{2m} g^{i}_{m,r,k}\eta^{i}\ \ \xi^{r+1-m}\exp(-k\eta) \quad (39)$$

$$\frac{\partial\phi_{m-1}}{\partial\xi}=\sum_{k=0}^{m}\sum_{r=0}^{2m-3}(r-m+2)\Psi^{i}_{m-1,r,k}\eta^{i}\xi^{r-m+1}\exp(-k\eta) \quad (40)$$

where

$$w^{i}_{m,r,k}=(i+1)\Lambda^{i+1}_{m,r,k}d^{i+1}_{m,r,k}-k\Lambda^{i}_{m,r,k}d^{i}_{m,r,k} \quad (41)$$

$$g^{i}_{m,r,k}=(i+1)(i+2)\Lambda^{i+2}_{m,r,k}d^{i+2}_{m,r,k}-2k(i+1)\Lambda^{i+1}_{m,r,k}d^{i+1}_{m,r,k}+k^{2}\Lambda^{i}_{m,r,k}d^{i}_{m,r,k} \quad (42)$$

Now

$$G_m=hR_m \quad (43)$$

$$= h\Big[\phi''_{m-1} + \frac{1}{2} P_r \eta (1-\xi)\phi'_{m-1}$$

$$- P_r(1-\xi)\xi \frac{\partial \phi_{m-1}}{\partial \xi} + \alpha P_r \xi \phi_{m-1} + \sum_{s=0}^{m-1}\big[P_r \xi f_s \phi'_{m-1-s} - 2P_r \xi f'_s \phi_{m-1-s} + P_r E_c f''_s f''_{m-1-s}$$

$$+ K f''_{m-1-s} \sum_{j=0}^{s} f''_{s-j}\ f'_j - K f_{m-1-s} \sum_{j=0}^{s} f''_{s-j}\ f'_j$$

$$+ M_n \xi\ f'_s\ f'_{m-1-s}\big] \qquad (44)$$

where $f$ can be calculated from Adel Rashed [24], we assume that $f(\eta,\xi)$ can be expressed by a set of functions $\left\{\xi^k \eta^j \exp(-n\gamma\eta) \middle| k \ge 0, n \ge 0, j \ge 0)\right\}$ in the form

$$f_m(\eta,\xi) = \sum_{k=0}^{m+1}\sum_{r=0}^{m} \psi_{m,r,k}(\eta)\ \ \xi^{-r} \exp(-k\gamma\eta) \qquad (45)$$

$m \ge 1$

where

$$\left.\begin{array}{l} \psi_{m,r,0} = b^0_{m,r,0},\ m \ge 1,\ k = 0 \\ \psi_{m,0,k} = b^0_{m,0,k},\ m \ge 1,\ k \ge 1,\ r = 0 \\ \psi_{m,0,0} = b^0_{m,0,0},\ m \ge 1,\ k = r = 0 \\ \psi_{m,r,k}(\eta) = \displaystyle\sum_{i=0}^{2m-(k+r)} \lambda^i_{m,r,k} b^i_{m,r,k} \eta^i \\ m \ge 1,\ 1 \le k \le m+1,\ 1 \le r \le m \end{array}\right\} \qquad (46)$$

where

$$\lambda^i_{m,r,k} = \left\{\begin{array}{ll} 0, & if \quad m = k = r = 0, \quad i > 0 \\ 0, & if \quad m > 0,\ k = 0, \quad i \ge 1 \\ 0, & if \quad m > 0,\ k \ge 1, r = 0, \quad i \ge 1 \\ 0, & if \quad m > 0,\ k = r = 0, \quad i \ge 1 \\ 0, & if \quad k > m+1 \\ 0, & if \quad r > m \\ 0, & if \quad \mathrm{i} > 2\mathrm{m} - (\mathrm{r}+\mathrm{k}) \\ 1, & \text{otherwies} \end{array}\right\} \qquad (47)$$

And

$$f'_m(\eta,\xi) = \sum_{k=0}^{m+1}\sum_{r=0}^{m}\sum_{i=0}^{2m-(k+r)} a^i_{m,r,k}\eta^i\ \ \xi^{-r}\exp(-k\gamma\eta) \qquad (48)$$

$$f''_m(\eta,\xi) = \sum_{k=0}^{m+1}\sum_{r=0}^{m}\sum_{i=0}^{2m-(k+r)} c^i_{m,r,k}\eta^i\ \ \xi^{-r}\exp(-k\gamma\eta) \qquad (49)$$

where

$$a^i_{m,r,k} = (i+1)\lambda^{i+1}_{m,r,k} b^{i+1}_{m,r,k} - (k\gamma)\lambda^i_{m,r,k} b^i_{m,r,k} \qquad (50)$$

$$c^i_{m,r,k} = (i+1)(i+2)\lambda^{i+2}_{m,r,k} b^{i+2}_{m,r,k}$$

$$-2(k\gamma)(i+1)\lambda^{i+1}_{m,r,k}b^{i+1}_{m,r,k}+(k\gamma)^2\lambda^{i}_{m,r,k}b^{i}_{m,r,k} \qquad (51)$$

Now

$$f'_s f'_{m-s-1}=\left[\sum_{k1=0}^{s+1}\sum_{r1=0}^{s}\sum_{i1=0}^{2s-(k1+r1)}a^{i1}_{s,r1,k1}\eta^{i1}\xi^{-r1}\exp(-k1\,\eta\gamma)\right]$$
$$\times\left[\sum_{k2=0}^{m-s}\sum_{r2=0}^{m-s-1}\sum_{i2=0}^{2(m-s-1)-(k2+r2)}a^{i2}_{m-s-1,r2,k2}\eta^{i2}\xi^{-r2}\exp(-k2\,\eta\gamma)\right] \qquad (52)$$

which can be rewrite as

$$=\left[\sum_{k1=0}^{s+1}\sum_{k2=0}^{m-s}\exp\left(-(k1+k2)\,\eta\gamma\right)\right]$$
$$\times\left[\sum_{r1=0}^{s}\sum_{r2=0}^{m-s-1}\xi^{-(r1+r2)}\right]\times\left[\sum_{i1=0}^{2s+(k1+r1)}\ \sum_{i2=0}^{2(m-s-1)-(k2+r2)}a^{i1}_{s,r1,k1}a^{i2}_{m-s-1,r2,k2}\eta^{i1+i2}\right]$$
$$=\left[\sum_{k=0}^{m+1}\exp(-k\eta\gamma)\sum_{k1=\max\{0,k-m+s\}}^{\min\{m+1,k\}}\right]\times\left[\sum_{r=0}^{m-1}\xi^{-r}\sum_{r1=\max\{0,r-m+s+1\}}^{\min\{m,r\}}\right]$$
$$\times\left[\sum_{i=0}^{2(m-1)+2k1-k+2r1-r}\eta^{i}\sum_{i1=\max\{0,i-2(m-s-1)+(k-k1+r-r1)\}}^{\min\{2s+k1+r1,i\}}a^{i1}_{s,r1,k1}a^{i-i1}_{m-s-1,r-r1,k-k1}\right]$$

This finally gives

$$f'_s f'_{m-s-1}=\sum_{k=0}^{m+1}\exp(-k\eta\gamma)\sum_{r=0}^{m-1}\xi^{-r} \qquad (53)$$
$$\times\sum_{i=0}^{2(m-1)+2k1-k+2r1-r}\eta^{i}\quad Q^{i}_{m,r,k}$$

By the similar way, we can have

$$f''_s f''_{m-s-1}=\sum_{w=0}^{m-1}\eta^{w}\sum_{n=0}^{m+1}\exp(-n\gamma\eta)\sum_{\mathrm{i=max\{0,w\text{-}2(m\text{-}s\text{-}1)+k1+r1\}}}^{\mathrm{min\{2s\text{-}k\text{-}r,w\}}}$$
$$\times\sum_{\mathrm{k=max\{0,n\text{-}m+s\}}}^{\mathrm{min\{s+1,n\}}}\ \sum_{\mathrm{r=0}}^{\mathrm{s}}\sum_{\mathrm{r1=0}}^{\mathrm{m\text{-}s\text{-}1}}c^{i}_{m,r,k}\xi^{-r-r1} \qquad (54)$$

$$f''_{s-j}f'_j=\sum_{n=0}^{s+2}\exp(-n\gamma\eta)\sum_{w=0}^{2s-(k+k1+r+r1)}\eta^{w}\sum_{\mathrm{k1=max\{0,n\text{-}j\text{-}1\}}}^{\mathrm{min\{s\text{-}j+1,n\}}}$$
$$\times\sum_{i1=\max\{0,w-2s+2j+k+r\}}^{\min\{2j-k1-r1,w\}}\ \sum_{\mathrm{r=0}}^{\mathrm{s\text{-}j}}\sum_{\mathrm{r1=0}}^{\mathrm{j}}c^{i}_{s-j,r,k}a^{i1}_{j,r1,k1}\xi^{-r-r1} \qquad (55)$$

$$f''_{m-s-1}\sum_{j=0}^{s}f''_{s-j}f'_j=\sum_{j=0}^{s}\ \sum_{n1=0}^{m+2}\exp(-n1\gamma\eta)$$
$$\times\sum_{w2=0}^{2(m-1-k-r)-(k1+r1)}\eta^{w2}\Pi^{i}_{m,r,k} \qquad (56)$$

$$f_{m-1-s}\sum_{j=0}^{s}f''_{s-j}f'_j=\sum_{j=0}^{s}\ \sum_{n2=0}^{m+2}\exp(-n2\gamma\eta)$$
$$\times\sum_{w2=0}^{2m-2-(k+k1+k2+r+r1+r2)}\eta^{w2}\mathrm{E}^{i}_{m,r,k} \qquad (57)$$

Substitution equations (48-49) and (53-57) into (44), we obtain

$$G_m = h\Bigg[\sum_{k=0}^{m}\sum_{r=0}^{2m-3}\sum_{i=0}^{2m-2} g_{m-1,r,k}^{i}\eta^{i}\xi^{r-m+2}\exp(-k\eta)$$
$$+\frac{1}{2}P_r\eta(1-\xi)\sum_{k=0}^{m}\sum_{r=0}^{2m-3}\sum_{i=0}^{2m-2} w_{m-1,r,k}^{i}\eta^{i}\xi^{r-m+2}\exp(-k\eta)$$
$$-P_r\xi(1-\xi)\sum_{k=0}^{m-1}\sum_{r=0}^{2m-5}\sum_{i=0}^{2m-4}\Lambda_{m-2,r,k}^{i}d_{m-2,r,k}^{i}\eta^{i} \times(r-m+3)\xi^{r-m+2}\exp(-k\eta) \quad +\alpha P_r\xi\sum_{k=0}^{m}\sum_{r=0}^{2m-3}\sum_{i=0}^{2m-2}\Lambda_{m-1,r,k}^{i}d_{m-1,r,k}^{i}\eta^{i}\xi^{r-m+2}\exp(-k\eta)$$
$$+\sum_{s=0}^{m-1}\Bigg[P_r\xi f_s\sum_{k=0}^{m-s}\sum_{r=0}^{2m-2s-3}\sum_{i=0}^{2(m-1-s)} w_{m-1-s,r,k}^{i}\eta^{i}\xi^{r-m-s+2}\exp(-k\eta) \quad -2P_r\xi f_s'\sum_{k=0}^{m-s}\sum_{r=0}^{2m-3-2s} \times\sum_{i=0}^{2(m-1-s)}\Lambda_{m-1-s}^{i}d_{m-1-s,r,k}^{i}\eta^{i}\xi^{r-m-s+2}\exp(-k\eta)$$
$$+P_rE_c f_s'' f_{m-s-1}'' + K\ f_{m-s-1}''\sum_{j=0}^{s} f_{s-j}'' f_j'$$
$$-Kf_{m-s-1}\sum_{j=0}^{s} f_{s-j}'' f_j' \ + M_n\xi f_s'\ f_{m-s-1}\Bigg] \qquad (58)$$

which can be rewritten as:

$$G_m = \sum_{k=0}^{m}\exp(-k\eta)\sum_{i=0}^{2m-2}\eta^{i}A_{m,r,k}^{i}$$
$$+\sum_{k=0}^{m}\exp(-k\eta)\sum_{i=0}^{2m-2}\eta^{i+1}B_{m,r,k}^{i}$$
$$+\sum_{k=0}^{m-1}\exp(-k\eta)\sum_{i=0}^{2m-4}\eta^{i}D_{m,r,k}^{i} + \sum_{k=0}^{m}\exp(-k\eta)\sum_{i=0}^{2m-2}\eta^{i}F_{m,r,k}^{i}$$
$$+\sum_{s=0}^{m-1}\sum_{k1=0}^{s+1}\sum_{k=0}^{m-s}\exp(-\eta(\gamma k1+k))\sum_{w=0}^{2m-2-k1-r1}\eta^{w}\Delta_{m,r,k}^{i}$$
$$+\sum_{s=0}^{m-1}\sum_{k1=0}^{s+1}\sum_{k=0}^{m-s}\ \exp(-\eta(k1\gamma+k))\sum_{w=0}^{2m-2-k1-r1}\eta^{w}\Omega_{m,r,k}^{i}$$
$$+\sum_{s=0}^{m-1}\sum_{w=0}^{2m-2-(k+k1+r+r1)}\eta^{w}\sum_{n=0}^{m+1}\ \exp(-n\gamma\eta)\Gamma_{m,r,k}^{i}$$
$$+\sum_{s=0}^{m-1}hK\sum_{j=0}^{s}\sum_{n1=0}^{m+2}\exp\big(-n1\gamma\eta\big) \times\sum_{w2=0}^{2(m-1-k-r)-(k1+r1)}\eta^{w2}\Pi_{m,r,k}^{i}$$
$$-\sum_{s=0}^{m-1}hK\sum_{j=0}^{s}\sum_{n2=0}^{m+2}\exp(-n2\gamma\eta) \times\sum_{w2=0}^{2m-2-(k+k1+k2+r+r1+r2)}\eta^{w2}\mathrm{E}_{m,r,k}^{i}$$
$$+\sum_{s=0}^{m-1}hM_n\sum_{k=0}^{m+1}\exp(-k\eta\gamma)\sum_{r=0}^{m-1}\xi^{-r}$$

$$\times \sum_{i=0}^{2(m-1)+2k1-k+2r1-r} \eta^{i} \quad Q^{i}_{m,r,k} \qquad (59)$$

where

$$A^{i}_{m,r,k} = h\sum_{r=0}^{2m-3} c^{i}_{m-1,r,k}\xi^{r-m+2} \qquad (60)$$

$$B^{i}_{m,r,k} = \frac{h}{2}P_r(1-\xi)\sum_{r=0}^{2m-3} w^{i}_{m-1,r,k}\xi^{r-m+2} \qquad (61)$$

$$D^{i}_{m,r,k} = -hP_r\xi(1-\xi)\sum_{r=0}^{2m-5} d^{i}_{m-2,r,k}(r-m+3)\xi^{r-m+2} \qquad (62)$$

$$F^{i}_{m,r,k} = h\alpha P_r\sum_{r=0}^{2m-3} d^{i}_{m-1,r,k}\xi^{r-m+3} \qquad (63)$$

$$\Delta^{i}_{m,r,k} = hP_r \sum_{i=\max\{0,w-2s+k1+r1\}}^{\min\{2(m-1-s),w\}} \sum_{r1=0}^{s}$$

$$\times \sum_{\mathrm{r=0}}^{\mathrm{2(m\text{-}s)\text{-}3}} w^{i}_{m-1-s,r,k} b^{i1}_{m1,r1,k1}\xi^{r-r1-m-s+3} \qquad (64)$$

$$\Omega^{i}_{m,r,k} = -2hP_r \sum_{i=\max\{0,w-2s+k1+r1\}}^{\min\{m-1-s,w\}} \sum_{r1=0}^{s}$$

$$\times \sum_{\mathrm{r=0}}^{\mathrm{2m\text{-}3\text{-}2s}} a^{i1}_{s,r1,k1}\Lambda^{i}_{m-1-s,r,k} d^{i}_{m-1-s,r,k}\xi^{r-r1-m-s+3} \qquad (65)$$

$$\Gamma^{i}_{m,r,k} = hP_rE_c \sum_{\mathrm{i=\max\{0,w\text{-}2(m\text{-}s\text{-}1)+k1+r1\}}}^{\mathrm{\min\{2s\text{-}k\text{-}r,w\}}}$$

$$\times \sum_{\mathrm{k=\max\{0,n\text{-}m+s\}}}^{\mathrm{\min\{s+1,n\}}} \sum_{\mathrm{r=0}}^{\mathrm{s}} \sum_{\mathrm{r1=0}}^{\mathrm{m\text{-}s\text{-}1}} c^{i}_{m,r,k}\xi^{-r-r1}\Gamma^{i}_{m,r,k} \qquad (67)$$

$$\Pi^{i}_{m,r,k} = \sum_{w=\max\{0,w2-2(m-s-1)+(k+r)\}}^{\min\{2s-(k+k1+r+r1,w2)} \sum_{n=\max\{0,n2-m+s\}}^{\min\{s+2,n1\}} \sum_{\mathrm{k1=\max\{0,n\text{-}j\text{-}1\}}}^{\mathrm{\min\{s\text{-}j+1,n\}}} \times \sum_{i1=\max\{0,w-2s+2j+k+r\}}^{\min\{2j-k1-r1,w\}} \sum_{\mathrm{r=0}}^{\mathrm{s\text{-}j}} \sum_{\mathrm{r1=0}}^{\mathrm{j}}$$

$$\times \sum_{r2=0}^{m-s-1} c^{i}_{s-j,r,k} c^{i2}_{m-s-1,r2,k2} a^{i1}_{j,r1,k1}\xi^{-r-r1-r2} \qquad (68)$$

$$\mathrm{E}^{i}_{m,r,k} = \sum_{n=\max\{0,n2-m+s\}}^{\min\{s+2,n2\}} \sum_{r2=0}^{m-1-s} \sum_{w=\max\{0,w2-2(m-1-s)+(k2+r2)\}}^{\min\{2s-(k+k1+r+r1),w2\}}$$

$$\left(\lambda^{i2}_{m-1-s,r2,k2} b^{i2}_{m-1-s,r2,k2}\xi^{-r2}\right)$$

$$\sum_{\mathrm{k1=\max\{0,n\text{-}j\text{-}1\}}}^{\mathrm{\min\{s\text{-}j+1,n\}}} \sum_{i1=\max\{0,w-2s+2j+k+r\}}^{\min\{2j-k1-r1,w\}} \sum_{\mathrm{r=0}}^{\mathrm{s\text{-}j}} \sum_{\mathrm{r1=0}}^{\mathrm{j}} c^{i}_{s-j,r,k} a^{i1}_{j,r1,k1}\xi^{-r-r1} \qquad (69)$$

$$Q^{i}_{m,r,k} = \sum_{k1=\max\{0,k-m+s\}}^{\min\{m+1,k\}} \sum_{r1=\max\{0,r-m+s+1\}}^{\min\{m,r\}}$$

$$\sum_{i1=\max\{0,i-2(m-s-1)+(k-k1+r-r1)\}}^{\min\{2s+k1+r1,i\}} a_{s,r1,k1}^{i1} a_{m-s-1,r-r1,k-k1}^{i-i1} \qquad (70)$$

In order to obtain the general solution of equation (59), we will start with an ordinary differential equation look like the mth order equation

$$y''(\eta) - y(\eta) = \eta^q \xi^r \exp(-k\eta) \qquad (71)$$

To find the particular solution of this equation we use the formula

$$\int \eta^q \exp(-k\eta) d\eta = -\exp(-k\eta) \sum_{j=0}^{q} \frac{q!}{j!} \frac{\eta^j}{k^{q-j+1}} \qquad (72)$$

The particular solution of (71) is

$$y_p = \exp(-k\eta)\xi^r \sum_{j=0}^{q} \mu_{k,j}^q \, \eta^j \qquad (73)$$

where

$$\mu_{k,j}^q = \frac{1}{2}\frac{q!}{j!}\left(\frac{1}{(k-1)^{q-j+1}} - \frac{1}{(k+1)^{q-j+1}}\right) \quad (74)$$

Applying the solution (73) on the differential equation (59), we obtain the following general solution

$$\begin{aligned}
&\phi_m - \chi_m \phi_{m-1} = \\
&\sum_{k=0}^{m} \exp(-k\eta) \sum_{i=0}^{2m-2} \sum_{j=0}^{i} \eta^j \mu_{k,j}^i A_{m,r,k}^i \\
&+ \sum_{k=0}^{m} \exp(-k\eta) \sum_{i=0}^{2m-2} \sum_{j=0}^{i+1} \eta^{j+1} \mu_{k,j}^{i+1} B_{m,r,k}^i \\
&+ \sum_{k=0}^{m-1} \exp(-k\eta) \sum_{i=0}^{2m-4} \sum_{j=0}^{i} \eta^j \mu_{k,j}^i D_{m,r,k}^i \\
&+ \sum_{k=0}^{m} \exp(-k\eta) \sum_{i=0}^{2m-2} \sum_{j=0}^{i} \eta^j \mu_{k,j}^i F_{m,r,k}^i \\
&+ \sum_{s=0}^{m-1} \sum_{k1=0}^{s+1} \sum_{k=0}^{m-s} \exp(-\eta(\gamma k1 + k)) \\
&\times \sum_{w=0}^{2m-2-k1-r1} \sum_{j=0}^{w} \mu_{\gamma k1+k,j}^w \eta^j \Delta_{m,r,k}^i \\
&+ \sum_{s=0}^{m-1} \sum_{k1=0}^{s+1} \sum_{k=0}^{m-s} \exp(-\eta(k1\gamma + k)) \\
&\times \sum_{w=0}^{2m-2-k1-r1} \sum_{j=0}^{w} \eta^j \mu_{k1\gamma+k,j}^w \Omega_{m,r,k}^i \\
&+ \sum_{s=0}^{m-1} \sum_{w=0}^{2m-2-(k+k1+r+r1)} \sum_{n=0}^{m+1} \exp(-n\gamma\eta) \\
&\times \sum_{j=0}^{w} \mu_{n\gamma,j}^w \eta^j \Gamma_{m,r,k}^i
\end{aligned}$$

$$+\sum_{s=0}^{m-1} hK \sum_{j=0}^{s} \sum_{n1=0}^{m+2} \exp\left(-n1\gamma\eta\right)$$
$$\times \sum_{w2=0}^{2(m-1-k-r)-(k1+r1)} \sum_{j=0}^{w2} \mu_{n1\gamma,j}^{j} \eta^{j} \Pi_{m,r,k}^{i}$$
$$-\sum_{s=0}^{m-1} hK \sum_{j=0}^{s} \sum_{n2=0}^{m+2} \exp(-n2\gamma\eta)$$
$$\times \sum_{w2=0}^{2m-2-(k+k1+k2+r+r1+r2)} \sum_{j=0}^{w2} \mu_{n2\gamma,j}^{w2} \eta^{j} \mathrm{E}_{m,r,k}^{i}$$
$$+\sum_{s=0}^{m-1} hM_{n} \sum_{k=0}^{m+1} \exp(-k\eta\gamma) \sum_{r=0}^{m-1} \xi^{-r}$$
$$\times \sum_{i=0}^{2(m-1)+2k1-k+2r1-r} \sum_{j=0}^{i} \mu_{k\gamma,j}^{i} \eta^{j} Q_{m,r,k}^{i}$$
$$+C_1^m \exp(\eta) + \mathrm{C}_2^{\mathrm{m}} \exp(-\eta) \qquad (75)$$

**Convergence of the solution**

From the homotopy analysis method, as long as the series solution is convergent, it should converge to one of the solutions of original equation. The convergence and rate of the approximation for the HAM strongly depend upon the value of the auxiliary parameter h, as pointed out by Liao [25, 28]. So we have a family of solution expressions in the auxiliary parameter h, and the physical quantities also depend upon h. So, regarding h as an independent variable, it is easy to plot curves of these kinds of quantities versus h by means of the so-called h-curve. If the solution is unique, all of them converge to the same value and therefore there exists a horizontal line segment in the h-curve, and if we set h any value in the horizontal line segment we quite sure that the corresponding solution series converge.

Fig (1) portrays the h-curve of $\theta_{\eta\eta}(0,\xi)$. The range for admissible value of h is $-1.2 \le h \le 1.0$ we see that series converges in the whole region of $\eta$ when $h = -0.3$, this value of $h$ lie in the admissible range of h.

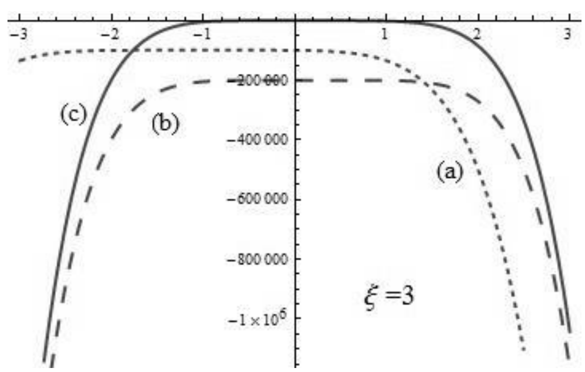


Fig. (1) The h-curves of $\theta_{\eta\eta}(0,\xi)$ obtained by the 6th-order approximation of the HAM, when $\xi = 3$

(a) $\mathrm{M_n} = K = 0,\ \alpha = 0, \mathrm{Pr} = 1, \mathrm{Ec} = 0.01$

(b) $\mathrm{M_n} = K = 1,\ \alpha = 0, \mathrm{Pr} = 1, \mathrm{Ec} = 0.2$

(c) $M_n = K = 0,\ \alpha = 0.1, Pr = 10, Ec = 0.2$

**Result and conclusion**

We have studied the effects of different dimensionless numbers that governing the energy equation.

Figures (2-7), illustrate the effect of time $\xi$, internal heat parameter $\alpha$, Eckert number $E_c$, Prandtl number $P_r$, viscoelastic parameter $K$ and magnetic parameter $M_n$ respectively.

All the results are made using Wolfram Mathematica 8 package.

**Effects of time $\xi$ :**

To study the effects of time $\xi$ on the energy equation, we keep magnetic parameter $M_n$, viscoelastic parameter $K$ ,internal heat parameter $\alpha$, Prandtl number $P_r$ and Eckert number $E_c$ fixed at 1, 1, 0, 1, 0.01 respectively, and we give time $\xi$ for values 1, 1.25, 1.5, 2, 2.5, 3, 3.5 the following result is made:

As time $\xi$ increases, there is small decreasing in the temperature. See Fig. (2).

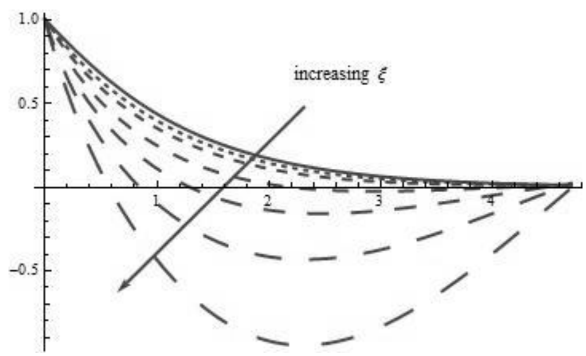


Fig.(2) The approximation solution of $\theta(\eta,\xi)$ obtained by the 6th-order of the HAM, when $M_n = 1,\ K = 1,\ \alpha = 0,\ Pr = 1,\ Ec = 0.01$, and $\xi = 1,\ 1.25,\ 1.5,\ 2,\ 2.5,\ 3,\ 3.5$

**Effects of internal heat parameter $\alpha$ :**

To study the effects of time $\alpha$ on the energy equation, we keep magnetic parameter $M_n$, viscoelastic parameter $K$, time $\xi$, Prandtl number $P_r$ and Eckert number $E_c$ fixed at 0, 0, $\pi/4$, 1, 0.01 respectively, and we give internal heat parameter $\alpha$ for values -1, -0.5, -0.1, 0, 0.1, 0.5, 1 the following result is made:

As internal heat parameter $\alpha$ increases, there is small increasing in the temperature. See Fig. (3).

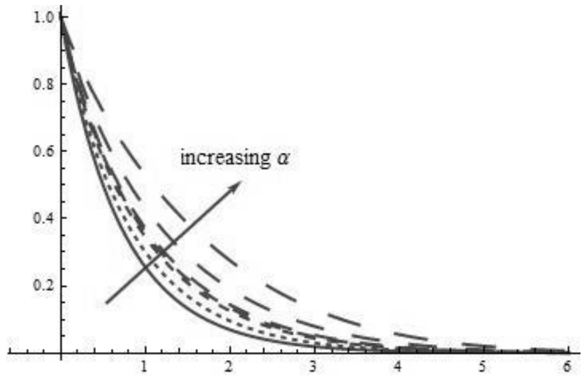


Fig.(3) The approximation solution of $\theta(\eta,\xi)$ obtained by the 6th-order of the HAM, when $M_n = 0$, $K = 0$, $\xi = \pi/4$, $Pr = 1$, $Ec = 0.01$, and $\alpha = -1, -0.5, -0.1, 0, 0.1, 0.5, 1$

**Effects of parameter** $Ec$ **:**

To study the effects of Eckert number $E_c$ on the energy equation, we keep magnetic parameter $M_n$, viscoelastic parameter $K$, time $\xi$, Prandtl number $P_r$ and internal heat parameter $\alpha$ fixed at 1, 0, 1, 1, 0 respectively, and we give Eckert number $E_c$ for values -0.5, -0.2, -0.1, 0, 0.1, 0.2, 0.5, 1 the following result is made:

As Eckert number $E_c$ increases, there is small increasing in the temperature. See Fig. (4).

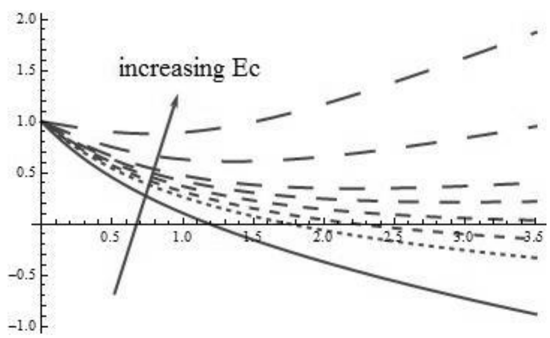


Fig.(4) The approximation solution of $\theta(\eta,\xi)$ obtained by the 6th-order of the HAM, when $M_n = 1$, $K = 0$, $\xi = 1$, $Pr = 1$, $\alpha = 0$, and $Ec = -0.5, -0.2, -0.1, 0, 0.1, 0.2, 0.5, 1$

**Effects of parameter** $Pr$ **:**

To study the effects of Prandtl number $P_r$ on the energy equation, we keep magnetic parameter $M_n$, viscoelastic parameter $K$, time $\xi$, Eckert number $E_c$ and internal heat parameter $\alpha$ fixed at 10, 1, 1, 0.01, 0.1 respectively, and we give Prandtl number $P_r$ for values 1, 2, 3, 4 the following result is made:

As Prandtl number $P_r$ increases, there is small change in the temperature. See Fig. (5).

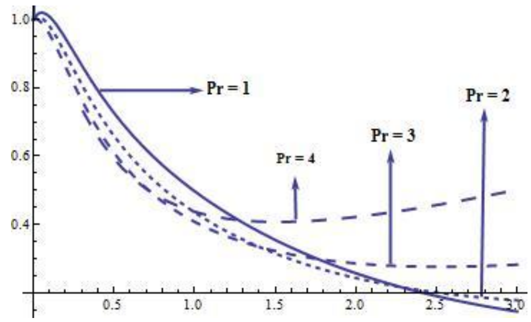


Fig.(5) The approximation solution of $\theta(\eta,\xi)$ obtained by the 6th-order of the HAM, when $M_n = 10$, $K = 1$, $\xi = 1$, $Ec = 0.01$, $\alpha = 0.1$, and $Pr = 1, 2, 3, 4$

these values fit with many materials, such such as polymer solutions or melts, drilling mud, certain oils and greases and many other emulsions.

**Effects of viscoelastic parameter $K$ :**

To study the effects of viscoelastic parameter $K$ on the energy equation, we keep magnetic parameter $M_n$, time $\xi$ , internal heat parameter $\alpha$, Prandtl number $P_r$ and Eckert number $E_c$ fixed at 0, 1, 0, 1, 0.01 respectively, and we give viscoelastic parameter $K$ for values 0, 0.5, 1, 2, 3 the following result is made:

As viscoelastic parameter $K$ increases, there is small decreasing in the temperature. See Fig. (6).

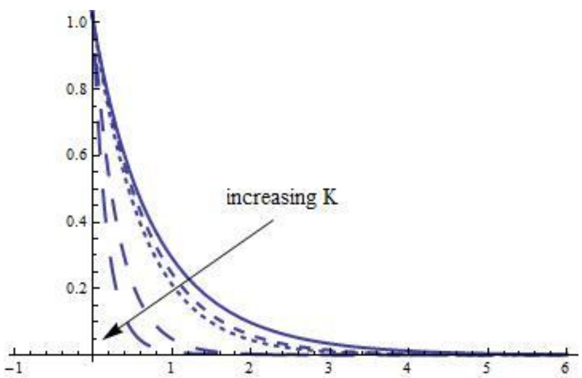


Fig.(6) The approximation solution of $\theta(\eta,\xi)$ obtained by the 6th-order of the HAM, when $M_n = 0$, $\xi = 1$, $Pr = 1$, $\alpha = 0$, $E_c = 0.01$ and $K = 0,\ 0.5,\ 1,\ 2,\ 3$

**Effects of magnetic parameter $M_n$ :**

To study the effects of magnetic parameter $M_n$ on the energy equation, we keep viscoelastic parameter $K$, time $\xi$ , internal heat parameter $\alpha$, Prandtl number $P_r$ and Eckert number $E_c$ fixed at 0, 1, 0, 1, 0.01 respectively, and we give magnetic parameter $M_n$ for values 0, 0.5, 1, 2, 3, 4, 5 the following result is made:

As magnetic parameter $M_n$ increases, there is small increasing in the temperature. See Fig. (7).

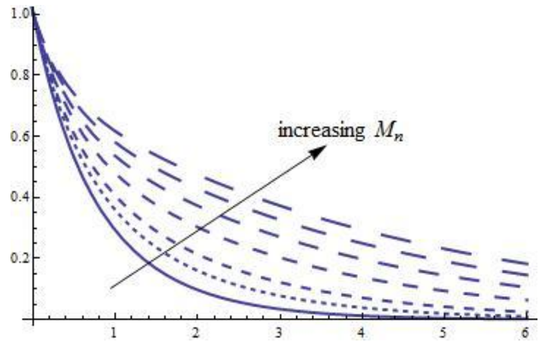


Fig.(7) The approximation solution of $\theta(\eta,\xi)$ obtained by the 6th-order of the HAM, when $K=0$, $\xi=1$, $\Pr=1$, $\alpha=0$, $E_c=0.01$ and $M_n=0, 0.5, 1, 2, 3, 4, 5$

**Note:** in fig. (6) and (7), when $\xi=1$ we recover the steady state as obtained by [19] fig. (3a), (3b) p. 4433.

**References**

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