Transcritical & Pitchfork Bifurcation in Differential Algebraic Equations Without Reduction

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Abstract

In this paper, the system of *differential algebraic equations* (DAEs) are considered and sufficient conditions to make the origin undergoes a transcortical or a pitchfork bifurcation are given. Those conditions talk about index one and index two DAE systems, and gotted by applying two strategies: depending on differential part and depending on constraint. We generalize the work of [Stephen,1990] and apply his strategy on index one and two DAEs. His work applied on *Ordinary differential equations* (ODEs) with:

1. equilibrium solution are dealt with instead of solution, and
2. the shape of bifurcation diagram is looked for instead of looking for the stability of equilibrium solution.

Depending on this we can determine which kind of bifurcation will occur in the system of differential algebraic equations. Investigate bifurcation points depending on only the constraint equation in [Ian,1992] had considered here and upgraded to go with DAEs of index one and two.

Keywords. Differential Algebraic Equations; DAEs; Bifurcation; Pitchfork; Transcritical.

الملخص

تناول هذا البحث موضوع التفرع في نظام المعادلات التفاضلية الجبرية. تم وضع شروط لجعل نقطة الاصل تقع تحت تأثير نوعان من التفرع هما (transcritical) و (pitchfork). تم العمل على جعل هذه الشروط تتناسب مع النظام ذو الدليل ١ والدليل ٢، كذلك; تم تطبيق طريقتين مختلفتين للحصول على هذه الشروط . هاتان الطريقتان هما الاعتماد على الجزء التفاضلي والطريقة الثانية الاعتماد على القيد.

يعتبر البحث تعميما للطريقة المتبعة في المصدر [Wiggins,1990] الذي طبق على نظام المعادلات التفاضلية الاعتيادية حيث تتلخص طريقة عمله كما يلي:

١. الاعتماد على الحل في النقطة الحرجة بدلا من الحل العام.
٢. دراسة نوع التفرع من خلال شكل مخطط التفرع وليس الاستقرارية،

هنا تم تعميم هذه الفكرة لكي تتناسب مع نظام المعادلات التفاضلية الجبرية بالاعتماد على الجزء التفاضلي مرة وأخرى بالاعتماد على القيد.

1 Introduction

The main difference between ordinary differential equation and differential algebraic equation systems is that the system output in the later may not be smooth because of the constraint which means in this case that system output can not be a (manifold), and hence system may not have a solution. The *semi-explicit* formula for differential algebraic equation has more attentions in literature because of its importance in applications, it has the form [Harlad, 1994].

$$\begin{aligned} \dot{x} &= f(x, y, t) \\ 0 &= g(x, y, t), \end{aligned} \tag{1}$$

where $t \in \mathcal{I} \subset \mathcal{R}$ and $\dot{x}, x, y \in \mathcal{R}^n$.

2 Bifurcation

Bifurcation theory is the study of equation with multiple solutions and the mechanism of equilibrium points when created and destroyed as parameter varies. One and two dimensional projections are the most important. They fall under the category of problems called *bifurcation at a simple eigenvalue.*

To get an idea of what this topic is all about, let's go back to elementary algebra and look at the quadratic function $f(x) = x^2 + x + c$, where c is a constant. We should realize that the *zeros* of this function depend on the parameter c. To see this, let's consider the following example taken from [Henry, 2009]:

$$x^2 + x + c = (x + \frac{1}{2})^2 + (c - \frac{1}{4}). \tag{2}$$

Clearly, the item $(x+\frac{1}{2})^2$ is always nonnegative, so that if $c > \frac{1}{4}$ the expression in (2) is always greater than zero, and the quadratic equation $x^2 + x + c = 0$ has no real solution. If $c = \frac{1}{4}$ then the only solution is $x = -\frac{1}{2}$. Finally, if $c < \frac{1}{4}$ then it has two solutions $x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$. The important point in this example is that $\frac{1}{4}$ is the value of the parameter c at which the nature of the solutions of the quadratic equation changes. We say that $c = \frac{1}{4}$ is a *bifurcation point* because as c decreases through $\frac{1}{4}$, the solution $x = 0$ splits into two solutions(see Figure 1).

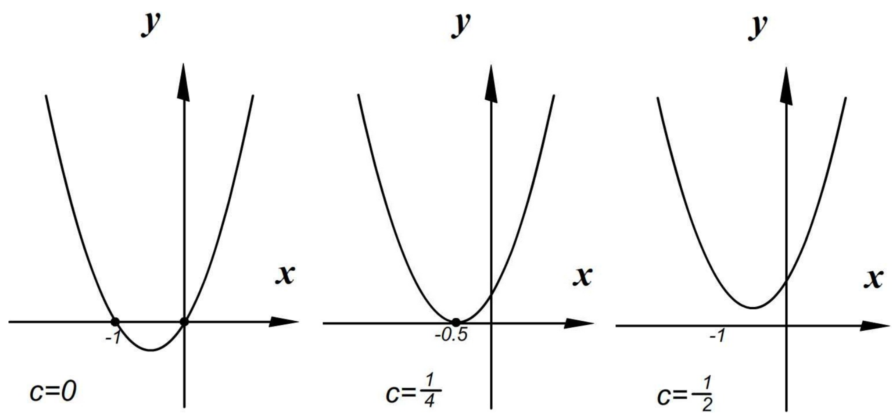

Figure 1: Behavior of $f(x) = x^2 + x + c$ when c changed.

2.1 Kinds of Bifurcation

It is conventional in the study of bifurcation to arrange things so that the bifurcation point is the origin point. Bifurcation has several kinds depending on branches number before and after the bifurcation point. Here we give definitions only for those kinds which is discussed in this paper. Other kinds can be founded in [Gerard,1990] and [Lawrence, 2001]. First we define bifurcation.

Def 1. [Gerard, 1990]**:** *We say that one solution of* (1) ***bifurcates*** *from another at* $\mu = \mu_0$ *if there are two distinct solutions* $x_1(\mu, t)$ *and* $x_2(\mu, t)$ *continuous in* μ *such that*

$$x_1(\mu_0, t) = x_2(\mu_0, t).$$

The point μ_0 *is called a* ***bifurcation point****.*

Def 2. [Stephen, 1990]**:** *If two curves of fixed points intersect at the origin in the* $\mu - x$ *plain, both existed on either side of* $\mu = 0$ *then the origin is called a* ***transcritical*** *bifurcation (TCB) point.*

Def 3. [Stephen, 1990]**:** *If two curves of fixed points intersect at the origin in the* $\mu - x$ *plain and only one exists in both sides of* $\mu = 0$*, moreover, the other curve of fixed points lays entirely to one side of* $\mu = 0$*, then the origin is called a* ***pitchfork*** *bifurcation (PFB) point.*

Remark 1. *The small circle over a function name means that function is evaluated at the origin point. For example J^0 is the Jacobian matrix evaluated at the origin.*

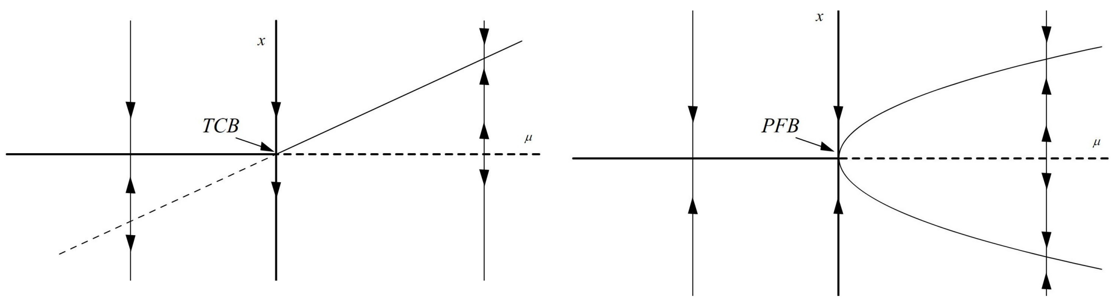

Figure 2: Some bifurcation kinds

To study bifurcation in differential algebraic equations we consider the semi-explicit formula

$$\begin{aligned} \dot{x} &= f(x, y) \\ 0 &= g(x, y), \end{aligned} \tag{3}$$

with x,$y \in \mathcal{R}$, and the *parameterized* semi-explicit DAE

$$\begin{aligned} \dot{x} &= f(x, y, \mu) \\ 0 &= g(x, y, \mu), \end{aligned} \tag{4}$$

where x,y,$\mu \in \mathcal{R}$.

3 General Assumptions

Consider the semi-explicit DAE system (3) and the parameterized differential algebraic equation (4). Any critical point in (3) should satisfy the constraint equation. The constraint acts as a surface containing all the vector fields and hence the solution of DAE. We assume here there is always a neighbor for the bifurcation point where both f and g are continuous in their common points at that neighbor.

Let $\mathcal{S}$ be the surface of the critical (*equilibrium*) points of (4), where

$\mathcal{S} = \{(x, y, \mu) \in \mathcal{R}^2 \times \mathcal{R} : f(x, y, \mu) = g(x, y, \mu) = 0\}$.

We define a function F which is equivalent to the set $\mathcal{S}$ when F equals zero:

$$F(x, y, \mu) = \begin{bmatrix} f(x, y, \mu) \\ g(x, y, \mu) \end{bmatrix} = \mathbf{0}. \tag{5}$$

See that the function F is in vector form to avoid the change of the original functions. Now the study of bifurcation of equilibrium solutions of (4) is equivalent to the study of singular points of the surface (5) [Gerard, 1990]. The small circle over function name means it is evaluated at the origin.

Assumption 1. *The origin is an equilibrium point i.e.,* $f^0 = g^0 = 0$.

Bifurcation point should be non-hyperbolic. The reverse is not always true. For example in the one dimensional ODE

$$\dot{x} = x^3,$$

the origin is nonhyperbolic but it is not a bifurcation point [Robert,2011]. The non hyperbolic DAE should has a singular *Jacobian* (J^0) matrix, i.e., $Det(J^0) = 0$. The higher index DAE of two variables should has $Det(g_y^0) = 0$, in this case the index will be greater than one. If we let either $g_x^0 = 0$ or $f_y^0 = 0$ together with $g_y^0 = 0$ then J^0 will has a row or column with zero values which make it singular. For this purpose the following assumption is necessary:

Assumption 2. $f_x^0 = g_x^0 = 0$.

The following lemma gives guarantee for the point (0,0,0) in (4) to be a bifurcation point. The proof can be found in [Gerard, 1990] (and in [Daniel,1983] for bifurcation around the y-axis).

Lemma 1. *Suppose that* $f_{xx}^0 \neq 0$ *or* $f_{yy}^0 \neq 0$*, and assumption* 2 *holds, then* (0,0,0) *is a bifurcation point of* (4)*.*

From lemma 1 we put:

Assumption 3. $f_{xx}^0 \neq 0$

For index two case we use different asuumption to go with it

Assumption 4. $g_{xx}^0 = g_{xy}^0 = 0$

Assumption 4 includes $g_{yx}^0 = 0$ because the function g is continuous at (0,0,0), hence $g_{yx}^0 = g_{xy}^0 = 0$. Depending on assumption 4 we can state our modified lemma as follows:

Lemma 2. *Let* (3) *be of index* 2 *and suppose that assumptions* 2 *and* 4 *are hold. Then*

$$\frac{dy}{dx}|_0 = 0.$$

Proof. First, we differentiate the constraint equation $g(x, y) = 0$ implicitly two times:

$$g_x\dot{x} + g_y\dot{y} = 0.$$

$$g_x\ddot{x} + (g_{xx}\dot{x} + g_{xy}\dot{y})\dot{x} + g_y\ddot{y} + (g_{yx}\dot{x} + g_{yy}\dot{y})\dot{y} = 0.$$

Evaluate at the origin and apply the given conditions and that $g_y^0 = 0$:

$$g_{yy}^0\dot{y}^2|_0 = 0.$$

Since (3) is of index 2 then $g^0_{yy} \neq 0$ which implies:

$$\frac{dy}{dt}|_0 = 0.$$

Then by chain rule:

$$\frac{dy}{dx}|_0 = 0.$$

□

4 Transcritical Bifurcation (TCB)

To understand this bifurcation type better, let's describe it by taking an example.

Example 1. *Consider the parameterized one dimensional ordinary differential equation:* $\dot{x} = \mu x - x^2$.
Here $f(x, \mu) = \mu x - x^2$. There are two critical points: $x = 0$ and $x = \mu$. Those two critical points represent two linear curves of critical points in $x - \mu$ plain. Since both curves are linear then they should pass the origin from one side to another. To find stability of those two curves we consider the relation between $\dot{x}$ and $f(x)$ while changing the parameter μ. Then we can see that each curve will change its stability case after pass the origin point. Then, the origin point should be a transcritical bifurcation point if there are two curves of critical points passing it from one side to another.

5 Transcritical Bifurcation in DAEs

Example 1 shows that in TCB there should be two curves (surfaces in two dimensional sight) passing through the origin and that $F_\mu = 0$. That means we can not use the implicit function theorem, so we need to apply a different strategy. Hence we write the function F given in (5) as follows:

$$F(x, y, \mu) = \begin{bmatrix} xM(x, y, \mu) \\ xN(x, y, \mu) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

Where the functions M and N are defined as follows:

$$M(x, y, \mu) = \begin{cases} \frac{f(x,y,\mu)}{x} & \text{if } x \neq 0 \\ f_x & \text{if } x{=}0 \end{cases} \tag{7}$$

and

$$N(x, y, \mu) = \begin{cases} \frac{g(x,y,\mu)}{x} & \text{if } x \neq 0 \\ g_x & \text{if } x{=}0 \end{cases} \tag{8}$$

See that $x = 0$ is a surface of equilibrium points passing through the origin from side to side of $\mu = 0$. Hence, all what we need to do is finding another surface, in $M = 0$ or $N = 0$, passing through the origin from side to side of $\mu = 0$, i.e., we need to prove that $\mu^0_x \neq 0$.

5.1 TCB of Index One DAEs

In this subsection we consider (3) with index one, which is a DAE with $g_y \neq 0$, i.e., g_y has an inverse. First we give the following helpful lemma:

Lemma 3. *Let* (3) *be of index one and the assumption* 2 *holds. Then*

$$\frac{dy}{dx}|_0 = 0.$$

Proof. If we implicitly differentiate the constraint we get:

$$g_x\dot{x} + g_y\dot{y} = 0 \Rightarrow g_y\frac{dy}{dt} = -g_x\frac{dx}{dt}.$$

Since g_y^{-1} exists then we can write:

$$\frac{dy}{dt} = -g_y^{-1}g_x\frac{dx}{dt} \tag{9}$$

By *chain rule* we can put:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx},$$

Then with (9) we get:

$$\frac{dy}{dx} = -g_y^{-1}g_x\frac{dx}{dt}\frac{dt}{dx}.$$

Chain rule cancels the last two terms, then we evaluate the result at the origin to apply assumption 2. We lastly get:

$$\frac{dy}{dx}|_0 = 0.$$

□

Theorem 1. *Consider the index one DAE* (3) *with the surface of equilibrium points* (5). *Assume that the assumption* 2 *is satisfied. If*

1. $f_\mu = 0$,
2. $f^0_{x\mu} \neq 0$ *and*
3. $f^0_{xx} \neq 0$,

then (0,0,0) *is a transcritical bifurcation point.*

Proof. Consider the function $M(x, y, \mu)$ given in (7). Since $f^0_{x\mu} \neq 0$ which by definition of the function M means $M_\mu(x, y, \mu) \neq 0$. Then by implicit function theorem

$$M(x, y, \mu(x, y)) = 0,$$

which can be differentiate with respect to x to get

$$M_x(x, y, \mu) + M_y(x, y, \mu)\frac{dy}{dx} + M_\mu(x, y, \mu)\left(\mu_x(x, y) + \mu_y(x, y)\frac{dy}{dx}\right) = 0.$$

If we evaluate that in the origin, and because of assumption 2, then we can use lemma 3 and get:

$$f^0_{xx}(x,y,\mu) + f^0_{x\mu}(x,y,\mu)\mu^0_x(x,y) = 0.$$

Finally, by conditions 2 and 3 above we get

$$\mu^0_x(x,y) \neq 0.$$

□

5.2 TCB of Index Two DAEs

The following theorem can be proved by depending on lemma 2 instead of lemma 3.

Theorem 2. *Consider the index two DAE* (3) *with the surface of equilibrium points* (5)*. Assume that the assumptions* 2 *and* 4 *are satisfied. If*

1. $f_\mu = 0$,
2. $f^0_{x\mu} \neq 0$ *and*
3. $f^0_{xx} \neq 0$,

then (0,0,0) *is a transcritical bifurcation point.*

5.3 Depending On The Constraint Only

In this subsection we generalize the work of I. Dobson in [Ian, 1992] by depending on the fact that the constraint *should* have all the DAE points. Consider (3), algebraically we can find a function h with $h(0,0) = 0$ as follows:

$$g(x,y) = 0$$

$$\dot{x} = f(x,y) = h(g(x,y)).$$

Dobson assume that the Jacobian Dh is invertible. Here we consider the assumption 3 so we can use lemma 1. If we consider (4) then we can write:

$$\begin{aligned} \dot{x} &= h(x,y,\mu) \\ 0 &= g(x,y,\mu). \end{aligned} \tag{10}$$

Hence, with assumption 3 and with equation (10), the study of bifurcation with the constraint in (4) is equivalent to the study of bifurcation in the whole system of (4). For this we have the following theorems on index one and two. The proofs are exactly like the previous theorems but apply steps on the function g instead of f.

Theorem 3. *Let* (3) *has index one. Consider the surface of equilibrium points* (5)*. Assume that the assumptions* 2 *and* 3 *are satisfied. If*

1. $g_\mu = 0$,

2. $g^0_{x\mu} \neq 0$ *and*

3. $g^0_{xx} \neq 0$,

then (0,0,0) *is a transcritical bifurcation point.*

Theorem 4. *Let* (3) *has index two. Consider the surface of equilibrium points* (5). *Assume that the assumptions* 2 *and* 4 *are satisfied. If*

1. $g_\mu = 0$,

2. $g^0_{x\mu} \neq 0$ *and*

3. $g^0_{xx} \neq 0$,

then (0,0,0) *is a transcritical bifurcation point.*

6 Pitchfork Bifurcation (PFB)

In this kind of bifurcation the critical point which is a pitchfork bifurcation point will divided into three critical points while changing the parameter μ to a certain direction and will combine in one point if the parameter μ is changed to the opposite direction.

Example 2. *Consider the parameterized one dimensional vector field:*
$\dot{x} = \mu x - x^3$.
There are two curves of critical points passing through the origin, one is $x = 0$ and the other is $\mu = x^2$. The first one exists on both sides of $\mu = 0$ while the second one exists on only one side. The second curve should satisfy $\mu^0_x = 0$ and $\mu^0_{xx} \neq 0$.

6.1 Pitchfork Bifurcation in DAEs

To get a pitchfork bifurcation at the origin we need to find two surfaces of critical points passing through the origin, one from side to side of $\mu = 0$ and the other should lay entirely in one side of $\mu = 0$. To achieve our goals we recall the functions M and N in equations (7) and (8) respectively. The plain $x = 0$ is passing the origin from side to side with $\mu = 0$. Hence all what we need do is finding another surface coming from $M = 0$ or $N = 0$ with the two attributes:

$$\mu^0_x = 0, \tag{11}$$

and

$$\mu^0_{xx} \neq 0. \tag{12}$$

6.2 PFB of Index One DAEs

Applying same steps in TCB, lemma 3 can be seen in those theorems again.

Theorem 5. *Let* (3) *has index one. Consider the surface of equilibrium points* (5). *Assume that the assumption* 2 *is satisfied. If*

1. $f_\mu = 0$,

2. $f^0_{x\mu} \neq 0$,

3. $f^0_{xx} = 0$ *and*

4. $f^0_{xxx} + (f^0_{xy} + f^0_{x\mu}\mu^0_y)\frac{d^2y}{dx^2}|_0 \neq 0$

then (0,0,0) *is a pitchfork bifurcation point.*

Proof. Consider the function $M(x, y, \mu)$ given in (7). Since $f^0_{x\mu} \neq 0$ which by definition of the function M means $M_\mu(x, y, \mu) \neq 0$. Then by implicit function theorem

$$M(x, y, \mu(x, y)) = 0. \tag{13}$$

Differentiation (13) with respect to x yeilds:

$$M_x(x, y, \mu) + M_y(x, y, \mu)\frac{dy}{dx} + M_\mu(x, y, \mu)\left(\mu_x(x, y) + \mu_y(x, y)\frac{dy}{dx}\right) = 0.$$

If we evaluate that in the origin, and because of assumption 2, then lemma 3 gives:

$$f^0_{xx}(x, y, \mu) + f^0_{x\mu}(x, y, \mu)\mu^0_x(x, y) = 0.$$

Now, by conditions 2 and 3 above

$$\mu^0_x(x, y) = 0$$

It remains to establish that $\mu^0_x x(x, y) \neq 0$. To do so we differentiate (13) two times with respect to x:

$$M_{xx} + M_{xy}\frac{dy}{dx} + M_{x\mu}(\mu_x + \mu_y\frac{dx}{dy}) + M_y\frac{d^2y}{dx^2}+$$

$$\left[M_{yx} + M_{yy}\frac{dy}{dx} + M_{y\mu}(\mu_x + \mu_y\frac{dy}{dx})\right]\frac{dy}{dx} + M_\mu\left[\mu_{xx} + \mu_{xy}\frac{dy}{dx} + \mu_y\frac{d^2y}{dx^2} + (\mu_{yx} + \mu_{yy}\frac{dy}{dx})\frac{dy}{dx}\right]+$$

$$\left[M_{\mu x} + M_{\mu y}\frac{dy}{dx} + M_{\mu\mu}(\mu_x + \mu_y\frac{dy}{dx})\right]\left[\mu_x + \mu_y\frac{dy}{dx}\right] = 0.$$

If we evaluate at the origin then lemma 3 will make it shorter

$$\mu^0_{xx} = -\left[f^0_{xxx} + (f^0_{xy} + f^0_{x\mu}\mu^0_y)\frac{d^2y}{dx^2}|_0\right](f^0_{x\mu})^{-1}.$$

Hence by conditions 2 and 4 above we get $\mu^0_{xx} \neq 0$. □

6.3 PFB of Index Two DAEs

The following theorem can be proved in same steps above but depending on lemma 2 instead of lemma 3.

Theorem 6. *Let* (3) *has index two. Consider the surface of equilibrium points* (5)*. Assume that the assumptions* 2 *and* 4 *are satisfied. If*

1. $f_\mu = 0$,
2. $f^0_{x\mu} \neq 0$,
3. $f^0_{xx} = 0$ *and*
4. $f^0_{xxx} + (f^0_{xy} + f^0_{x\mu}\mu^0_y)\frac{d^2y}{dx^2}|_0 \neq 0$

then (0,0,0) *is a pitchfork bifurcation point.*

6.4 Depending on the Constraint Only

We see in TCB case how can we depend only on the constraint. Here we recall same strategy and give two theorems with index one then index two. The proofs have the same steps as the two theorems above but steps should applied on the function g instead of f.

Theorem 7. *Let* (3) *has index one. Consider the surface of equilibrium points* (5)*. Assume that the assumptions* 2 *and* 3 *are satisfied. If*

1. $g_\mu = 0$,
2. $g^0_{x\mu} \neq 0$,
3. $g^0_{xx} = 0$ *and*
4. $g^0_{xxx} + (g^0_{xy} + g^0_{x\mu}\mu^0_y)\frac{d^2y}{dx^2}|_0 \neq 0$

then (0,0,0) *is a pitchfork bifurcation point.*

Theorem 8. *Let* (3) *has index two. Consider the surface of equilibrium points* (5)*. Assume that the assumptions* 2 *and* 4 *are satisfied. If*

1. $g_\mu = 0$,
2. $g^0_{x\mu} \neq 0$,
3. $g^0_{xx} = 0$ *and*
4. $g^0_{xxx} + (g^0_{xy} + g^0_{x\mu}\mu^0_y)\frac{d^2y}{dx^2}|_0 \neq 0$

then (0,0,0) *is a pitchfork bifurcation point* [Daniel, 1983].

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