On Rings whose Simple Singular R-Modules are GP-Injective

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الملخص

في هذا البحث ندرس الحلقات التي تكون مقاساتها اليمنى البسيطة المنفردة غامرة من النمط – GP . برهنا أنه في حلقة كوازي- ديو التي تكون مقاساتها اليمنى البسيطة المنفــردة غامرة من النمط-GP فأن كل مثالي أيمن مختزل يكون قابلا للجمع المباشر. كما بينا أن الحلقة شبه ألا بدالية تكون حلقة منتظمة ضعيفة مختزلة إذا كانت مقاساتها اليمنى البــسيطة المنفــردة غامرة من النمط- GP .

ABSTRACT

In this work we give a characterization of rings whose simple singular right R-modules are Gp-injective .We prove that if R is a quasi-duo ring whose simple singular right R-modules are Gp-injective, then any reduced right ideal of R is a direct summand. We also consider that a zero commutative ring with every simple singular left R-module is Gp-injective

1. Introduction:

Throughout this paper, R denotes an associative ring with identity, and all modules are unitary right R- modules. Recall that: (1) A right R-module M is called general right principally injective (briefly right Gp-injective) if for any $0 \neq a \in R$ there exists a positive integer n, such that $a^n \neq 0$ and any right R-homomorphism of a^nR into M extends to one of R into M;(2) R is called reduced if R has no non-zero nilpotent elements; (3) R is right (left) quasi-duo ring if every maximal right (left) ideal of R is an ideal of R; (4) A ring R is called semi-prime if 0 is the only nilpotent ideal ; (5) for any element a in R we define a right annihilator of a by $r(a)=\{x \in R:ax=0\}$ and a left annihilator of a, l(a) is similarly defined.

2. Rings whose simple singular modules are GP-Injective:

In this section, we study rings whose simple singular right R-modules are Gp-injective.

We begin this section with the following result.

<u>Proposition 2-1:</u>

Let R be a qusi-duo ring, with every simple singular right R-modules is Gp-injective. Then any reduced right ideal of R is a direct summand.

<u>Proof:</u> Let I=aR be a reduced principal right ideal of R. We shall show that aR+r(a)=R. if not, there exists a maximal right ideal M of R such that $aR+r(a) \subseteq M$ .Now, M is essential right ideal of R, if not, then there exists a non-zero right ideal L of R such that $M \cap L=0$. Then $aRL \subseteq ML \subseteq M \cap L=0$, implies that $L \subseteq r(a) \subseteq M$, so $M \cap L=L=0$, and this is a contradiction.

So M must be essential right ideal of R. Therefore R/M is Gp-injective. Then there exists a positive integer n such that any R-homomorphism of a^nR into R/M extends to one of R into R/M . let $f:a^nR \rightarrow R/M$ be defined by $f(a^nr)=r+M$. f is a well-defined R-homomorphism. Indeed , let $r_1,r_2 \in R$ such that $a^nr_1=a^nr_2$. Then $a^nr_1-a^nr_2=0$, implies that $a^n(r_1-r_2)=0$, so $r_1-r_2 \in r(a^n)$, since I is reduced. Therefore $r(a^n)=r(a)$,this implies that $r_1-r_2 \in r(a) \subseteq M$. Hence, $r_1+M=r_2+M$. Now R/M is Gp-injective,so there exists $c \in R$ such that $1+M=f(a^n) =ca^n+M$. Hence , $1-ca^n \in M$, since $a^n \in M$ and R is a quasi-duo ring , then $ca^n \in M$ and so $1 \in M$. This contradicts $M \neq R$.

Therefore aR+r(a)=R. In particular ar+c=1, for some $r \in R$ and $c \in r(a)$, whence $a^2r=a$. if we set $d=ar^2 \in I$, then $a=a^2d$. clearly $(a-ada)^2=0$, since I is reduced, thus a =ada , and hence I=eR, where e=ad is an idempotent element. Thus I is a direct summand.

Proposition 2-2:

Let R be a semi-prime ring with every simple singular right R-module is Gp-injective . Then every right ideal of R is an idempotent .

Proof: For any right ideal I of R, suppose there exists an element b in I , such that $b\notin I^2$. Then $bR\neq(bR)^2$. Since R is a semi-prime ring, then $(bR)^2$ is essential in bR. By zorn's lemma , the set of right ideals J such that $(bR)^2\subseteq J\subset bR$ has a maximal member L. Then bR/L is a simple singular, and therefore is Gp-injective. Now, let $f:bR\to bR/L$ is the canonical homomorphism defined by f(br)=br+L for all ring R , since bR/L is Gp-injective, so there exists $c\in R$, such that f(br)=(bc+L)br. Then f(b)=(bc+L)b=b+L , which implies that b+L=bcb+L. Hence; $b-bcb\in L$, whence it follows that $b\in L$. Thus $bR\subset L$ and this is a contradiction. Therefore $I=I^2$.

3-Zero Commutative Rings

In this section we introduce the notion of a zero commutative ring in order to study the connection between rings whose simple singular right R-modules are Gp-injective and other rings.

Definition3-1:

A ring R is called zero commutative (briefly ZC)if for $a,b\in R$, ab=0 if ba=0.

We shall begin this section with the following result.

Lemma 3-2:

Let R be a ZC ring. Then RaR+l(a) is an essential left ideal of R.

Proof: Given $a\in R$, assume that $[RaR+l(a)]\cap I=0$, where I is a right ideal of R. Then $aI\subseteq I\cap RaR=0$, so $I\subseteq r(a)\subseteq l(a)$. Hence, I=0; where RaR+l(a)is an essential left ideal of R

Lemma 3-3:

Let R be a ZC ring with every simple singular left R-module is Gp-injective, then R is reduced.

Proof: Let $a^2=0$. suppose that $a\neq0$. By lemma (3-2), l(a)is an essential left ideal of R. since $a\neq0$, $l(a)\neq R$. Thus, there exists a maximal essential left ideal M of R containing L(a), therefore R/M is Gp-injective. So any R-homomorphism of Ra intoR/M extends to one of R into R/M. Let $f:Ra\to R/M$ be defined by f(ra)=r+M. Clearly, f is a well-defined R-homomorphism . Thus 1+M=f(a)=ac+M. Hence, 1-ac$\in$M and so 1$\in$M, which is a contradiction. Hence a=0, and so R is reduced.

Definition3-4:

A ring R is said to be right weakly regular if for all a in R , there exists b in RaR such that a=ab .
Now, we give the main result.

Proposition 3-5:

If R is ZC and every simple singular left R-module is Gp-injective, then R is a reduced weakly regular ring.

Proof: By Lemma (3-3) , R is a reduced ring. We shall show that RaR+l(a)=R for any a$\in$R. Suppose that there exists b$\in$R such that RbR+l(b)$\neq$R . Then there exists a maximal left ideal M of R containing RbR+l(b). By Lemma (3-2), M must be essential in R. Therefore R/M is Gp-injective. So there exists a positive integer n such that any R-homomrphism of Rb^n into R/M extends to one of R into R/M. let $f:Rb^n \rightarrow R/M$ be defined by $f(rb^n)=r+M$. Since R is a reduced ring, f is a well- R-homomorphism. Now, R/M is Gp-injective , so there exists c$\in$R such that $1+M=f(b^n)=b^nc+M$. Hence $1-b^nc\in M$ and so 1$\in$M, which is a contradiction. Therefore RaR+l(a)=R for any a$\in$R. Hence R is a left weakly regular ring. Since R is reduced, then RaR+r(a)=R, implies that R is a right weakly regular ring. Therefore R is a weakly regular ring.
Kimand Nam in [2] proved that. Rings whose simple right R-modules are Gp-injective are always semi-prime. But in general rings whose simple singular right R-modules are Gp-injective need not be semi-prime.

Proposition 3-6:

Let R be a ZC ring, and every simple singular left R-module is Gp-injective, then R is a semi-prime ring .

Proof: From Lemma (3-3), R is a reduced ring and then R is a semi-prime ring

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