The Fuzzy Closed BH-Algebra With Respect To an Element

By
Hussein Hadi Abbass
University of Kufa\ College of Education for girls\ Department of Mathematics
Msc_hussien@yahoo.com
Hasan Mohammed Ali Saeed
University of Kufa\ College of Engineering \ Department of Material Engineering
hasan.shlaka@uokufa.edu.iq

Abstract
In this paper, we define the concepts of a fuzzy closed ideal with respect to an element of a BH-algebra and a fuzzy closed BH-algebra with respect to an element of BH-algebra. We stated and proved some theorems which determine the relation between these notion and the notions of fuzzy ideals in a BH-algebra.
Keywords: BCH and BH-algebra, p-semisimple, medial, associative, ideal, fuzzy ideal, closed ideal, fuzzy closed ideal, subalgebra, fuzzy subalgebra, BCA-part, medial part,.

جبر BH المغلق ضبابياً بالنسبة لعنصر

حسين هادي عباس النعماني
جامعة الكوفة / كلية التربية للبنات / قسم الرياضيات
msc_hussien@yahoo.com
حسن محمد علي الاسدي
جامعة الكوفة / كلية الهندسة / قسم هندسة المواد
hasan.shlaka@uokufa.edu.iq

المستخلص :
عرفنا في بحثنا هذا مفهومي المثالي الضبابي المغلق بالنسبة لعنصر في جبر (BH) وجبر (BH) المغلق ضبابياً بالنسبة لعنصر في جبر (BH) حيث اعطينا وبرهنّا عدد من المبرهنات التي حددت العلاقة بين المفاهيم أعلاه وبعض أنواع المثاليات في جبر (BH)

1. Introduction

The notion of BCH-algebras [9] was formulated first in 1983 by Q. P. Hu and X. Li as a generalization of BCK/ BCI-algebras [4, 14] which are generalization of the concept of set-theoretic difference and propositional calculus. After that, many mathematical papers have been published investigating some algebraic properties of BCK\BCI\BCH- algebras and their relation with other universal structures including lattices and Boolean algebras. In 1991, M. A. Chaudhry introduced the notions of BCA-part of a BCH-algebra, medial part of a BCH-algebra, subalgebra of a BCH-algebra, ideals and closed ideals in a BCH-algebra [6]. In 1998, Jun et al introduced the notion of BH-algebra, which is a generalization of BCH-algebras [12]. then, Y. B. Jun, E. H. Roh, H. S. Kim and Q. Zhang discussed more properties on BH-algebras [8, 11, 12]. In 2011, H. H. Abbass and H. M. Saeed introduced the notions of (a closed ideal and closed BCH-algebra) with respect to an element of a BCH-algebra[3].

On the other hand, we shall mention the development of a fuzzy set, fuzzy subalgebra, fuzzy ideals, fuzzy closed ideals.

In 1965, L. A. Zadeh introduced the notions of a Fuzzy subset of a set as a method for representing uncertainty in real physical world. Since then its application have been growing rapidly over many disciplines [5]. In 1999, Y. B. Jun introduced the notion of Fuzzy closed ideals in BCH-algebras [13]. In 2001, Q. Zhang, E. H. Roh and Y. B. Jun studied the fuzzy theory in BH-algebras [10].

In this paper, we introduce the notion of a fuzzy closed ideal and a fuzzy closed BH-algebra with respect to an element of a BH-algebra. We prove some theorems and give some examples to show that the relations between this notion and the notions of ideals in a BH-algebra.

2. PRELIMINARIES

In this section we give some basic concept about BCH-algebra, , p-semi simple BCH-algebra ,medial BCH-algebra , associative BCH-algebra , BCA-part of BCH-algebra , medial part of BCH-algebra and (subalgebra , ideal , closed ideal , quasi-associative ideal) in BCH-algebra with some theorems, propositions and examples.

Definition (2.1) : [9]

A BCH-algebra is an algebra (X,*,0), where X is nonempty set, * is a binary operation and 0 is a constant , satisfying the following axioms: for all x, y, z$\in$ X:

i. x*x = 0,
ii. x*y = 0 and y*x = 0 imply x = y,
iii. (x*y)*z = (x*z)*y.

Definition (2.2) : [9]

In any BCH-algebra X, a partial order $\leq$ is defined by putting x $\leq$ y if and only if x*y = 0.

Proposition (2.3) : [6]

In a BCH-algebra X, the following holds for all x, y, z $\in$ X,

1. x * 0 = x ,
2. (x * (x * y)) * y = 0,
3. 0 * (x * y) = (0 * x) * (0 * y) ,
4. 0 * (0 * (0 * x)) = 0 * x ,
5. x $\leq$ y implies 0 * x = 0 * y.

Definition (2.4) : [12]

A BH-algebra is a nonempty set X with a constant 0 and a binary operation * satisfying the following conditions:

i. x * x = 0, $\forall$ x$\in$ X.
ii. x * y = 0 and y * x = 0 imply x = y, $\forall$ x, y$\in$ X.
iii. x *0 = x, $\forall$ x$\in$ X.

Definition (2.5) : [1]

A BCH-algebra X that satisfying in condition if 0 * x = 0 then x = 0, for all x$\in$X is called a ***P-semisimple BCH-algebra.***

we generalize the notion of p-semisimple BCH-algebra to a BH-algebra

Definition (2.6) :

A BH-algebra X that satisfying the condition if $0*x = 0$ then $x = 0$, for all $x \in X$ is called a ***P*-semisimple BH-algebra.**

Definition (2.7) : [7]

A BCH-algebra X is called ***medial*** if: $x * (x * y) = y$, for all $x , y \in X$.

we generalize the notion of medial BCH-algebra to a BH-algebra

Definition (2.8) :

A BH-algebra X is called ***medial*** if: $x*(x*y) = y$, for all $x, y \in X$.

Definition (2.9) : [1]

A BCH-algebra X is called an ***associative BCH-algebra*** if:$(x*y)*z = x*(y*z)$, for all $x, y, z \in X$.

we generalize the notion of associative BCH-algebra to a BH-algebra

Definition (2.10) :

A BCH-algebra X is called an ***associative BH-algebra*** if:$(x*y)*z = x*(y*z)$, for all $x, y, z \in X$.

Definition (2.11) : [6]

Let X be a BCH-algebra . Then the set $X_+ = \{ x \in X : 0 * x = 0 \}$ is called ***the BCA-part*** of X

Remark (2.12) : [6]

The BCA-part X_+ of X is a nonempty Since $0 *0 = 0$ gives $0 \in X_+$. Further the BCA-part of a BCH-algebra may coincide with the BCH-algebra, but not necessarily with a BCK-algebra.

we generalize the notion of BCA-part of BCH-algebra to a BH-algebra

Definition (2.13) :

Let X be a BH-algebra. Then the set $X_+ = \{ x \in X : 0*x = 0 \}$ is called ***the BCA-part*** of X.

Definition (2.14) : [6]

Let X be a BCH-algebra . Then the set $med(X) = \{x \in X : 0*(0*x) = x\}$ is called ***the medial part of X***.

Remark (2.15) : [6]

The medial part med(X) of X is nonempty Since $0*(0*0) = 0$ gives $0 \in med(X)$.

Theorem (2.16) : [7]

Let X be a BCH-algebra . Then $x \in med(X)$ if and only if $x*y = 0*(y*x)$ for all $x ,y \in X$

we generalize the notion of medial part of BCH-algebra to a BH-algebra

Definition (2.17) :

Let X be a BH-algebra . Then the set $med(X) = \{x \in X : 0*(0*x) = x\}$ is called ***the medial part of X***.

<u>Definition (2.18) :</u> [12]

Let X a BH-algebra and $S \subset X$. Then S is called a ***subalgebra*** of X if $x*y \in S$ for all $x, y \in S$.

<u>Definition (2.19) :</u> [10]

A fuzzy set B in a BH-algebra X is said to be a fuzzy subalgebra of X if it satisfies: $B(x * y) \geq \min\{B(x), B(y)\}$, $\forall x, y \in X$.

<u>Definition (2.20) :</u> [10]

Let X be a BH-algebra and $\phi \neq I \subset X$. Then I is called an ***ideal*** of X if it satisfies:

i. $0 \in I$.

ii. $x*y \in I$ and $y \in I$ imply $x \in I$.

<u>Definition (2.21) :</u> [13]

A fuzzy subset A of a BH-algebra X is said to be a ***fuzzy ideal*** if and only if:

i. For any $x \in X$, $A(0) \geq A(x)$.

ii. For any $x, y \in X$, $A(x) \geq \min\{A(x*y), A(y)\}$.

<u>Definition (2.22) :</u> [7]

Let X be a BCH-algebra and $I \subset X$ be an ideal. Then I is called a ***closed ideal*** of X if $0*x \in I$ for all $x \in I$.

<u>Definition (2.23) :</u> [13]

A ***fuzzy ideal*** A of a BCH-algebra X is said to be ***closed*** if $A(0*x) \geq A(x)$, for all $x \in X$.

<u>Definition (2.24) :</u> [3]

Let X be a BCH-algebra and I be an ideal of X. Then I is called a ***closed ideal with respect to an element b∈X, denoted b-closed ideal of X***, if $b*(0*x) \in I$ for all $x \in I$.

<u>Definition (2.25) :</u> [3]

Let X be a BCH-algebra and $b \in X$. Then X is called a ***closed BCH-algebra with respect to b***, or ***b-closed BCH-algebra***, if and only if every proper ideal is closed ideal with respect to b.

3. <u>THE MAIN RESULTS</u>

In this section we define the notions of a fuzzy closed ideal and a fuzzy closed BH-algebra with respect to an element of a BH-algebra. For our discussion, we will link this notion with other types of ideals which mentioned in preliminaries.

<u>Definition (3.1) :</u>

Let X be a BH-algebra and A be a fuzzy ideal of X, then A is called a ***fuzzy closed ideal with respect to an element b***$\in X$, denoted by a ***fuzzy b-closed ideal*** of X, if $A(b*(0*x)) \geq A(x)$, $\forall x \in X$.

Example (3.2) :

Consider the BH-algebra X = {0, 1, 2, 3} with the following operation table.

*	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	3
3	3	3	3	0

We define the fuzzy set A by

$$A(x) = \begin{cases} 0.5 & \text{if } x = 0 \\ 0.3 & \text{if } x \neq 0 \end{cases}$$

Then A is a ***fuzzy 0-closed ideal***, Since

1- A is a fuzzy ideal, since

i. $A(0) = 0.5 \geq A(x), \forall x \in X$

ii. $A(0) = 0.5 \geq \min\{A(0*0), A(0)\} = 0.5$, $A(1) = 0.3 \geq \min\{A(1*0), A(0)\} = 0.3$

$A(0) = 0.5 \geq \min\{A(0*1), A(1)\} = 0.3$, $A(1) = 0.3 \geq \min\{A(1*1), A(1)\} = 0.3$

$A(0) = 0.5 \geq \min\{A(0*2), A(2)\} = 0.3$, $A(1) = 0.3 \geq \min\{A(1*2), A(2)\} = 0.3$

$A(0) = 0.5 \geq \min\{A(0*3), A(3)\} = 0.3$, $A(1) = 0.3 \geq \min\{A(1*3), A(3)\} = 0.3$

$A(2) = 0.3 \geq \min\{A(2*0), A(0)\} = 0.3$, $A(3) = 0.3 \geq \min\{A(3*0), A(0)\} = 0.3$

$A(2) = 0.3 \geq \min\{A(2*1), A(1)\} = 0.3$, $A(3) = 0.3 \geq \min\{A(3*1), A(1)\} = 0.3$

$A(2) = 0.3 \geq \min\{A(2*2), A(2)\} = 0.3$, $A(3) = 0.3 \geq \min\{A(3*2), A(2)\} = 0.3$

$A(2) = 0.3 \geq \min\{A(2*3), A(3)\} = 0.3$, $A(3) = 0.3 \geq \min\{A(3*3), A(3)\} = 0.3$

Therefore, A is a ***fuzzy ideal*** of X

2- $A(0*(0*x)) \geq A(x), \forall x \in X$, since

$A(0*(0*0)) = A(0*0) = A(0) = 0.5 \geq A(0) = 0.5$, $A(0*(0*1)) = A(0*0) = A(0) = 0.5 \geq A(1) = 0.3$

$A(0*(0*2)) = A(0*0) = A(0) = 0.5 \geq A(2) = 0.3$, $A(0*(0*3)) = A(0*0) = A(0) = 0.5 \geq A(3) = 0.3$

Theorem (3.3) :

Let X be a BH-algebra. If $X=X_+$, then the following conditions are equivalent:

(1) A is a fuzzy closed ideal of X

(2) A is a fuzzy 0-closed ideal of X.

Proof

(1) ⇒ (2)

Let A be a fuzzy closed ideal of X

⇒ A is a fuzzy ideal of X [By definition of a fuzzy closed ideal]

⇒ To prove that A is a fuzzy 0-closed ideal.

Let $x \in X$

$A(0*(0*x)) = A(0*0)$ [Since $0*x = 0$, $\forall x\in X_+$]

$= A(0)$ [Since $x*x = 0$, $\forall x\in X$]

But $A(0) \geq A(x)$, $\forall x\in X$ [Since A is a fuzzy ideal] $\Rightarrow A(0*(0*x)) \geq A(x)$

Therefore, A is a fuzzy 0-closed ideal of X.

(2) $\Rightarrow$ (1)

Let A be a fuzzy 0-closed ideal of X

$\Rightarrow$ A is a fuzzy ideal [By definition of a fuzzy 0-closed ideal]

$\Rightarrow$ To prove that A is a fuzzy closed ideal

Let $x\in X$

$A(0*x) = A(0)$ [Since $0*x = 0$, $\forall x\in X_+$]

But $A(0) \geq A(x)$, $\forall x\in X$ [Since A is a fuzzy ideal] $\Rightarrow A(0*x) \geq A(x)$

Therefore, A is a fuzzy closed ideal of X. ■

Definition (3.4):

Let X be a BH-algebra and $b\in X$. Then X is called a *fuzzy closed BH-algebra with respect to the element b*, or *fuzzy b-closed BH-algebra*, if and only if every fuzzy ideal is a fuzzy closed ideal with respect to b.

Example (3.5) :

Let the BH-algebra X = {0, 1, 2, 3} with the following operation table

*	0	1	2	3
0	0	0	0	0
1	1	0	3	3
2	2	0	0	2
3	3	0	0	0

Then, X is a fuzzy 0-closed BH-algebra, because
every fuzzy ideal define on X is a fuzzy 0-closed ideal. Since
If A is a fuzzy ideal of X, then $A(0*(0*x)) = A(0*0) = A(0) \geq A(x)$, $\forall x\in X$.

Theorem (3.6) :

Every medial BH-algebra is a fuzzy 0-closed BH-algebra.

Proof

Let X be a medial BH-algebra and A be a fuzzy ideal of X

To prove that A is a fuzzy 0-closed ideal of X

Let $x\in X$. Then

$A(0*(0*x)) = A(x)$ [Since X is medial $\Rightarrow x*(x*y)=y$, $\forall x, y\in X$]

$\Rightarrow A(0*(0*x)) \geq A(x)$

$\Rightarrow$ A is a fuzzy 0-closed ideal of X.

Therefore,

X is a fuzzy 0-closed BH-algebra. ■

Theorem (3.7) :

Let X be a BH-algebra. If $X=X_+$. Then X is a fuzzy 0-closed BH-algebra.

Proof

Let A be a fuzzy ideal of X $\Rightarrow$

To prove that A is a fuzzy 0-closed ideal.

Let $x \in X$

$A(0*(0*x)) = A(0*0)$ [Since $0*x = 0, \forall x \in X_+$]

$= A(0)$ [Since $x*x = 0, \forall x \in X$]

But $A(0) \geq A(x), \forall x \in X$ [Since A is a fuzzy ideal]

$\Rightarrow A(0*(0*x)) \geq A(x)$

$\Rightarrow$ A is a fuzzy 0-closed ideal of X

Therefore,

A is a fuzzy 0-closed BH-algebra. ■

Remark(3.8):

If $X \neq X_+$, then the above theorem is not necessary to be true as in the following example

Example (3.9) :

Consider the ***BH-algebra*** X = {0, 1, 2, 3} with the following operation table

*	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

Then $X \neq X_+$ which is not fuzzy 0-closed BH-algebra, because if A a fuzzy set defined as follows:

$$A(x) = \begin{cases} 1 & , \text{if } x = 0, 1 \\ 0 & \text{if } x = 2, 3 \end{cases}$$ Then A is fuzzy ideal, Since

i- $A(0) \geq A(x), \forall x \in X$.

ii- $A(x) \geq \min\{A(x*y), A(y)\}, \forall x, y \in X$. Where

$A(0) = 1 \geq \min\{A(0*0), A(0)\} = A(0) = 1$, $A(1) = 1 \geq \min\{A(1*0), A(0)\} = A(1) = 1$

$A(0) = 1 \geq \min\{A(0*1), A(1)\} = A(3) = 0$, $A(1) = 1 \geq \min\{A(1*1), A(1)\} = A(1) = 1$

$A(0) = 1 \geq \min\{A(0*2), A(2)\} = A(2) = 0$, $A(1) = 1 \geq \min\{A(1*2), A(2)\} = A(2) = 0$

$A(0) = 1 \geq \min\{A(0*3), A(3)\} = A(3) = 0$, $A(1) = 1 \geq \min\{A(1*3), A(3)\} = A(3) = 0$

$A(2) = 0 \geq \min\{A(2*0), A(0)\} = A(2) = 0$, $A(3) = 0 \geq \min\{A(3*0), A(0)\} = A(3) = 0$

$A(2) = 0 \geq \min\{A(2*1), A(1)\} = A(2) = 0$, $A(3) = 0 \geq \min\{A(3*1), A(1)\} = A(3) = 0$

$A(2) = 0 \geq \min\{A(2*2), A(2)\} = A(2) = 0$, $A(3) = 0 \geq \min\{A(3*2), A(2)\} = A(2) = 0$

$A(2) = 0 \geq \min\{A(2*3), A(3)\} = A(3) = 0$, $A(3) = 0 \geq \min\{A(3*3), A(3)\} = A(3) = 0$

But A is not fuzzy 0-closed ideal, Since

$A(0*(0*1)) = A(0*3) = A(2) = 0 < A(1) = 1$.

Theorem (3.10) :

Let X = X_+ be a BH-algebra and $b\in X$ such that for any fuzzy ideal A of X and A(b) = A(0). Then X is a fuzzy b-closed BH-algebra.

Proof

Let A be a fuzzy ideal and $b\in X$

To prove that A is a fuzzy b-closed ideal of X

Let $x\in X$, then

$A(b*(0*x)) = A(b*0)$ [Since $0*x=0, \forall x\in X_+$]

$A(b*(0*x)) = A(b)$ [Since $x*0=x, \forall x\in X$]

$= A(0)$ [Since A(b) = A(0)]

$\geq A(x)$ [Since A is a fuzzy ideal of X]

$\Rightarrow$ A is a fuzzy b-closed ideal of X

Therefore, X is a fuzzy b-closed BH-algebra. ■

Remark (3.11) :

If there exist $b\in X$ such that A(b) < A(0). Then the theorem(3.10) is not necessary to be true as in the following example

Example (3.13):

consider the BH-algebra X={0, 1, 2, 3} in example(3.2). then the fuzzy ideal

$$A = \begin{cases} 0.5 & \text{if } x = 0 \\ 0.3 & \text{if } x \neq 0 \end{cases}$$

$A(1) \neq A(0)$

$\Rightarrow$ A is not fuzzy 1-closed ideal. since

A(1*(0*0) A(1*0) = A(1) = 0.3 < A(0) = 0.5.

Remark (3.14) :

If $X \neq X_+$, then theorem(3.10) is not necessary to be true as in the following example.

Example (3.15):

consider the BH-algebra X={0, 1, 2, 3} in example(3.9). then the fuzzy ideal

$$A(x) = \begin{cases} 1 & , \text{if } x = 0,1 \\ 0 & \text{if } x = 2,3 \end{cases}$$

is not fuzzy 1-closed ideal. Since

A(1*(0*1)) = A(1*3) = A(2) = 0 < A(1) = 1.

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