Properties of Estimation for Pareto Distribution with some Applications

Samer M. Saleh
Dept. of Mathematics
College of Science/Thi-Qar University
Akram M. Al-Abood
Dept. Of Mathematics and Comp. Application
College of Science/Al-Nahrain University

Abstract

In this paper we consider the Pareto distribution of two parameters , since it have many applications in economics (income and distribution of wealth), human population and many other fields. Estimation of the distribution parameters is obtained by two methods namely the modified moments method and the maximum likelihood method .

Approximation to the mean and variance of the estimators is made theoretically by utilizing Taylor series expansion up to second order derivative and results assessed practically by using Monte Carlo simulation. Two applications related to Pareto distribution are discussed namely earthquake and dilution of concentration in the liquid.
Keywords: Pareto Distribution, Estimation, modified moments method

الملخص:

في هذا البحث تم عتماد توزيع باريتو ذو المعلمتين وذالك لوفرة التطبيقات الاقتصاديه (توزيع العائد وتوزيع الربح) والتطبيقات الانسانيه وفي مجالات اخرى . تقدير معلمات التوزيع حصلنا عليها بأستخدام طريقتين هي طريقة العزوم المطوره وطريقة مقدر قمة دالة الترجيح . كما تم حساب تقريبات لمعدل متباين المقدرات باستخدام متسلسلات تايلر لغاية الرتبه الثانيه كما تم تقويم النتائج باستخدام طريقة مونتي كارلو للمحاكات كما تم نطبيق النتائج على بيانات عمليه تمثل الهزات الارضيه ونموذج وتخفيف التركيز في سائل .

1.Introduction

Pareto distribution is a power law probability distribution that coincides with social, scientific, geophysical and many other types of observable phenomena, [1].It is a continuous distribution bounded on the lower side and it has two parameters, shape and mode and it is a highly skewed distribution. It is a decreasing function and it has a finite value at the minimum value. It is a heavy tailed distribution meaning that a random variable following a Pareto distribution can have extreme values.The mode parameter for Pareto distribution sets the position of the “left edge” of the probability density function. The only outcomes that can be observed from this distribution are greater than or equal to the value of the mode parameter. Changes in the mode simply shift the boundary to the left or right. Pareto distribution named after the Italian economist and sociologist Vilfredo Pareto (1848-1923). It was proposed first by Pareto in (1897) at the university of Lausanne as a model for the distribution of incomes, [3].

Pareto discovered that at the high wealth range the wealth are distributed according to a power-law distribution. The parameters of Pareto distribution may change across societies. Pareto claimed that the wealth distribution obeys this general distribution law, which became known as the Pareto distribution or Pareto law. Pareto originally used his distribution to describe the allocation of wealth among individuals since it seemed to show rather well the way that a larger portion of the wealth of any society is owned by a smaller percentage of the people in that society.

This idea is sometimes expressed more simply as the Pareto principle or the "80-20 rule" which says that 20% of the population controls 80% of the wealth, [2]. This distribution is not limited to describe wealth or income, but many situations in which an equilibrium is found in the distribution of the "small" to the "large" observations. It has been used to study the tensile strength of nylon carpet fibers [6], the city population, occurrence of natural resources, the insurance risk, the business failures, the stock price fluctuations, internet traffic, the wind speed and oil field locations, [7].

Also, it has application in military areas and suitable for approximating the right tails of distribution with positive skewness, [15].

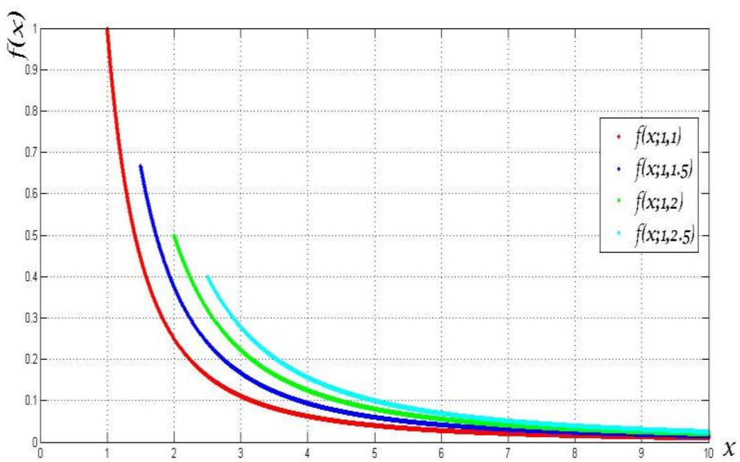
Harris C. in 1968, [8] used this distribution in determining times of maintenance service, Pickands J. in 1975, [13] was apparently the first who used Pareto distributions in the analysis of extreme flood events, Daragahi N. in 1989, [4] advocated the use of Pareto distribution for annual maxima of the wind speed and that of maximum floods of the Feather river, Smith R. in 1989, applied Pareto distribution to the study of ozone levels in the upper atmosphere, [14], Eldesoky E. in 2006, [5] derived some recurrence relation of single and product moments of order statistics from Pareto distribution.

2 Some Mathematical and Statistical Properties of Pareto Distribution [10]:

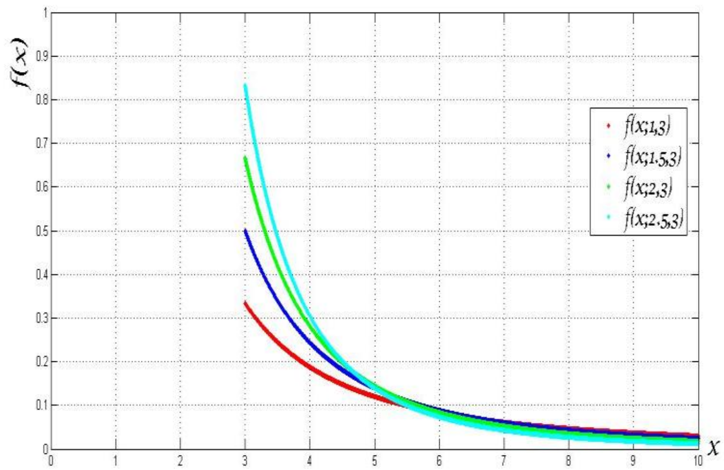
A continuous r.v. X is said to have Pareto distribution, denoted by X~ Par(α, β), if X has the following p.d.f.

$$f(x;\alpha,\beta) = \begin{cases} \alpha\beta^{\alpha}x^{-(\alpha+1)}, & \beta \le x < \infty \\ 0, & \text{e.w.} \end{cases} \quad \ldots\ldots (1)$$

where $\alpha > 0$, $\beta > 0$ known as the shape and scalar parameters respectively. The Pareto distribution depends on two parameters α and β and a variety of p.d.f. shapes can be generated by fixing the value α and letting β vary or fixing β and letting α vary. Figure (1) and figure (2) show some Pareto p.d.f. with fixed α and β varying and with fixed β and α varying respectively



Figure(1)Pareto p.d.f's with α=1 and β=1,1.5,2,2.5



Figure(2) Pareto p. d. f^{,s} with β=3 and α=1,1.5,2,2.5

The c.d.f. of Pareto distribution is given by:

$$F(x;\alpha,\beta) = \int_{\beta}^{x} f(t;\alpha,\beta)dt = \begin{cases} 0 & x \le \beta \\ 1-\left(\frac{\beta}{x}\right)^{\alpha} & \beta < x < \infty \\ 1 & x \to \infty \end{cases} \quad \ldots\ldots(2)$$

For Pareto distribution the r^{th} moment about the origin is given by

$$\mu_r = E(X^r) = \int_{\beta}^{\infty} x^r f(x;\alpha,\beta)dx$$

$$= \frac{\alpha\beta^r}{(\alpha-r)}, \quad \alpha > r \ldots\ldots(3)$$

Setting r=1,2 in the above equation one can have the following first two moments about the origin,[12]

$$\mu_1 = E(X) = \frac{\alpha\beta}{(\alpha-1)}, \ \alpha > 1, \mu_2' = E(X^2) = \frac{\alpha\beta^2}{(\alpha-2)}, \ \alpha > 2$$

The moment generating function of Pareto distribution does not exists. So direct expectation approach could be use to find the moments and higher degree moments and hence mean and variance for Pareto distribution is respectively given by

$$\mu = \frac{\alpha\beta}{\alpha-1}, \quad \alpha > 1 \ldots\ldots(4)$$

$$Var(X) = \sigma^2 = \frac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)}, \quad \alpha > 2 \ldots\ldots(5)$$

3 Estimation of Parameters for Pareto Distribution:

3.1 Estimation of Parameters by the Modified Moments method, [9]

For Pareto distribution, we have two unknown parameters α and β. So we take a r.s. of size n from Par(α,β) and let Y_1 to be the first order statistic of the sample, then according to the order statistics theory the first order statistic Y_1 has p.d.f.

$$g_1(y_1) = \begin{cases} n\alpha\beta^{n\alpha} y_1^{-(n\alpha+1)}, & \beta \le y_1 < \infty \\ 0\,, & \text{e.w.} \end{cases} \quad \ldots\ldots(6)$$

This shows that $Y_1 \sim Par(n\alpha, \beta)$, therefore

$$E(Y_1) = \frac{n\alpha\beta}{(n\alpha-1)} \quad \ldots\ldots(7)$$

Next, to apply the modified moments method, we set $\mu_1' = M_1$ and $E(Y_1) = Y_1$, at $\alpha = \hat{\alpha}$, $\beta = \hat{\beta}$ which leads to

$$\frac{\hat{\alpha}\hat{\beta}}{\hat{\alpha}-1} = \bar{X} \quad \ldots\ldots(8)$$

$$\frac{n\hat{\alpha}\hat{\beta}}{n\hat{\alpha}-1} = Y_1 \quad \ldots\ldots(9)$$

By solving the above equations, one can yield the following estimators:

$$\hat{\alpha} = \frac{n\bar{X}-Y_1}{n(\bar{X}-Y_1)} \quad \ldots\ldots(10)$$

$$\hat{\beta} = \frac{(n\hat{\alpha}-1)Y_1}{n\hat{\alpha}} \quad \ldots\ldots(11)$$

3.2 Estimation of Parameters by the Maximum Likelihood Method

For Pareto distribution, let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from Par(α,β), in this case, the likelihood function is

$$L(\alpha, \beta, \underline{x}) = f(\underline{x}, \alpha, \beta) = \prod_{i=1}^{n} f(x_i, \alpha, \beta)$$

$$= \alpha^n \beta^{n\alpha} \prod_{i=1}^{n} x_i^{-(\alpha+1)}.$$

Therefore

$$\ln L(\alpha, \beta, \underline{x}) = n\ln\alpha + n\alpha\ln\beta - (\alpha+1)\sum_{i=1}^{n} \ln x_i.$$

By differentiating the above equation with respect to α and then with respect to β, and by setting the resulting equations equal to zero one can get:

$$\frac{\partial \ln L}{\partial \beta}=\frac{n\alpha}{\beta}=0 \text{ at } \alpha=\hat{\alpha}\ ,\beta=\hat{\beta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

$$\frac{\partial \ln L}{\partial \alpha}=\frac{n}{\alpha}+n\ln\beta-\sum_{i=1}^{n}\ln x_i=0 \text{ at } \alpha=\hat{\alpha}\ ,\beta=\hat{\beta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (13)$$

Unfortunately $\frac{\partial \ln L}{\partial \beta}=\frac{n\alpha}{\beta}=0$ provide no solution so, we must select a value close to β as possible which maximize the likelihood function.

If $\beta<x_i<\infty\ \forall\, i=1,2,\ldots,n$, then $\beta<Y_1<Y_2<\cdots<Y_n<\infty$.

Therefore we can choose $Y_1=\min\{X_1,X_2,\ldots,X_n\}$ to be an estimator of β, that is

$$\hat{\beta}=Y_1=\min\{X_1,X_2,\ldots,X_n\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

Thus by substituting eq.(14) into eq.(13) one can get:

$$\hat{\alpha}=\frac{n}{\sum_{i=1}^{n}\ln\left(\frac{x_i}{\hat{\beta}}\right)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

4 Some Related Theorems to Pareto Distribution

We shall give some relations depending on Pareto distribution which we need in the work later.

1- If the r.v. X ~Par(α,β) then the r.v. Y = ln X has p.d.f.

$$g(y;\alpha,\beta)=\begin{cases}\alpha\beta^{\alpha}e^{-\alpha y}, & \ln\beta\le y<\infty\\ 0, & \text{e.w.}\end{cases}.$$

Proof:

Since X~Par(α,β) then the p.d.f. of X is by eq.(1).Then the function Y = lnX defines a one-to-one transformation that map the space $A=\{x:\beta\le x<\infty\}$ onto the space $B=\{y:\ln\beta\le y<\infty\}$ with the inverse transform $x=e^y$ and $J=\frac{\partial x}{\partial y}=e^y$ then the p.d.f. of Y is

$$g(y;\alpha,\beta)=f(e^y)|J|$$

and hence

$$g(y;\alpha,\beta)=\begin{cases}\alpha\beta^{\alpha}e^{-\alpha y}, & \ln\beta\le y<\infty\\ 0, & \text{e.w.}\end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Remark

The c.d.f. of Y is given by

$$G(y;\alpha,\beta)=\begin{cases}0, & y\le \ln\beta \\ 1-\left(\frac{\beta}{e^{y}}\right)^{n\alpha}, & \ln\beta<y<\infty \\ 1, & y\to\infty\end{cases} \quad \ldots\ldots(17)$$

and the m.g.f. is given by

$$M_Y(t)=E(e^{ty})=\frac{\alpha\beta^{t}}{\alpha-t} \qquad t<\alpha \quad \ldots\ldots(18)$$

with

$$\mu_Y=\ln\beta+\frac{1}{\alpha} \quad \ldots\ldots(19)$$

and

$$var(Y)=\frac{1}{\alpha^{2}} \quad \ldots\ldots(20)$$

2-Let $X_1, X_2, \ldots, X_n$ be a r.s. of size n from Par(α,β). Then :-

1-The first order statistics $Y_1=\min(X_1,X_2,\ldots,X_n)\sim Par(n\alpha,\beta)$.

2-The limiting distribution of Y_1 is Deg(β).

Proof:

1- The p.d.f. and c.d.f. of Pareto sample are given by eq.(1) and eq.(2) respectively and from order statistic theory one can have:

$$g_1(y_1)=\begin{cases}n\alpha\beta^{n\alpha}y_1^{-(n\alpha+1)}, & \beta\le y_1\le\infty \\ 0, & e.w.\end{cases}$$

that is $Y_1\sim Par(n\alpha,\beta)$.

2-Since $Y_1\sim Par(n\alpha,\beta)$, the c.d.f. of Y_1 is

$$G_1(y_1)=\begin{cases}0 & y_1\le\beta \\ 1-\left[\frac{\beta}{y_1}\right]^{n\alpha} & \beta<y_1<\infty \\ 1 & y_1\to\infty\end{cases}$$

Therefore

$$\lim_{n\to\infty}G_1(y_1)=\begin{cases}0 & y_1\le\beta \\ 1 & y_1>\beta\end{cases}$$

This shows that the limiting distribution of Y_1 is Deg(β).

Remark

We note that,

1-The mean and variance of r.v. $\hat{\beta} = Y_1$ that is described eq.(6) are respectively

$$E(Y_1) = \frac{n\alpha\beta}{(n\alpha-1)} \quad \ldots\ldots(21)$$

$$Var(Y_1) = \frac{n\alpha\beta^2}{(n\alpha-1)^2(n\alpha-2)} \quad \ldots\ldots(22)$$

2-Since The limiting distribution of Y_1 is Deg(β) then Y_1 converge stochastically to β.

4 Approximation to the Mean and Variance of the Modified Moments Estimators:

We shall approximate the mean and variance of the estimators that obtained by using the modified moments method by considering the mean and variance of the Taylor series expansion of the function g (X,Y) at point (μ_X, μ_Y) up to second order which is given by, [11]:

$$E[g(X,Y)] = g(\mu_X, \mu_Y) + \frac{1}{2}Var[X]\frac{\partial^2}{\partial x^2}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}} + \frac{1}{2}Var[Y]\frac{\partial^2}{\partial Y^2}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}} + Cov[X,Y]\left[\frac{\partial^2}{\partial X\,\partial Y}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right] \ldots\ldots(23)$$

$$Var[g(X,Y)] = Var[X]\left[\frac{\partial}{\partial x}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right]^2 + Var[Y]\left[\frac{\partial}{\partial Y}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right]^2 + 2Cov[X,Y]\left[\frac{\partial}{\partial x}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right]\left[\frac{\partial}{\partial Y}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right] \ldots\ldots(24)$$

By setting $X = \overline{X}$ and $Y = Y_1$ in eq.(23) and eq.(24), one can get

$$E\left[g(\overline{X}, Y_1)\right] \cong g(\mu_{\overline{X}}, \mu_{Y_1}) + \frac{1}{2}Var[\overline{X}]\frac{\partial^2}{\partial \overline{X}^2}g(\overline{X}, Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}} + \frac{1}{2}Var[Y_1]\frac{\partial^2}{\partial {Y_1}^2}g(\overline{X}, Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}} + Cov[\overline{X}, Y_1]\left[\frac{\partial^2}{\partial \overline{X}\,\partial Y_1}g(\overline{X}, Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}\right] \ldots\ldots(25)$$

and

$$\mathrm{Var}\left[g\left(\overline{X}, Y_1\right)\right] =$$
$$\mathrm{Var}[\overline{X}]\left[\frac{\partial}{\partial \overline{X}} g\left(\overline{X}, Y_1\right)\Big|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}}\right]^2 + \mathrm{Var}[Y_1]\left[\frac{\partial}{\partial Y_1} g\left(\overline{X}, Y_1\right)\Big|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}}\right]^2 +$$
$$2\mathrm{Cov}[\overline{X}, Y_1]\left[\frac{\partial}{\partial \overline{X}} g\left(\overline{X}, Y_1\right)\Big|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}}\right]\left[\frac{\partial}{\partial Y_1} g\left(\overline{X}, Y_1\right)\Big|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}}\right] \ldots.(26)$$

We have the modified method estimators given by eq.(10) and eq.(11) which is:

$$\hat{\alpha} = \frac{n\overline{X} - Y_1}{n(\overline{X} - Y_1)}$$

$$\hat{\beta} = \frac{(n-1)\overline{X}Y_1}{n\overline{X} - Y_1}$$

Let $g\left(\overline{X}, Y_1\right) = \hat{\alpha} = \frac{n\overline{X} - Y_1}{n\left(\overline{X} - Y_1\right)}$

It is known that

$$\mu_{\overline{X}} = \mu = \frac{\alpha\beta}{(\alpha-1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(27)$$

and

$$\mathrm{Var}[\overline{X}] = \frac{\sigma^2}{n} = \frac{\alpha\beta^2}{n(\alpha-1)^2(\alpha-2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(28)$$

From eq.(21) and eq.(27) one can have:

$$g\left(\mu_{\overline{X}}, \mu_{Y_1}\right) = \frac{n\left[\frac{\alpha\beta}{(\alpha-1)}\right] - \left[\frac{n\alpha\beta}{(n\alpha-1)}\right]}{n\left[\frac{\alpha\beta}{(\alpha-1)} - \frac{n\alpha\beta}{(n\alpha-1)}\right]} = \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(29)$$

On the other hand

$$\frac{\partial g\left(\overline{X}, Y_1\right)}{\partial \overline{X}} = -\frac{(n-1)Y_1}{n(\overline{X} - Y_1)^2}$$

$$\frac{\partial g\left(\overline{X}, Y_1\right)}{\partial \overline{X}}\Big|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}} = -\frac{(n-1)}{n}\left[\frac{\frac{n\alpha\beta}{n\alpha-1}}{n\left(\frac{\alpha\beta}{\alpha-1} - \frac{n\alpha\beta}{n\alpha-1}\right)^2}\right] = -\frac{(n\alpha-1)(\alpha-1)^2}{(n-1)\alpha\beta}, \quad \ldots\ldots\ldots\ldots\ldots(30)$$

$$\frac{\partial^2 g\left(\overline{X}, Y_1\right)}{\partial \overline{X}^2} = \frac{2(n-1)Y_1}{n(\overline{X} - Y_1)^3}$$

$$\frac{\partial^2 g\left(\overline{X}, Y_1\right)}{\partial \overline{X}^2}\Big|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}} = \frac{2(n-1)\left[\frac{n\alpha\beta}{(n\alpha-1)}\right]}{n\left[\frac{\alpha\beta}{(\alpha-1)} - \frac{n\alpha\beta}{(n\alpha-1)}\right]^3} = \frac{2(\alpha-1)^3(n\alpha-1)^2}{(n-1)^2\alpha^2\beta^2}, \quad \ldots\ldots\ldots\ldots(31)$$

$$\frac{\partial g(\overline{X}, Y_1)}{\partial Y_1} = \frac{(n-1)}{n}\left[\frac{\overline{X}}{(\overline{X}-Y_1)^2}\right]$$

$$\left.\frac{\partial g(\overline{X}, Y_1)}{\partial Y_1}\right|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}} = \frac{(n-1)}{n}\left[\frac{\frac{\alpha\beta}{\alpha-1}}{\left(\frac{\alpha\beta}{\alpha-1}-\frac{n\alpha\beta}{n\alpha-1}\right)^2}\right] = \frac{(\alpha-1)(n\alpha-1)^2}{n(n-1)\alpha\beta}, \quad \text{....................(32)}$$

$$\frac{\partial^2 g(\overline{X}, Y_1)}{\partial Y_1^2} = \frac{2(n-1)\overline{X}}{n(\overline{X}-Y_1)^3}$$

$$\left.\frac{\partial^2 g(\overline{X}, Y_1)}{\partial Y_1^2}\right|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}} = \frac{2(n-1)\left[\frac{\alpha\beta}{(\alpha-1)}\right]}{n\left[\frac{\alpha\beta}{(\alpha-1)}-\frac{n\alpha\beta}{(n\alpha-1)}\right]^3} = \frac{2(\alpha-1)^2(n\alpha-1)^3}{n(n-1)^2\alpha^2\beta^2} \quad \text{....................(33)}$$

and

$$\frac{\partial^2 g(\overline{X}, Y_1)}{\partial \overline{x}\, \partial Y_1} = \frac{(n-1)}{n}\left[\frac{-2\overline{X}}{(\overline{X}-Y_1)^3}+\frac{1}{(\overline{X}-Y_1)^2}\right] = \frac{(n-1)}{n}(\overline{X}-Y_1)^{-2}\left[1-\frac{2\overline{X}}{\overline{X}-Y_1}\right]$$

$$= \frac{-(n-1)}{n}(\overline{X}-Y_1)^{-2}\left[\frac{\overline{X}+Y_1}{\overline{X}-Y_1}\right]$$

$$\left.\frac{\partial^2 g(\overline{X}, Y_1)}{\partial \overline{x}\, \partial Y_1}\right|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}} = \frac{-(n-1)}{n}\left[\frac{\alpha\beta}{(\alpha-1)}-\frac{n\alpha\beta}{(n\alpha-1)}\right]^{-2}\left[\frac{\frac{\alpha\beta}{(\alpha-1)}+\frac{n\alpha\beta}{(n\alpha-1)}}{\frac{\alpha\beta}{(\alpha-1)}-\frac{n\alpha\beta}{(n\alpha-1)}}\right] = \frac{-(\alpha-1)^2(n\alpha-1)^2(2n\alpha-n-1)}{n\alpha^2\beta^2(n-1)^2} \text{.....................(34)}$$

In general $\overline{X}$ converge stochastically to $\mu_{\overline{X}}$ and we prove that Y_1 converge stochastically to β. So $\overline{X}Y_1$ converge stochastically to $\mu_{\overline{X}}\beta$, [23], therefore

$$E[\overline{X}Y_1] \cong \mu_{\overline{X}}\beta = \frac{\alpha\beta^2}{\alpha-1} \quad \text{..........................(35)}$$

and hence

$$Cov[\overline{X}, Y_1] \cong E[\overline{X}Y_1] - E[\overline{X}]E[Y_1] = \frac{\alpha\beta^2}{\alpha-1} - \frac{\alpha\beta}{(\alpha-1)}\frac{n\alpha\beta}{(n\alpha-1)}$$

$$= \frac{-\alpha\beta^2}{(\alpha-1)(n\alpha-1)} \quad \text{.............................(36)}$$

By substituting eq.'s(22), (28),(29),(31), (33), (34) and (36) in eq.(25), one can get the approximation to the expected value of $\hat{\alpha}$ which is given by:

$$E[\hat{\alpha}] \cong E\left[\frac{n\overline{X}-Y_1}{n(\overline{X}-Y_1)}\right] = \alpha + \frac{1}{2}\left[\frac{\alpha\beta^2}{n(\alpha-1)^2(\alpha-2)}\right]\left[\frac{2(\alpha-1)^3(n\alpha-1)^2}{(n-1)^2\alpha^2\beta^2}\right] +$$

$$\frac{1}{2}\left[\frac{n\alpha\beta^2}{(n\alpha-1)^2(n\alpha-2)}\right]\left[\frac{2(\alpha-1)^2(n\alpha-1)^3}{n(n-1)^2\alpha^2\beta^2}\right]+$$

$$\left[\frac{-\alpha\beta^2}{(\alpha-1)(n\alpha-1)}\right]\left[\frac{-(\alpha-1)^2(n\alpha-1)^2(2n\alpha-n-1)}{n\alpha^2\beta^2(n-1)^2}\right]$$

$$=\alpha+\frac{(\alpha-1)(n\alpha-1)}{\alpha(n-1)^2}\left[\frac{n\alpha-1}{n(\alpha-2)}+\frac{\alpha-1}{n\alpha-2}+\frac{2n\alpha-n-1}{n}\right]\dots\dots\dots\dots\dots\dots\dots\dots.(37)$$

Moreover

$$\lim_{n\to\infty} E(\widehat{\alpha})=\alpha.$$

This shows that $\widehat{\alpha}$ is asymptotically unbiased estimator of α.

By substituting eq.'s (28), (30), (22), (310) and (36) in eq.(26), one can obtain the approximation to variance of $\widehat{\alpha}$ which is given by:

$$Var[\widehat{\alpha}]\cong Var\left[\frac{n\bar{x}-y_1}{n(\bar{x}-y_1)}\right]$$

$$=\left[\frac{\alpha\beta^2}{n(\alpha-1)^2(\alpha-2)}\right]\left[-\frac{(n\alpha-1)(\alpha-1)^2}{(n-1)\alpha\beta}\right]^2+$$

$$\left[\frac{n\alpha\beta^2}{(n\alpha-1)^2(n\alpha-2)}\right]\left[\frac{(\alpha-1)(n\alpha-1)^2}{n(n-1)\alpha\beta}\right]^2+$$

$$2\left[\frac{-\alpha\beta^2}{(\alpha-1)(n\alpha-1)}\right]\left[-\frac{(n\alpha-1)(\alpha-1)^2}{(n-1)\alpha\beta}\right]\left[\frac{(\alpha-1)(n\alpha-1)^2}{n(n-1)\alpha\beta}\right]$$

and hence

$$Var[\widehat{\alpha}]=\frac{(n\alpha-1)^2(\alpha-1)^2}{n\alpha(n-1)^2}\left[\frac{1}{\alpha-2}+\frac{1}{n\alpha-2}+2\right]\dots\dots\dots\dots\dots\dots\dots\dots\dots(38)$$

Next, let

$$g(\bar{X},Y_1)=\hat{\beta}=\frac{(n-1)\bar{X}Y_1}{n\bar{X}-Y_1}$$

From eq.(21), eq.(27) and the above equation one can have:

$$g(\mu_{\bar{X}},\mu_{Y_1})=\frac{(n-1)\left(\frac{\alpha\beta}{\alpha-1}\right)\left(\frac{n\alpha\beta}{n\alpha-1}\right)}{\left(\frac{n\alpha\beta}{\alpha-1}\right)-\left(\frac{n\alpha\beta}{n\alpha-1}\right)}=\beta\ \dots\dots\dots\dots\dots\dots(39)$$

On the other hand

$$\frac{\partial g(\overline{X}, Y_1)}{\partial \overline{X}} = -\frac{(n-1)Y_1^2}{(n\overline{X} - Y_1)^2}$$

$$\left.\frac{\partial g(\overline{X}, Y_1)}{\partial \overline{X}}\right|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}} = -\frac{(n-1)\left[\frac{n\alpha\beta}{n\alpha-1}\right]^2}{\left[\frac{n\alpha\beta}{\alpha-1} - \frac{n\alpha\beta}{n\alpha-1}\right]^2} = -\frac{(\alpha-1)^2}{(n-1)\alpha^2}, \ldots\ldots(40)$$

$$\frac{\partial^2 g(\overline{X}, Y_1)}{\partial \overline{X}^2} = \frac{2n(n-1)Y_1^2}{\left[n\overline{X} - Y_1\right]^3}$$

$$\left.\frac{\partial^2 g(\overline{X}, Y_1)}{\partial \overline{X}^2}\right|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}} = \frac{2n(n-1)\left[\frac{n\alpha\beta}{n\alpha-1}\right]^2}{\left[\frac{n\alpha\beta}{\alpha-1} - \frac{n\alpha\beta}{n\alpha-1}\right]^3} = \frac{2(n\alpha-1)(\alpha-1)^3}{n(n-1)^2\alpha^4\beta}, \ldots\ldots(41)$$

$$\frac{\partial g(\overline{X}, Y_1)}{\partial Y_1} = \frac{n(n-1)\overline{X}^2}{\left[n\overline{X} - Y_1\right]^2}$$

$$\left.\frac{\partial g(\overline{X}, Y_1)}{\partial Y_1}\right|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}} = \frac{n(n-1)\left[\frac{\alpha\beta}{\alpha-1}\right]^2}{\left[\frac{n\alpha\beta}{\alpha-1} - \frac{n\alpha\beta}{n\alpha-1}\right]^2} = \frac{(n\alpha-1)^2}{n(n-1)\alpha^2}, \ldots\ldots(42)$$

$$\frac{\partial^2 g(\overline{X}, Y_1)}{\partial Y_1^2} = \frac{2n(n-1)\overline{X}^2}{\left[n\overline{X} - Y_1\right]^3}$$

$$\left.\frac{\partial^2 g(\overline{X}, Y_1)}{\partial Y_1^2}\right|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}} = \frac{2n(n-1)\left[\frac{\alpha\beta}{\alpha-1}\right]^2}{\left[\frac{n\alpha\beta}{\alpha-1} - \frac{n\alpha\beta}{n\alpha-1}\right]^3} = \frac{2(\alpha-1)(n\alpha-1)^3}{n^2(n-1)^2\alpha^4\beta} \ldots\ldots(43)$$

and

$$\frac{\partial^2 g(\overline{X}, Y_1)}{\partial \overline{x}\,\partial Y_1} = \frac{-2n(n-1)\overline{X}Y_1}{\left[n\overline{X} - Y_1\right]^3}$$

$$\left.\frac{\partial^2 g(\overline{X}, Y_1)}{\partial \overline{x}\,\partial Y_1}\right|_{\substack{\mu_{\overline{X}} \\ \mu_{Y_1}}} = \frac{-2n(n-1)\frac{n\alpha^2\beta^2}{(n\alpha-1)(\alpha-1)}}{\left[\frac{n\alpha\beta}{\alpha-1} - \frac{n\alpha\beta}{n\alpha-1}\right]^3} = \frac{-2(\alpha-1)^2(n\alpha-1)^2}{n(n-1)^2\alpha^4\beta} \ldots\ldots(44)$$

Substituting eq.'s(28), (36), (22), (39), (41), (43) and (44) in eq.(25), one can get the approximation to the expected value of $\hat{\beta}$ which is given by:

$$E[\hat{\beta}] \cong E\left[\frac{(n-1)\overline{X}Y_1}{n\overline{X} - Y_1}\right] = \beta + \frac{1}{2}\left[\frac{\alpha\beta^2}{n(\alpha-1)^2(\alpha-2)}\right]\left[\frac{2(n\alpha-1)(\alpha-1)^3}{n(n-1)^2\alpha^4\beta}\right] +$$

$$\frac{1}{2}\left[\frac{n\alpha\beta^2}{(n\alpha-1)^2(n\alpha-2)}\right]\left[\frac{2(\alpha-1)(n\alpha-1)^3}{n^2(n-1)^2\alpha^4\beta}\right] +$$

$$\left[\frac{-\alpha\beta^2}{(\alpha-1)}\right]\left[\frac{-2(\alpha-1)^2(n\alpha-1)^2}{n(n-1)^2\alpha^4\beta}\right].$$

hence

$$E[\hat{\beta}] = \beta + \frac{\beta(\alpha-1)(n\alpha-1)}{n\alpha^3(\alpha-2)(n-1)^2}\left[\frac{1}{\alpha-2}+\frac{1}{n\alpha-2}+2\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (45)$$

Moreover

$$\lim_{n\to\infty} E(\hat{\beta}) = \beta$$

This shows that $\hat{\beta}$ is asymptotically unbiased estimator of β.

By substituting eq.'s (28), (40), (22), (42), and (36) in eq.(26), one can obtain:

$$\mathrm{Var}[\hat{\beta}] \cong \mathrm{Var}\left[\frac{(n-1)\bar{X}Y_1}{n\bar{X}-Y_1}\right] = \left[\frac{\alpha\beta^2}{n(\alpha-1)^2(\alpha-2)}\right]\left[-\frac{(\alpha-1)^2}{(n-1)\alpha^2}\right]^2 + \left[\frac{n\alpha\beta^2}{(n\alpha-1)^2(n\alpha-2)}\right]\left[\frac{(n\alpha-1)^2}{n(n-1)\alpha^2}\right]^2 + 2\left[\frac{-\alpha\beta^2}{(\alpha-1)(n\alpha-1)}\right]\left[-\frac{(\alpha-1)^2}{(n-1)\alpha^2}\right]\left[\frac{(n\alpha-1)^2}{n(n-1)\alpha^2}\right]$$

hence

$$\mathrm{Var}[\hat{\beta}] = \frac{\beta^2}{n\alpha^3(n-1)^2}\left[\frac{(\alpha-1)^2}{(\alpha-2)}+\frac{(n\alpha-1)^2}{(n\alpha-2)}+2(\alpha-1)(n\alpha-1)\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (46)$$

5 Approximation to the Mean and Variance of the Maximum Likelihood Estimators

We have the maximum likelihood estimators given by eq. (14) and eq.(15) which is:

$$\hat{\beta} = Y_1 = \min\{X_1, X_2, \ldots, X_n\}$$

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \ln\left(\frac{x_i}{\hat{\beta}}\right)}$$

From the previous work the first order statistic $Y_1 \sim \mathrm{Par}(n\alpha, \beta)$

Therefore

$$E[\hat{\beta}] = E[Y_1] = \frac{n\alpha\beta}{(n\alpha-1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47)$$

Moreover

$$\lim_{n\to\infty} E[\hat{\beta}] = \lim_{n\to\infty} \frac{n\alpha\beta}{(n\alpha-1)} = \beta.$$

This shows that $\hat{\beta}$ is asymptotically unbiased estimator for β.

Moreover

$$\mathrm{Var}[\hat{\beta}] = \mathrm{Var}[Y_1] = \frac{n\alpha\beta^2}{(n\alpha-1)^2(n\alpha-2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (48)$$

Since

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \ln\left(\frac{x_i}{\hat{\beta}}\right)} = \frac{n}{\sum_{i=1}^{n} \ln(x_i) - n\ln y_1} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} \ln(x_i) - \ln y_1}.$$

By setting $Z_i = \ln X_i$ and $W = \ln Y_1$ one can get:

$$\hat{\alpha} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} z_i - w} = \frac{1}{\bar{z} - w}$$

By setting $X = \bar{Z}$ and $Y = W$ in eq.(23) and eq.(24), one can get:

$$E[g(\bar{Z}, W)] \cong g(\mu_{\bar{Z}}, \mu_W) + \frac{1}{2}\mathrm{Var}[\bar{Z}] \frac{\partial^2}{\partial \bar{Z}^2} g(\bar{Z}, W)\Big|_{\substack{\mu_{\bar{Z}} \\ \mu_W}} +$$

$$\frac{1}{2}\mathrm{Var}[W] \frac{\partial^2}{\partial W^2} g(\bar{Z}, W)\Big|_{\substack{\mu_{\bar{Z}} \\ \mu_W}} +$$

$$\mathrm{Cov}[\bar{Z}, W] \left[\frac{\partial^2}{\partial \bar{X}\,\partial Y_1} g(\bar{Z}, W)\Big|_{\substack{\mu_{\bar{Z}} \\ \mu_W}}\right] \ldots\ldots\ldots\ldots\ldots\ldots (49)$$

and

$$\mathrm{Var}[g(\bar{Z}, W)] = \mathrm{Var}[\bar{Z}] \left[\frac{\partial}{\partial \bar{Z}} g(\bar{Z}, W)\Big|_{\substack{\mu_{\bar{Z}} \\ \mu_W}}\right]^2 + \mathrm{Var}[W] \left[\frac{\partial}{\partial W} g(\bar{Z}, W)\Big|_{\substack{\mu_{\bar{Z}} \\ \mu_W}}\right]^2 +$$

$$2\mathrm{Cov}(\bar{Z}, W) \left[\frac{\partial}{\partial \bar{Z}} g(\bar{Z}, W)\Big|_{\substack{\mu_{\bar{Z}} \\ \mu_W}}\right] \left[\frac{\partial}{\partial W} g(\bar{Z}, W)\Big|_{\substack{\mu_{\bar{Z}} \\ \mu_W}}\right] \ldots\ldots\ldots\ldots\ldots (50)$$

By using eq.(19) and eq.(20), one can obtain:

$$\mu_{\bar{Z}} = \ln\beta + \frac{1}{\alpha}, \ldots\ldots\ldots\ldots\ldots (51)$$

$$\mathrm{Var}(\bar{Z}) = \frac{1}{n\alpha^2}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52)$$

$$\mu_W = \ln\beta + \frac{1}{n\alpha}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

and

$\mathrm{Var}(W) = \frac{1}{n^2\alpha^2}$ ……………………………………...………(54)

W has c.d.f as given by equation (17), that is

$$H(w) = \begin{cases} 0 & w \le \ln\beta \\ 1 - \left(\frac{\beta}{e^w}\right)^{n\alpha} & \ln\beta < w < \infty \\ 1 & w \to \infty \end{cases} \quad \text{………………………...(55)}$$

This shows that the r.v. W is converges stochastically to lnβ.

Let $g(\bar{z}, w) = \hat{\alpha} = \frac{1}{\bar{z} - w}$

From eq.(51) and eq.(52), one can obtain:

$$g(\mu_{\bar{z}}, \mu_w)\Big|_{\substack{\mu_{\bar{z}} \\ \mu_w}} = \frac{1}{\ln\beta + \frac{1}{\alpha} - \ln\beta + \frac{1}{n\alpha}} = \frac{n\alpha}{n-1} \quad \dots\dots\dots\dots\dots\dots\dots (56)$$

On the other hand

$$\frac{\partial g(\bar{Z}, W)}{\partial \bar{Z}} = \frac{-1}{(\bar{Z} - W)^2}$$

$$\frac{\partial g(\bar{Z}, W)}{\partial \bar{Z}}\Bigg|_{\substack{\mu_{\bar{z}} \\ \mu_w}} = \frac{-1}{(\ln\beta + \frac{1}{\alpha} - \ln\beta + \frac{1}{n\alpha})^2} = \frac{-n^2\alpha^2}{(n-1)^2}, \quad \dots\dots\dots\dots\dots (57)$$

$$\frac{\partial^2 g(\bar{Z}, W)}{\partial \bar{Z}^2} = \frac{2}{(\bar{Z} - W)^3}$$

$$\frac{\partial^2 g(\bar{Z}, W)}{\partial \bar{Z}^2}\Bigg|_{\substack{\mu_{\bar{z}} \\ \mu_w}} = \frac{2}{(\ln\beta + \frac{1}{\alpha} - \ln\beta + \frac{1}{n\alpha})^3}$$
$$= \frac{2n^3\alpha^3}{(n-1)^3}, \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (58)$$

$$\frac{\partial g(\bar{Z}, W)}{\partial W} = \frac{1}{(\bar{Z} - W)^2}$$

$$\frac{\partial g(\bar{Z}, W)}{\partial W}\Bigg|_{\substack{\mu_{\bar{z}} \\ \mu_w}} = \frac{1}{(\ln\beta + \frac{1}{\alpha} - \ln\beta + \frac{1}{n\alpha})^2} = \frac{n^2\alpha^2}{(n-1)^2}, \quad \dots\dots\dots\dots\dots (59)$$

$$\frac{\partial^2 g(\bar{Z}, W)}{\partial W^2} = \frac{2}{(\bar{Z} - W)^3}$$

$$\left.\frac{\partial^2 g(\overline{Z},W)}{\partial W^2}\right|_{\substack{\mu_{\overline{Z}}\\ \mu_w}} = \frac{2}{(\ln\beta+\frac{1}{\alpha}-\ln\beta+\frac{1}{n\alpha})^3} = \frac{2n^3\alpha^3}{(n-1)^3}, \ldots\ldots\ldots\ldots (60)$$

and

$$\frac{\partial^2 g(\overline{Z},W)}{\partial\overline{Z}\,\partial W} = \frac{-2}{(\overline{Z}-W)^3}$$

$$\left.\frac{\partial^2 g(\overline{Z},W)}{\partial\overline{Z}\,\partial W}\right|_{\substack{\mu_{\overline{Z}}\\ \mu_w}} = \frac{-2}{(\ln\beta+\frac{1}{\alpha}-\ln\beta+\frac{1}{n\alpha})^3} = \frac{-2n^3\alpha^3}{(n-1)^3} \ldots\ldots\ldots\ldots (61)$$

Since $\overline{Z}$ converge stochastically to $\mu_{\overline{Z}}$ and W converge stochastically to lnβ. So $\overline{Z}W$ converge stochastically to $\mu_{\overline{Z}}\ln\beta$, therefore

$$E[\overline{Z}W] \cong \mu_{\overline{Z}}\ln\beta = \left[\ln\beta+\frac{1}{\alpha}\right]\ln\beta$$

hence

$$Cov[\overline{Z},W] = E[\overline{Z}W]-E[\overline{Z}]E[W] \cong \left(\ln\beta+\frac{1}{\alpha}\right)\ln\beta-\left(\ln\beta+\frac{1}{\alpha}\right)\left(\ln\beta+\frac{1}{n\alpha}\right)$$

hence

$$Cov[\overline{Z},W] = \frac{-(\alpha\ln\beta+1)}{n\alpha^2} \ldots\ldots\ldots\ldots (62)$$

By substituting eq.'s(52), (54), (56), (58), (60), (61) and (62) in eq.(49), one can get

$$E[\hat{\alpha}] \cong E[g(\overline{Z},W)]$$
$$= \frac{n\alpha}{n-1}+\frac{1}{2}\left[\frac{1}{n\alpha^2}\right]\left[\frac{2n^3\alpha^3}{(n-1)^3}\right]+\frac{1}{2}\left[\frac{1}{n^2\alpha^2}\right]\left[\frac{2n^3\alpha^3}{(n-1)^3}\right] + \left[\frac{-(\alpha\ln\beta+1)}{n\alpha^2}\right]\left[\frac{-2n^3\alpha^3}{(n-1)^3}\right]$$

hence

$$E[\hat{\alpha}] = \alpha\left[1+\frac{1}{n-1}\right] + \frac{n\alpha(3n+1+2n\alpha\ln\beta)}{(n-1)^3} \ldots\ldots\ldots\ldots (63)$$

Moreover

$$\lim_{n\to\infty} E(\hat{\alpha}) = \alpha.$$

This shows that $\hat{\alpha}$ is asymptotically unbiased estimator of α.

By substituting eq.'s (52),(54),(57),(59) and (62) in eq.(50), one can get:

$$\mathrm{Var}[\hat{\alpha}] = \mathrm{Var}\left[g\left(\overline{Z}, W\right)\right]$$

$$= \left[\frac{1}{n\alpha^2}\right]\left[\frac{-n^2\alpha^2}{(n-1)^2}\right]^2 + \left[\frac{1}{n^2\alpha^2}\right]\left[\frac{n^2\alpha^2}{(n-1)^2}\right]^2 + 2\left[\frac{-(\alpha\ln\beta+1)}{n\alpha^2}\right]\left[\frac{-n^2\alpha^2}{(n-1)^2}\right]\left[\frac{n^2\alpha^2}{(n-1)^2}\right]$$

hence

$$\mathrm{Var}[\hat{\alpha}] = \frac{n^2\alpha^2(3n + 2n\alpha\ln\beta + 1)}{(n-1)^4} \quad \ldots \ldots (64)$$

6 Procedure for Generating Random Variates of Pareto Distribution:

We shall consider one procedure for generating random variates from Pareto distribution depend on the inverse transform method.

The c.d.f. of Pareto distribution is given by eq.(2), by setting u=F(x;α, β) this implies that

$$x = \frac{\beta}{(1-u)^{\frac{1}{\alpha}}}$$

and can be summarized by giving the following algorithm:

Algorithm(P-1):

1- Read α, β where α, β >0.
2- Generated U from $U\left(0,1\right)$.
3- Set $X = \frac{\beta}{(1-U)^{\frac{1}{\alpha}}}$.
4- Deliver X as a r.v. generated from Par(α,β).
5- Stop.

7 Some Applications Related the to Pareto Distribution

7.1 Earthquake

In this application we consider the magnitude of earthquake in Japan where each r.v. is the magnitude of earthquake with the assumption that the r.v.'s follow Pareto distribution. For this, we calculate $\hat{\alpha}$ and $\hat{\beta}$ by two methods and then using Chi-square test to find the value of Y and then compare it with the tabulated values of Chi-square distribution to find the values of significant level in which our assumption is true.

Japan:

For this application the r.v. is the magnitude of earthquake in Japan as shown in table(1), where n=52.

Table(1) represents the magnitude of earthquakes in Japan.

8.4	7.9	8.3	7.1	7.5	8.25	8.5	7.8	8	8
8.6	8.4	8.4	6.9	7	7.1	8	6.6	8.5	8.3
7.6	8.4	7.2	6.8	8.1	7.1	7.6	7.7	6.8	7.5
6.9	7	7.2	8.3	8.1	6.9	6.6	6.9	6.9	7.1
6.5	6.6	7	7.4	7.2	9	6.8	7.1	6.5	6.5
6.6	6.8								

(i) The maximum likelihood method

For this method, the estimation of the parameters are

$\hat{\alpha} = 7.4407, \ \hat{\beta} = 6.500$

The test statistic for this application is Y=3.5209 with 4 deg. of freedom. This value of Y is compared with the tabulated critical values of a given significance level of Chi-square test. Then we get table (2).

Table (2) represents the Critical values of a given significance level of Chi-square test and decisions for Finland.

Significance Level	0.20	0.10	0.05	0.02	0.01
Critical Value	5.9886	7.7794	9.4877	11.668	13.277
Decision	Accept	Accept	Accept	Accept	Accept

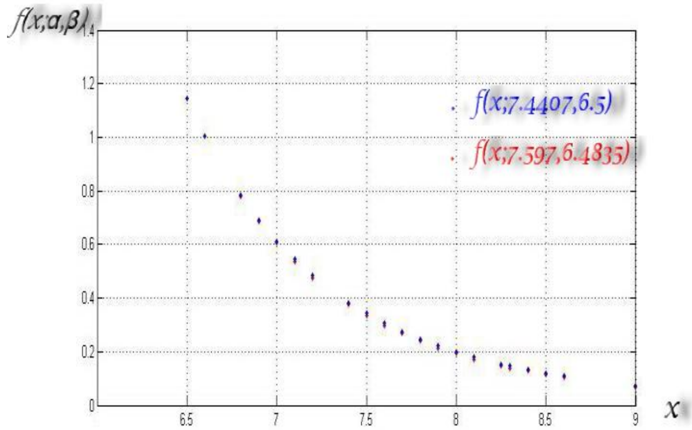
(ii) The modified moments method

For this method, the estimation of the parameters are

$\hat{\alpha} = 7.597, \ \hat{\beta} = 6.4835$

The test statistic for this application is Y=4.1536 with 4 deg. of freedom. This value of Y is compared with the tabulated critical values of a given significance level of Chi-square test. Then we get table (2).

The following figure shows the values of the p.d.f. at the values of r.v.'s $X_1, X_2, \dots, X_{52}$ that are given in table(1) .



Figure(3) the values of p.d.f. at specific values of $r.v.^{,s}$.

7.2 Dilution of Concentration in the Liquid

In an chemical experiment let n represent the number of doing the experiment of dilution of concentration in the liquid and consider the r.v. to be the new concentration.

Let k_i, $(i = 0,1,2, \dots, n)$ be the concentration and v_i represent the volume. The dilution experiment is based on the general dilution law which can be written as:

$$k_i = \frac{v_{i-1} k_{i-1}}{v_i} \quad i = 1,2, \dots, n$$

where k_0 is a given initial value of dilution and v_0 is a given initial value of the volume. We consider the $r.v^{,s}$ to be the dilution of saltiness in any liquid. Each r.v. is the new dilution with assumption that the $r.v^{,s}$ follow Pareto distribution. For this, we calculate $\hat{\alpha}$ and $\hat{\beta}$ by two methods and then using Chi-square test to find the value of Y and then compare it with the tabulated values of Chi-square distribution to find the values of significant level.

For $k_0 = 6$ and $v_0 = 2$ and n=59 the $r.v^{,s}$ are as shown in table(3).

Table(3) represents the values of concentration in the liquid.

6	4	3	2.4	2	1.7143	1.5	1.3333	1.2	1.0909
1	0.9231	0.8571	0.8	0.75	0.7059	0.6667	0.6316	0.6	0.5714
0.5455	0.5217	0.5	0.48	0.4615	0.4444	0.4286	0.4138	0.4	0.3871
0.3750	0.3636	0.3529	0.3429	0.3333	0.3243	0.3158	0.3077	0.3	0.2927
0.2857	0.2791	0.2727	0.2667	0.2609	0.2553	0.25	0.2499	0.24	0.2353
0.2308	0.2264	0.2222	0.2182	0.2143	0.2105	0.2069	0.2034	0.2	

(i) The maximum likelihood method

For this method, the estimation of the parameters are

$\hat{\alpha} = 1.1145, \quad \hat{\beta} = 0.2$

The test statistic for this application is Y=0.25006 with 5 deg. of freedom. This value of Y is compared with the tabulated critical values of a given significance level of Chi-square test. Then we get table (4).

Table(4) represents the critical values of a given significance level of Chi-square test and decisions for Greece.

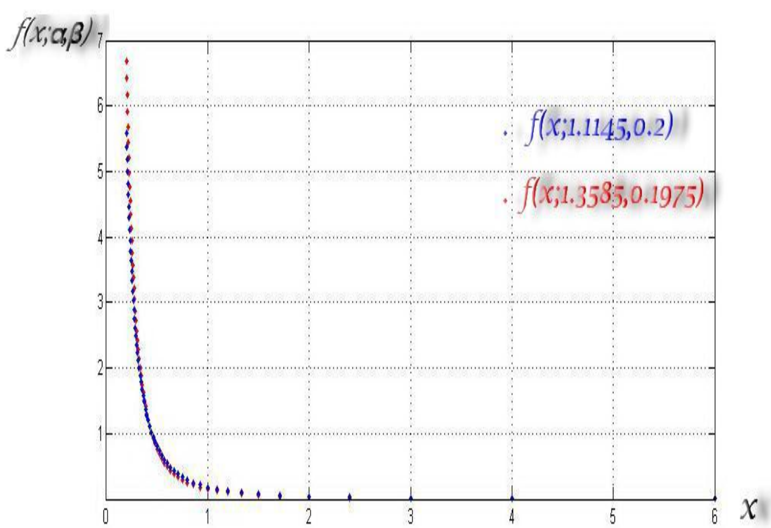
Significance Level	0.20	0.10	0.05	0.02	0.01
Critical Value	7.2893	9.2364	11.07	13.388	15.086
Decision	Accept	Accept	Accept	Accept	Accept

(ii) The modified moments method

For this method, the estimation of the parameters are

$\hat{\alpha} = 1.3585, \quad \hat{\beta} = 0.1975$

The test statistic for this application is Y=3.7149 with 5 deg. of freedom. This value of Y is compared with the tabulated critical values of a given significance level of chi-square test. Then we get table (4). The following figure shows the values of the p.d.f. at the values of $\mathrm{r.v.}^{,s}$ $X_1, X_2, \ldots, X_{59}$ that are given in table (3).



Fig(4) represents the values of p.d.f. at specific values of $\mathrm{r.v.}^{,s}$.

Conclusions

We can conclude from our study the following

1-The approximated mean and variance become more accurate if the higher order of approximation is used.

2-The earthquake and general dilution law show that both applications fit Pareto distribution for all significance level α.

3-This study can be extended for Pareto distribution of three parameters.

4-This study is applicable for any phenomena in which has an equilibrium of the distribution of small to large observations.

References

[1] Abdel Ghaly A., Attia A. and Aly H. , "Estimation of the Parameters of Pareto Distribution and the Reliability Function Using Accelerated Life Testing With Censoring", Communication Statistics, (1998), 27 (2):469-484.

[2] Arnold B., "Pareto Distributions", International Co- operative Publishing House, Maryland, (1983).

[3] Cirillo R., "The Economics of Vilfredo Pareto", Frank Case and Company Limited, London, (1979).

[4] Daragahi N., "On Tail Estimation An Improved Method", Math Geol, (1989), 21:829–842.

[5] Eldesoky E., "Order Statistic from Pareto Distribution", Journal of Applied Sciences, (2006), 10:2151-2157.

[6] Grimshaw, S. "Computing Maximum Likelihood Estimates for the Generalized Pareto Distribution", Technometrics, (1993), 35:185-191.

[7] Gyan P., "Some Estimators for the Pareto Distribution", Journal of Scientific Research, (2009), 2:236-247.

[8] Harris C., "The Pareto Distribution as a Queue Service Discipline", Operations Research, (1968), 16: 307-313.

[9] Johnson N. and Kotz S., "Continuous Univariate Distributions", John Wiley and Sons, New York, (1970).

[10] Lester D. "Probability and Mathematical Statistics", Harper and Row Publishers, New York, (1974).

[11] Levy H., "Approximating Expected Utility by a Function of Mean and Variance", American Economic Review, (1979), 69:308–317.

[12] Miyazima S., Lee Y., Nagamine T., and Miyajima H. "Power-law Distribution of Family Names in Japanese Societies", Physica, (2000), 278: 282-288.

[13] Pickands J., "Statistical Inference Using Extreme Order Statistics", Ann Stat, (1975), 3:119–131.

[14] Smith, R. "Extreme Value Analysis of Environmental Time Series: An Application to Trend Detection in Ground-Level Ozone", Statistical Science, (1989), 4:367-393.

[15] Steindle J., " Random Processes and the Growth of Firms a Study of the Pareto Law", Hafner, New York, (1965).