ON SOME REFLECTION MATROIDS

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Abstract:

The aim of this paper is to construct the reflection matroids that related to some types of reflection groups. The Coxetermatroid of the complexification of the Coxeter arrangements$(A(I_2(n)), n = 4)$,$(A(A_n), n \geq 3)$, $(A(B_n), n \geq 3)$, $(A(C_n), n \geq 3)$and$(A(D_n)$, $n = 4)$ were constructed. As well as, the reflection matroids of the complex reflection arrangements $A(G_{24}), A(G_{25})$, $A(G_{26})$ and $A(G_{27})$were computed.

KeyWords: Reflection arrangement, Hypersolvable arrangement, Supersolvable arrangement, Matroid theory, Reduced Homological group and, Cohen Macaulay ring.

الملخص:

الهدف من هذا البحث هو بناء الماترويداتالأنعكاسية العائدة لبعض الأنواع من الزمر الأنعكاسية .

الماترويدكوكستر للترتيبات كوكستر العقدية $(A(I_2(n)), n = 4)$, و $(A(A_n), n \geq 3)$ و $(A(B_n), n \geq 3)$ و $(A(C_n), n \geq 3)$ و $(A(D_n),\ n = 4)$ اعطى بنائها. كذلك الماترويدات العقدية الأنعكاسية للترتيبات العقدية الأنعكاسية $A(G_{24})$ و $A(G_{25})$ و$A(G_{26})$ و $A(G_{27})$ تم حسابها .

Introduction*:*

The study of polytopes appears to interest more different kinds of peoples than any other branch of Geometry. In (1948), Coxeter introduced the symmetry groups of the Platonic solid. He started by introducing two-dimensional polygons and three-dimensional polyhedra. He then gave a rigorous combinatorial definition of "regularity" and uses it to show that there are no other convex regular polyhedra apart from the five Platonic solid. The concept of "regularity" was extended to non-convex shapes such as star polygons, star polyhedra and then to tessellations, honeycombs and to polytopes in higher dimensions. Coxeter introduced and used the groups of reflections (that became known as Coxeter groups) to ensure his work. By a reflection he means an endomorphism on a finite dimensional Euclidean space, $s: \mathbb{R}^n \to \mathbb{R}^n$ that has a finite order and its fixed point set $\{x \in \mathbb{R}^n | s(x) = x\} = H_s$ is a hyperplane of $\mathbb{R}^n$, (i.e. H_s is a subspace of $\mathbb{R}^n$ has a codimension 1).A Coxeter group $G$is a discrete subgroup that is generated by a set of reflections of the general linear group $GL(\mathbb{R}^n)$.

The concept of a "reflection" has been extended to unitary space by Shephard (1953). A reflection in unitary space is a congruent transformation of finite period that leaves invariant every point of a certain prime (hyperplane), and it is characterized by the property that all but one of the characteristic roots of the matrix of transformation are equal to unity. The remaining root, if the reflection is of period *m*, is a primitive m^{th}root of unity. In (1954), Shaphard and Todd completely classified the finite complex reflection and published a list of all finite irreducible complex reflection groups (up to conjugucy). Furthermore, Shaphard and Todd determined the degree of the reflection groups, using the invariant theory of the corresponding collineation groups in the primitive case. In (1967),Coxeter gave presentations for all the n-dimensional finite reflection groups and presented number of graphs connected with complex reflection groups in attempt to systematize the results of Shaphard and Todd.

In 1889, Roberts gave a structure of an arrangement as an arbitrary finite set of lines in the plane and the number of regions that remained if we remove those lines from the plane was calculated. The field of hyperplane arrangements becomes increasingly popular during the next century. By a hyperplane arrangement (or for shorten arrangement) A, we mean a finite set of hyperplanes of a finite dimensional vector space V. One of the most essential problems in the topological studies in the field of arrangements is the computation of the homotopy type of the complement of an arrangement, $M(A) = V\setminus \bigcup_{H\in A} H$. The complement in a complex space had been studied by Fadell,Fox and Neuwirth (1962), in a connection of the Braid arrangement which forms a special kind of Coxeter arrangement $A(S)$ that associated to the symmetric group S, (i.e. $A(S)$ the set of all reflecting hyperplanes related to a certain root system that produce the Coxeter group S). They gave a presentation of the cohomology ring of the complement $M(A(S))$ as generators and relations.

In (1973), Brieskorn replaced the symmetric group and the Braid arrangement by a finite Coxeter group G and the complexification of its reflection arrangement in order to generalized the previous work and gave a presentation to the cohomological ring $H^*(M(A(G)))$ of the complement $M(A(G))$.Orlik and Solomon (1980) generalized Brieskorn results to construct a graded algebra A associated to any complex arrangement A and their description involves the intersection lattice $L(A) = \{X \subseteq V | X = \bigcup_B H \text{ and } B \subseteq A\}$of A which is partially orderedby the inclusions and ranked by $rk(X) = \text{codim}(X) = \dim(V) - \dim(X)$. They proved that A is isomorphic to the cohomological ring $H^*(M(A))$ of the complement $M(A)$.

For a given total order ⊴on the hyperplanesof A, a circuit $C \subseteq A$ is a minimal (with respect to inclusion) dependent set of A. We call $C = C\setminus H$ a broken circuit of C, if H is the smallest hyperplane in C via ⊴and by an NBC base $B \subseteq A$ we mean B contains no broken circuit.

Many basic facts about the linear arrangement A and their intersection boset $L(A)$, are best understand from the more general viewpoint of the matroid theory. A matroid is a pair $M = (A, \Delta)$, where A is a finite set and Δ is a non-empty collection of subsets of A called independent sets such that Δ forms a simplicial complex and every induced subcomplex of Δ is a pure, i.e. if $B \subseteq A$, the maximal elements of $\Delta \cap 2^B$ have the same cardinality, where $2^B = \{C \subseteq A \mid C \subseteq B\}$. With a finite matroid$M$ there associated several simplicial complexes that are interrelated in an appealing way. Such complexes are: the G-complex Δ, the broken circuit complex $NBC_{\trianglelefteq}(M)$ and the reduced broken circuit complex $\overline{NBC_{\trianglelefteq}}(M)$ via a fixed ordering $\trianglelefteq$ of the underlying set A of M. In particular, the broken circuit complex carries the chromatic properties of M. The homology of geometric lattice complexes was firstly determined by Folkman (1966). Orlik and Solomon in (1980), showed that the cohomology ring of the complement of a complex arrangement of hyperplanes can be described entirely in terms of the order homology of the geometric lattice of intersections. Accordingly, the geometric lattice homology is related to interesting applications of matroids within mathematics.

The aim of this paper is to study the reflection matroids of some complex reflection arrangements. This study will organize as follows:

- In section one we review some basic facts that we needed in our work.
- The hypersolvable class of arrangements, was originally introduced by Jambu and Papadima in (1998), and (2002) as a generalization of supersolvable (Stanly) class (1972). In section one, we looked more closely at a construction given in [1] of a partition of an arrangement A is called "hypersolvabe partitions",$\prod = (\prod_1, \prod_2, \ldots, \prod_\ell)$, in order to apply its structure on the supersolvable Coxeter arrangements.
- The Sections (2) and (3) are devoted to construct the reflection matroidsof $(A(I_2(n)), n = 3,4)$,$(A(A_n), n \geq 3)$, $(A(B_n), n \geq 3)$, $(A(C_n), n \geq 3)$, $(A(D_n), n = 4)$, $A(G_{24}), A(G_{25})$, $A(G_{26})$ and $A(G_{27})$. Appendix (1), included a Maple program to compute the lattice intersection that preserve the order that we fixed it to each one of their arrangements. Consequently, the G-complex, the broken circuit complex, the reduced broken circuit complex and free minimal resolution for each one of them are constructed. As well as, the Hilbert function and the h-vector of those matroids are computed by using a Maple program given in appendix (2) and we use the Maple program given in appendix (3) in order to compute the f-vector to the broken circuit complexes of $(A(I_2(n)), n \geq 3)$,$(A(A_n), n \geq 3)$, $(A(B_n), n \geq 3)$ and $(A(C_n), n \geq 3)$ as an application to theorem(1.3) that given in [3]. We mentioned that the programs in appendix (2) and (3) were introducedfirstly by Fadhil (2012).

(1) Preliminaries:

In this section we review some basic definitions and facts of the notion of "Matroids", so we will start with the following:

Definition (1.1): Al-Ta'ai et al .(2010)

A "finite" *matroid* is a pair $M = (A, \Delta)$, where A is a finite set and Δ is a collection of subsets of A, satisfying the following axioms:

1. $\Delta$is a non-empty (abstract) simplicial complex, i.e. $\Delta \neq \emptyset$ and if $\Delta' \in \Delta$ and $\Delta'' \subset \Delta'$, then $\Delta'' \in \Delta$.

2. Every induced sub-complex of Δ is a pure, i.e. if $B \subseteq A$, the maximal elements of $\Delta \cap 2^B$ have the same cardinality, where $2^B = \{C \subseteq A \mid C \subseteq B\}$.

The members of Δ are called *independent* sets of the matroid, the facets is said to be the bases of the matroid and we write $v \in M$ to mean $v \in A$. We call Δ a G-complex. Two matroids$M_1 = (A_1, \Delta_1)$ and $M_2 = (A_2, \Delta_2)$ are said to be *isomorphic* if there exists a bijection$\psi : A_1 \longrightarrow A_2$ such that $\{v_1, \ldots, v_k\} \in \Delta_1$ if, and only if, $\{\psi(v_1), \ldots, \psi(v_k)\} \in \Delta_2$.

A *circuit* $C \subseteq A$ is a minimal dependent set, but C becomes independent when we remove any point from it. If $B \subseteq A$, we define the rank of B by;

$$rk(B) = \max\{|B'| \,|\, B' \subseteq B \text{ and } B' \in \Delta\}.$$

In particular, $rk(\phi) = 0$ and we will define the following:

1. The *rank* of the matroid M itself by $rk(M) = rk(A) = \dim(\Delta) + 1 = |F|$, where F is a facet of M. The level of a matroid is $l(M) = |A| - rk(M)$-1.
2. A *k-flat* of M is a maximal subset of rank k. It has been noticed that, if B and B' are flats of a matroid M, then so is $B \cap B'$. We can defined the *closure* $\bar{B}$ of a subset $B \subseteq A$ to be the smallest flat containing B, i.e. $\bar{B} = \bigcap_{flats\ B' \supseteq B} B'$.
3. $L(M)$ for a matroid M to be *the poset of flats* of M, ordered by inclusions. Since $L(M)$ has a top element A, then $L(M)$ is a lattice, which we call the *lattice of flats* of M. It has been noticed that, $L(M)$ has a unique minimal element $\hat{0} = \emptyset$.
4. Define the *characteristic polynomial* $\chi_M(t)$ of M, by;

$$\chi_M(t) = \sum_{X \in L(M)} \mu(\hat{0}, x)\, t^{r - rk(X)};$$

where μ denotes the Möbius function of $L(M)$ and $r = rk(M)$.

5. We say Δ *factors*, if A has a partition $\Pi = (\Pi_1, \Pi_2)$ such that $\Delta = (\Delta_1 * \Delta_2)$, where for $i = 1,2$, the induced subcomplexes $\Delta_i = \Delta|\Pi_i = \{S \in \Delta \mid S \subseteq \Pi_i\}$ is the restriction of Δ to Π_i and the join $\Delta_1 * \Delta_2 = \{S_1 \cup S_2 | S_1 \in \Delta_1 \text{ and} S_2 \in \Delta_2\}$. We say Δ*factors completely* if A has a partition $\Pi = (\Pi_1, \dots, \Pi_r)$ into r nonempty sets such that Δ is a multiplejoin of the induced sub-complexes $\Delta = \Delta_1 * \cdots * \Delta_r$, where Δ_i is discrete 0-dimensional, i.e. $\Delta_i = \{\emptyset\} \cup \{\{v\} \mid v \in \Pi_i\}$, for $1 \le i \le r$.
6. The *f-vector* of Δ is a vector of integers $f = (f_0, f_1, \dots, f_\delta)$, where for $0 \le k \le \delta$, f_k is the number of the faces of Δ have $k + 1$ elements. It has been noticed that, $f_0 = |A| = n$. For a positive integers m, define:

$$H(\Delta, m) = \sum_{k=0}^{\delta} f_k \binom{m-1}{k};$$

where for $m = 0$ define $H(\Delta, 0) = 1$. Define the *h-vector* of Δ to be the vector of integers $h = (h_1, \dots, h_r)$ satisfied the following:

$$\sum_{m=0}^{\infty} H(\Delta, m) x^m = \frac{(1 + h_1 x + \cdots + h_r x^r)}{(1-x)^r}.$$

Knowing the f-vector of Δ is equivalent to know its h-vector.

7. The *Euler characteristic* of Δ is;

$$\chi(\Delta) = -f_{-1} + f_0 - f_1 + \cdots = (-1)^{r-1} f_\Delta(-1), \text{ where } f_{-1} = 1.$$

Definition (1.2): Al-Ta'ai et al .(2010)

For a matroid $M = (A, \Delta)$, with $|A| = n$, we can define a linear ordering on the vertices by making $A = \{1,2, \dots, n\}$ and $1 < 2 < \cdots < n$. We call a matroid with this linear ordering $\trianglelefteq$ an *ordered matroid* or *oriented matroid* and we denoted it by $M_{\trianglelefteq}$. Let us agree that every subset of A will linearly ordered by the lexicographic order (DegLex). It has been noticed that, the $\trianglelefteq$-lexicographic order on the facets of Δ forms a shelling of Δ. In particular, all matroid complexes are shellable.

A *broken circuit* of an ordered matroid $M_{\trianglelefteq}$, is a set $\bar{C} = C\backslash v$, where C is a circuit and v is the minimal element of C via $\trianglelefteq$. The broken circuit complex (or BC-complex) which is defined to be the simplicial complex;

$$NBC_{\trianglelefteq}(M) = \{B \subseteq A \mid B \text{ contains no broken circuit}\}.$$

For $0 \leq k \leq rk(M)$, set;

$$NBC_{\trianglelefteq}^{k}(M) = \{B \subseteq A \mid B \text{ contains no broken circuit and } |B| = k+1\};$$

to be the k^{th}- skeleton of $NBC_{\trianglelefteq}(M)$. It has been noticed that, if $f^{\Delta} = (f_0^{\Delta}, f_1^{\Delta}, \ldots, f_{r-1}^{\Delta})$ be the f-vector of $NBC_{\trianglelefteq}(M)$, then $|NBC_{\trianglelefteq}^{k}(M)| = f_k^{\Delta}$ and;

$$\chi_M(t) = f_{-1}^{\Delta} t^r - f_0^{\Delta} t^{r-1} + \cdots + (-1)^r f_{r-1}^{\Delta}; \text{ where } f_{-1}^{\Delta} = 1.$$

The family of all subsets of $A/\{1\}$ that contains no broken circuits is called the *reduced broken circuit complex* of $M_{\trianglelefteq}$ and denoted by $\overline{NBC}_{\trianglelefteq}(M)$.

Definition (1.3): Al-Ta'ai (2010)and Stanley(1974)

Let Δ be a "finite" (abstract) simplicial complex with vertex set $A = \{v_1, \ldots, v_n\}$. Let I_{Δ} be the homogenous ideal of the polynomial algebra in n-indeterminate, $\mathrm{A} = K[x_1, \ldots, x_n]$ generated by all squarefree monomials $x_{i_1}. \ldots . x_{i_k}$, such that $\{x_{i_1}, \ldots, x_{i_k}\}$ is non-faces of Δ, i.e. $I_{\Delta}$is generated by the "minimal" non-faces of Δ. The ring $\mathrm{A}_{\Delta} = \mathrm{A}/I_{\Delta}$ is a *standard K-algebra*. As a graded algebra $\mathrm{A}_{\Delta} = \sum_{m=0}^{\infty} \mathrm{A}_{\Delta}^{m}$, the *Hilbert function* $H(\mathrm{A}_{\Delta}, m)$ of A_{Δ} is defined by $H(\mathrm{A}_{\Delta}, m) = \dim(\mathrm{A}_{\Delta}^{m})$ and the *krull dimension* of A_{Δ} which is denoted by $\dim(\mathrm{A}_{\Delta})$ is one more than the maximal integer m such that $H(\mathrm{A}_{\Delta}, m) \neq 0$. The important point to note here is, $H(\mathrm{A}_{\Delta}, m) = H(\Delta, m)$. If;

$$0 \longrightarrow M_h \longrightarrow M_{h-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow A_{\Delta} \longrightarrow 0,$$

is the minimal finite free resolution of A_{Δ}, the j^{th}-*Betti number* of A_{Δ} is denoted by $\beta_i(\mathrm{A}_{\Delta}) = \beta_i = rk(M_i)$. The integer h which represents the largest integer i such that $\beta_i \neq 0$ is the *homological dimension* of A_{Δ} denoted by $hd_{\mathrm{A}_{\Delta}}$. If $hd_{\mathrm{A}_{\Delta}} = n - \dim(\mathrm{A}_{\Delta})$, we call A_{Δ} a *Cohen-Macaulay ring* and $\beta_{hd_{\mathrm{A}_{\Delta}}}$ is called the *type* of A_{Δ}. Since Δ is a G-complex, then for $0 \leq k \leq rk(M)$;

$$\beta_k(\mathrm{A}_{\Delta}) = \beta_k = \sum_{flats X \in L_k(M)} |\mu(\hat{0}, x)|;$$

where $|\mu(\hat{0}, x)|$ represents the length of the maximal chain from minimal flat $\hat{0}$ into X of $L(M)$.

Theorem (Björner) (1.1): Björner(1992)

For an ordered matroid $M_{\trianglelefteq}$ on a set A, the simplicial complexes Δ, $NBC(M)$, and $\overline{NBC}(M)$ have a canonical set of basic cycles for the reduced homology group,

$$\widetilde{H}_d(\Delta) \cong \begin{cases} \mathbb{Z}^q & \text{if } d = dim\Delta = rk(M) - 1 \\ 0 & \text{if } d \neq dim\Delta = rk(M) - 1 \end{cases};$$

where$q = (-1)^r \chi(\Delta)$ is equal the number of facets F of Δ such that $\mathcal{R}(F) = F$.

Theorem (Ziegler) (1.2): Björner(1992)

Let $M_{\trianglelefteq}$ be an ordered matroid of rank r. Then $\overline{NBC}(M)$ has top-dimensional reduced homology;

$$\widetilde{H}_{r-2}\left(\overline{NBC}(M_{\trianglelefteq})\right) \cong \mathbb{Z}^{\beta(M)};$$

where $\beta(M) = (-1)^r \chi\left(\overline{NBC}_{\trianglelefteq}(M)\right)$.

Definition (1.4): Al-Ta'ai (2010)

Let $A = \{H_1, \dots, H_n\}$ be a central r-arrangement of hyperplanes over $\mathbb{C}$. Define a matroid $M_A = (A, \Delta)$ on A by letting Δ to be the collection of all independent subarrangements of A. It has been noticed that, $L(A) \equiv L(M_A)$. Via a linear ordering $\trianglelefteq$, let:

$$NBC_{\trianglelefteq}(M_A) = \{B \subseteq A \mid B \text{ contains no broken circuit}\};$$

be the NBC-complex of M_A. Then;

$$\chi_{M_A}(t) = f_{-1}^{\Delta} t^r - f_0^{\Delta} t^{r-1} + \cdots + (-1)^r f_{r-1}^{\Delta};$$

where$r = rk(A) = \delta + 1$ and $f^{\Delta} = (f_0^{\Delta}, f_1^{\Delta}, \dots, f_{\delta}^{\Delta})$ be the f-vector of $NBC_{\trianglelefteq}(M_A)$ and $f_{-1} = 1$. Notice that,$h_r = \beta_{rk(M_A)}(\mathrm{A}_{\Delta}) = f_{r-1}^{\Delta}$ is the type of the Cohen-Macaulay ring A_{Δ} and it has a minimal free resolution;

$$0 \longrightarrow M_r \longrightarrow M_{r-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow \mathrm{A}_{\Delta} \longrightarrow 0;$$

where for $0 \leq k \leq r,\ rk(M_k) = \beta_k(A_{\Delta}) = |NBC_{\trianglelefteq}^k(M)| = f_{k-1}^{\Delta}$.

Definitions (1.5): Orlik and Terao(1992)

Let $\Pi = (\Pi_1, \dots, \Pi_r)$ be a partition of a central ℓ- arrangement A. Then ;

1. If $d_i = |\Pi_i|$, for $1 \leq i \leq r$. We call $d = (d_1, \dots, d_r)$ the exponent vector of Π.

2. Let $X \in L(A)$ and $\Pi = (\Pi_1, \dots, \Pi_r)$ be a partition of$A$. Then the inducedpartitionΠ_X is partition of A_X with blocks is the non-empty sub-sets$\Pi_i \cap A_X, 1 \leq i \leq r$.

3. $\Pi$is said to be of an arrangement A is said to be independent, if for every choice of hyperplanes $H_i \in \Pi_i$ for$1 \leq i \leq r$, the resulting r-hyperplanes are independent, i.e. $rk\{H_1 \cap \dots \cap H_r\} = r$.

4. Call $S = \{H_1, \dots, H_k\}$ a k-section of Π if, for each$1 \leq i \leq k$,$H_i \in \Pi_{m_i}$,where $1 \leq m_1 < \cdots < m_k \leq r$. It has been noticed that, if Π is independent, then all its k-sections are independent. By $S_{\Pi}^k(A)$ we denotethe set of all k-sectionsof$\Pi$and$S_{\Pi}(A) = \bigcup_{k=1}^r S_{\Pi}^k(A)$.

5. A partition $\Pi = (\Pi_1, \dots, \Pi_r)$ of A is said to be niceif;

i.Π is independent, and

ii. If $X \in L(A)\backslash V$. Then the induced partition Π_X contains a block that is a singleton.

Definition (1.6): Ali (2007)

Let A be a central ℓ-arrangement. A partition $\Pi = (\Pi_1, \dots, \Pi_r)$ of A is said to be hypersolvaplewith length $\ell(A) = r$, exponent vector,(or d-vector), $d = (d_1, \dots, d_r)$,(where $d_i = |\Pi_i|$ for, $1 \le i \le r$) and denoted by Hp Π, if $|\Pi_1| = 1$ (i.e. Π_1 is a singleton) and for fixed $2 \le j \le r$, Π_j satisfies the following properties:

(Jth***Closed property of*****Π**):For any $H_1, H_2 \in \Pi_1 \cup \dots \cup \Pi_j$, there is no hyperplane$H \in \Pi_{j+1} \cup \dots \cup \Pi_\ell$ such that $rk\{H_1, H_2, H\} = 2$.

(Jth***Complete property of*** **Π**):For each $H_1, H_2 \in \Pi_j$, there exists $H \in \Pi_1 \cup \dots \cup \Pi_{j-1}$ such that $rk\{H_1, H_2, H\} = 2$. It has been noticed that, from closed property of Π_{j-1}, the hyperplanene H is uniqe and we will denote it by$H = H_{1,2}$.

(Jth***Solvable property of*****Π**):If $H_1, H_2, H_3 \in \Pi_j$, then the hyperplanes, $H_{1,2}, H_{1,3}, H_{2,3} \in \Pi_1 \cup \dots \cup \Pi_{j-1}$, either $H_{1,2} = H_{1,3} = H_{2,3}$ or$rk\{H_{1,2}, H_{1,3}, H_{2,3}\} = 2$.

For $1 \le j \le r$, we define the rank of a block Π_j of Π as $rk(\Pi_j) = rk\left(\bigcap_{H \in \Pi_1 \cup \dots \cup \Pi_j} H\right)$. We call Π_j singular if $rk(\Pi_j) = rk(\Pi_{j-1})$ and we call it non-singular otherwise. An Hp Π is said to be supersolvable and denoted by Sp, if it is independent. Observe that $rk(\Pi_{j-1}) \le rk(\Pi_j)$ in general, and if $r \ge 3$, then every $\Pi_{i_1}, \Pi_{i_2}, \Pi_{i_3} \in \Pi$ are independent, where $1 \le i_1 < i_2 < i_3 \le r$ (see Coxeter(1949))

Proposition (1.1): Al-Ta'ai (2010)

Let A be an essential central complex ℓ-arrangement. Then A is a hypersolvable (supersolvable) if, and only if, A has an HP (SP), $\Pi = (\Pi_1, \dots, \Pi_\ell)$.

Definition (1.7): Al-Ta'ai(2010)

The matroidM_A is said to be hypersolvable (supersolvable) matroid if, A is hypersolvable (supersolvable) arrangement.

If A is hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, \dots, \Pi_\ell)$ and d-vector$d = (d_1, \dots, d_\ell)$. Let $NBC_\trianglelefteq(M_A)$ be the NBC-complex of the hypersolvable matroid M_A via the hypersolvable ordering $\trianglelefteq$ with f-vector, $f^\Delta = (f_0^\Delta, f_1^\Delta, \dots, f_\delta^\Delta)$. That is, we shall give the no broken circuit subarrangements the degree lexicographic (DegLex) order with respect the hypersolvable ordering. Where, by $NBC_\trianglelefteq(M_A)|_i = \{S \in NBC_\trianglelefteq(M_A) \mid S \subseteq \Pi_i\}$ we denote the restriction of $NBC_\trianglelefteq(M_A)$to Π_i, for $1 \le i \le \ell$.

For $1 \le k \le \ell$, let $S_\Pi^k(A) = \{S \subseteq A \mid S \text{ is a } k-\text{section of } \Pi\}$ and let $S_\Pi(A)|_k = \{\{H\} \mid H \in \Pi_k\}$ be the discrete 0-dimensional simplicial complex. Let$S_\Pi(A) = S_\Pi(A)|_1 * \cdots * S_\Pi(A)|_\ell$ be the multiple join of the complexes $S_\Pi(A)|_1, \dots, S_\Pi(A)|_\ell$. That is $S_\Pi(A) = \bigcup_{k=1}^{\ell} S_\Pi^k(A)$. We call $S_{\Pi}(A)$ a *hypersolvable partition complex* of the matroidM_{A} via the hypersolvable ordering. It has been noticed that, in general $S_{\Pi}(A)$ need not to be a subcomplex of the G-complex Δ of the matroid $S_{\Pi}(A)$. The important point to know here $NBC_{\trianglelefteq}(M_A)|_k = S_{\Pi}(A)|_k$, for $1 \le k \le \ell$, but $NBC_{\trianglelefteq}(M_A)$ and $S_{\Pi}(A)$ need not to be equal in general.

Theorem (1.3): Al-Ta'ai(2010)

Let A be a hypersolvable arrangement with $\text{Hp}\Pi = (\Pi_1, \dots, \Pi_\ell)$ and exponent vector $d = (d_1, \dots, d_\ell)$, via a fix hypersolvable ordering $\trianglelefteq$ on G. Then the following statements are equivalent:

1. $A$is supersolvable.

2. $NBC_{\trianglelefteq}(M_A) \equiv S_{\Pi}(A)$, i.e.; $NBC_{\trianglelefteq}(M_A) \equiv S_{\Pi}(A)|_1 * \cdots * S_{\Pi}(A)|_\ell$ is factored completely and;

$$f_{k-1}^{\Delta} = \sum_{i_1=1}^{\ell-k} \sum_{i_2=i_1+1}^{\ell-k+1} \dots \sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} d_{i_2} \dots d_{i_k},$$

for$1 \le k \le \ell$,where $f^{\Delta} = (f_0^{\Delta}, f_1^{\Delta}, \dots, f_{\ell-1}^{\Delta})$ be the f-vector of $NBC_{\trianglelefteq}(M_A)$ and $\chi_A(t) = \chi_{M_A}(t) = f_{-1}^{\Delta} t^{\ell} - f_0^{\Delta} t^{\ell-1} + \cdots + (-1)^{\ell} f_{\ell-1}^{\Delta}$;

$$\text{where} f_{-1}^{\Delta} = 1 \text{ and} h_r = \beta_{rk(M_A)}(\text{A}_\Delta) = f_{\ell-1}^{\Delta} = d_2 d_3 \dots d_\ell;$$

is the type of the Cohen-Macaulay ring A_Δ and it has a minimal free resolution,

$$0 \longrightarrow M_\ell \longrightarrow M_{\ell-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow \text{A}_\Delta \longrightarrow 0,$$

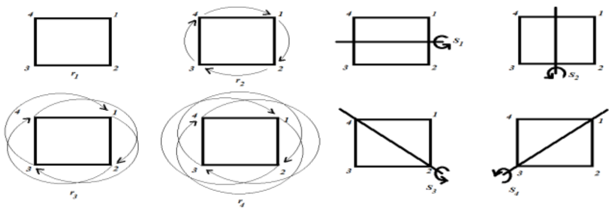
where for $0 \le k \le \ell,\ rk(M_k) = \beta_k(\text{A}_\Delta) = |NBC_{\trianglelefteq}^{k}(M)| = f_{k-1}^{\Delta}$.

(2) SomeCoxeter Reflection Matroids

In this section, we will compute the$G$-complex, the broken circuit complex and the reduced broken circuit complex for the complexification of the Coxeterarrangements $(A(I_2(m)), m = 4)$,$(A(A_n), n \ge 3)$, $(A(B_n), n \ge 3)$, $(A(C_n), n \ge 3)$, $(A(D_n), n = 4)$. For the defining polynomials of the Coxeter groups that we used in this section, see [4] and [7].We mentioned that, for simplicity we will write $1,2,3,\dots$ instead of the hyperplanes $H_1, H_2, H_3, \dots$ respectively, of the structure of G-complex.

(2.1): The ReflectionMatroid of $A(I_2(m)), m = 4$:

Let $V = \mathbb{R}^m$. Define $\mathfrak{D}_m = I_2(m)$ to be a dihederal group of symmetries of regular Polgon incuding both reflections and rotations. A regular Polgon with m −sides has $2m$ different symmetries, m-rotations; (through multiples $\frac{2\pi}{m}$) and m-reflections; (about the digonals of polygon). For $m = 4$, the dihederal group $\mathfrak{D}_4$ of the symmetries of the sequre as shown in figure (2.1):



Figure(2.1)

has four reflections, $S_1 = \begin{pmatrix} 1 & 2 & 34 \\ 2 & 1 & 43 \end{pmatrix} = (1\ 2)(3\ 4)$, $S_2 = \begin{pmatrix} 1 & 2 & 34 \\ 4 & 3 & 21 \end{pmatrix} = (1\ 4)(2\ 3)$, $S_3 = \begin{pmatrix} 1 & 2 & 34 \\ 3 & 2 & 14 \end{pmatrix} = (1\ 3)$ and $S_4 = \begin{pmatrix} 1 & 2 & 34 \\ 1 & 4 & 32 \end{pmatrix} = (2\ 4)$ which produce four rotations, $r_1 = \begin{pmatrix} 1 & 2 & 34 \\ 1 & 2 & 34 \end{pmatrix}$, $r_2 = \begin{pmatrix} 1 & 2 & 34 \\ 2 & 3 & 41 \end{pmatrix}$, $r_3 = \begin{pmatrix} 1 & 2 & 34 \\ 3 & 4 & 14 \end{pmatrix}$ and $r_4 = \begin{pmatrix} 1 & 2 & 34 \\ 4 & 1 & 23 \end{pmatrix}$. That is, $\mathfrak{D}_4$ is a subgroup of the 4th symmetric group $\mathbf{S}_4$ and its Coxeter arrangement $A(\mathfrak{D}_4)$ has a defining polynomial defined as follows:

$$Q(A(\mathfrak{D}_4)) = (x_1 - x_2 + x_3 - x_4)(x_1 - x_4 + x_2 - x_3)(x_1 - x_3)(x_2 - x_4).$$

In $\mathbb{C}^4$, the complexification of $A(\mathfrak{D}_4)$ is the complex arrangement$A_{\mathbb{C}}(\mathfrak{D}_4)$ which contains the four complex hyperplanes related to the reflections S_1, S_2, S_3 and S_4;

$$H_{S_1}^{\mathbb{C}} = ker\{x_1 - x_2 + x_3 - x_4\},\ H_{S_2}^{\mathbb{C}} = ker\{x_1 - x_4 + x_2 - x_3\},$$

$$H_{S_3}^{\mathbb{C}} = ker\{x_1 - x_3\} \text{ and } H_{S_4}^{\mathbb{C}} = ker\{x_2 - x_4\}.$$

$A_{\mathbb{C}}(\mathfrak{D}_4)$ is a non-essential supersolvable arrangement with $\text{Sp},\Pi = (\{H_{S_1}^{\mathbb{C}}\}, \{H_{S_2}^{\mathbb{C}}\}, \{H_{S_3}^{\mathbb{C}}, H_{S_4}^{\mathbb{C}}\})$and exponent vector $d = (1,1,2)$. The matriod of $A(\mathfrak{D}_4)$is $M_{A(\mathfrak{D}_4)} = (A(\mathfrak{D}_4), \Delta)$, where $\Delta = \bigcup_{k=0}^{2} \Delta_k$ such that;

$\Delta_0 = \{[1], [2], [3], [4]\}$, $\Delta_1 = \{[1,2], [1,3], [1,4], [2,3], [2,4], [3,4]\}$and;

$\Delta_2 = \{[1,2,3], [1,2,4], [1,3,4]\}$.That is the f-vector of $\Delta$is $f = (4,6,3)$. By easy calculation (see appendix (2)), we have $H(\Delta,0) = 1H(\Delta,1) = 4H(\Delta,2) = 10$, $H(\Delta,3) = 19$, $H(\Delta,4) = 31$,… and the h-vector of Δ is $h = (1,1,1)$.The rank of$M(A(\mathfrak{D}_4))$ is$rk(A(\mathfrak{D}_4)) = \dim(\Delta) + 1 = 3$. By applying theorem (1.3), we have $NBC_{\trianglelefteq}(M_{A(\mathfrak{D}_4)}) \equiv S_{\Pi}(A(\mathfrak{D}_4))$, i.e.;

$$NBC_{\trianglelefteq}(M_{A(\mathfrak{D}_4)}) \equiv S_{\Pi}(A(\mathfrak{D}_4))|_1 * S_{\Pi}(A(\mathfrak{D}_4))|_2 * S_{\Pi}(A(\mathfrak{D}_4))|_3 = \{[1]\} * \{[2]\} * \{[3], [4]\};$$

andthe f $-$vector of $NBC(M)$ is $f^{\Delta} = (1,4,5,2)$. That is,$\chi_{M_{A(\mathfrak{D}_4)}}(t) = t^3 - 4t^2 + 5t - 2$ and the Cohen-Macaulay ring $\mathrm{A}_{\Delta} = \sum_{m=0}^{\infty} \mathrm{A}_m^{rk(\mathrm{A}_m)} = \mathrm{A}_0^1 \oplus \mathrm{A}_1^4 \oplus \mathrm{A}_2^{10} \oplus \mathrm{A}_3^{19} \oplus \mathrm{A}_4^{31} \oplus$ ...hastype $h_3 = \beta_3(\mathrm{A}_{\Delta}) = f_2{}^{\Delta} = 2$and a minimal free resolution;

$$0 \to M_3^2 \to M_2^5 \to M_1^4 \to M_0^1 \to \mathrm{A}_{\Delta} \to 0;$$

that completely determined by the f-vector of $NBC_{\trianglelefteq}(M_{A(\mathfrak{D}_4)})$ and the homological dimension of A_{Δ} is $hd_{\mathrm{A}_{\Delta}} = n - dim\mathrm{A}_{\Delta} = 4 - 3 = 1$.Moreover, the reduced broken circuit complex will be, $\overline{NBC_{\trianglelefteq}}(M_{A(\mathfrak{D}_4)}) = \{[2]\} * \{[3], [4]\}$, and fromapplication of the theorems (1.1) and (1.2), we have;

$$\widetilde{H}_d(\Delta) \cong \begin{cases} 1 & \text{if } d = 2 \\ 0 & \text{if } d \neq 2 \end{cases} \text{and;} \widetilde{H}_1\left(\overline{NBC}(M_{A(\mathfrak{D}_4)})\right) = 1;$$

where$\beta(M_{A(\mathfrak{D}_4)}) = (-1)^3 \chi(\overline{NBC_{\trianglelefteq}}(M_{A(\mathfrak{D}_4)}))$ and $\chi\left(\overline{NBC_{\trianglelefteq}}(M_{A(\mathfrak{D}_4)})\right) = 0$.

(**2.2) The ReflectionMatroidOf $A(A_n), n \geq 3$:**

The group $A_n \cong \mathbf{S}_{n+1}$, where $\mathbf{S}_{n+1}$ is the $(n+1)^{th}$-symmetric group of O$(n+1,\mathbb{R})$ of $n+1 \times n+1$ orthogonal matrices. Make any permutation π acts on $\mathbb{R}^{n+1}$ by permuting the standared basis $e_1, e_2, \dots, e_{n+1}$. In particular, the transposition$(i \ \ j)$, $1 \leq i < j \leq n+1$ acts on $\mathbb{R}^{n+1}$ by sending $e_i - e_j$ in to its negative $-(e_i - e_j)$ and fixing pointwise the orthogonal complement $H_{e_i - e_j} = \{(x_1, \dots, x_{n+1}); \ x_i = x_j\}$. It is known that, the symmetric group $\mathbf{S}_{n+1}$ generated by the transpositions $\{(i \ \ j), 1 \leq i < j \leq n+1\}$. So $\mathbf{S}_{n+1}$ is a Coxeter group and is the corresponding reflection group. Thus, $A(A_n) = \left\{H_{e_i - e_j} | 1 \leq i < j \ \leq n+1\right\}$ is a Coxeter arrangement, where;

$$H_{e_i - e_j} = \{(x_1, \dots, x_{n+1}) | \ x_i = x_j\}, 1 \leq i < j \ \leq n+1.$$

Its complexification is the Braid arrangment $A_{\mathbb{C}}(A_n)$ of $\mathbb{C}^{n+1}$ that is a non-essential supersolvable arrangement with Sp;

$$\Pi = \left(\{H^{\mathbb{C}}_{e_1 - e_2}\}, \{H^{\mathbb{C}}_{e_1 - e_3}, H^{\mathbb{C}}_{e_2 - e_3}\}, \dots, \{H^{\mathbb{C}}_{e_1 - e_{n+1}}, \dots, H^{\mathbb{C}}_{e_n - e_{n+1}}\}\right);$$

and exponent vector $d = (d_1, \dots, d_n), = (1,2, \dots, n)$. If the Matriod of $A_{\mathbb{C}}(A_n)$is$M_{A(A_n)} = (A(A_n), \Delta)$, then the rank of $rk(M_{A(A_n)}) = rk(\ A(A_n)) = \dim(\Delta) + 1 = n$. By applying theorem (1.3), we have$NBC_{\trianglelefteq}(M_{A(A_n)}) \equiv S_{\Pi}(A(A_n))$, i.e.;

$$NBC_{\trianglelefteq}(M_{A(A_n)}) \equiv S_{\Pi}(A(A_n))|_1 * \cdots * S_{\Pi}(A(A_n))|_n = \{[1]\} * \{[2],[3]\} * \dots * \{[\tfrac{n(n-1)}{2} - n, \dots, nn-12\};$$

$$\text{and} f^{\Delta}_{k-1} = \sum_{i_1=1}^{n-k} \sum_{i_2=i_1+1}^{n-k+1} \dots \sum_{i_k=i_{k-1}+1}^{n} d_{i_1} d_{i_2} \dots d_{i_k}, \text{ for } 1 \leq k \leq n;$$

where$f^{\Delta} = (f^{\Delta}_0, \dots, f^{\Delta}_{n-1})$ be the f-vector of $NBC_{\trianglelefteq}(M_{A(A_n)})$. Then;

$$\chi_{A(A_n)}(t) = \chi_{M_{A(A_n)}}(t) = f^{\Delta}_{-1} t^n - f^{\Delta}_0 t^{n-1} + \dots + (-1)^n f^{\Delta}_{n-1};$$

where $f^{\Delta}_{-1} = 1$. $h_n = \beta_{rk(M_{A(A_n)})}(\mathrm{A}_{\Delta}) = f^{\Delta}_{n-1} = d_2 d_3 \dots d_n = n!$is the type of the Cohen-Macaulay ring A_{Δ} and it has a minimal free resolution;

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow \mathrm{A}_{\Delta} \longrightarrow 0;$$

where for $0 \leq k \leq n$, $rk(M_k) = \beta_k(\mathrm{A}_{\Delta}) = |NBC^k_{\trianglelefteq}(M_{A(A_n)})| = f^{\Delta}_{k-1}$.

For $n = 3$, the complexification of $A(A_3)$ is supersolvable with Sp;

$$\Pi = \left(\{H^{\mathbb{C}}_{x_1 - x_2}\}, \{H^{\mathbb{C}}_{x_1 - x_3}, H^{\mathbb{C}}_{x_2 - x_3}\}, \{H^{\mathbb{C}}_{x_1 - x_4}, H^{\mathbb{C}}_{x_2 - x_4}, H^{\mathbb{C}}_{x_3 - x_4}\}\right) = (\{1\}, \{2,3\}, \{4,5,6\}).$$

Thus the matroid$M_{A(A_3)} = (A(A_3), \Delta)$, is defined by letting;

$$\Delta = \cup_{k=0}^{2} \Delta_k, \text{ where } \Delta_0 = \{[1], \dots, [6]\}, \ \Delta_1 = \{[i,j] \ \ 1 \leq i < j \leq 6\} \text{ and;}$$

$$\Delta_2 = \{[i,j,k] \mid 1 \le i < j < k \le 6\} \setminus \{[1,2,3],[1,4,5],[2,4,6],[3,5,6]\}.$$

That is the f-vector of Δ is $f = (6,15,16)$ and by easy Maple calculation (see appendix (2)), we have $H(\Delta,0) = 1 H(\Delta,1) = 6 H(\Delta,2) = 21$, $H(\Delta,3) = 52$, $H(\Delta,4) = 99, \ldots$ and the h-vector of Δ is $h = (1,3,6,6)$.The BC-complex of $A(A_3)$is;

$$NBC_{\trianglelefteq}(M_{A(A_3)}) \equiv S_{\Pi}(A(A_3))|_1 * S_{\Pi}(A(A_3))|_2 * S_{\Pi}(A(A_3))|_3$$

$$=\{[1]\} * \{[2],[3]\} * \{[4],[5],[6]\},$$

and the f-vector of $NBC_{\trianglelefteq}(M_{A(A_3)})$ is $f^{\Delta} = (1,6,11,6)$ and;

$$\chi_{M_{A(A_3)}}(t) = (t-1)(t-2)(t-3) = t^3 - 6t^2 + 11t - 6.$$

The type of the Cohen-Macaulay ring $\mathrm{A}_{\Delta} = \sum_{m=0}^{\infty} \mathrm{A}_m^{rk(\mathrm{A_m})} = \mathrm{A}_0^1 \oplus \mathrm{A}_1^6 \oplus \mathrm{A}_2^{21} \oplus \mathrm{A}_3^{53} \oplus \mathrm{A}_4^{102} \oplus \ldots$, is $h_3 = \beta_3(\mathrm{A}_{\Delta}) = f_2^{\Delta} = 3! = 6$ and it has a minimal free resolution;

$$0 \to M_3^6 \to M_2^{11} \to M_1^6 \to M_0^1 \to \mathrm{A}_{\Delta} \to 0;$$

and the homological dimension of A_{Δ} is $hd_{\mathrm{A}_{\Delta}} = n - \dim \mathrm{A}_{\Delta} = 3$.The reduced broken circuit of $M_{A(A_3)}$ is given by;

$$\overline{NBC}_{\trianglelefteq}(M) \equiv S_{\Pi}(A(A_3))|_2 * S_{\Pi}(A(A_3))|_3 = \{[2],[3]\} * \{[4],[5],[6]\};$$

and as an application of the theorems (1.1) and (1.2), we have;

$$\widetilde{H}_d(\Delta) \cong \begin{cases} \mathbb{Z}^6 & \text{if } d = 2 \\ 0 & \text{if } d \neq 2 \end{cases} \text{and;} \widetilde{H}_1(\overline{NBC}(M_{A(A_3)}); \mathbb{Z}) \cong \mathbb{Z}^2;$$

where, $\beta(M) = (-1)^3 \chi(\overline{NBC}_{\trianglelefteq}(M_{A(A_3)}))$ and $\chi\left(\overline{NBC}_{\trianglelefteq}(M_{A(A_3)})\right) = -2$.

(2.3) The Reflection Matroidof $A(B_n), n \ge 2$:

Let $V = \mathbb{R}^n$ and let $e_1, e_2, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. The reflections S_{e_i} that sending an e_i to its negative $-e_i$ and fixing all other e_j, $1 \le i \neq j \le n$, i.e. $S_{e_i}: V \to V$ defined by :

$$S_{e_i}(e_j) = \begin{cases} -e_i & , \quad j = i \\ e_j & , \quad j \neq i \end{cases}$$

generte a group of order 2^n isomorphic to $(\frac{\mathbb{Z}}{2\mathbb{Z}})^n$ which intersects the n^{th}-symmetric group $\mathbf{S}_n$ trivially and normalized by $\mathbf{S}_n$. Notice that, the reflection hyperplanes H_{e_i} that orthogonal to e_i is $H_{e_i} = (x_1, \ldots, x_i, x_{i+1}, \ldots x_n) -$ hyperplane and the Coxeter arrangment $A((\frac{\mathbb{Z}}{2\mathbb{Z}})^n)$ will be the Boolean arrangment which is essential supersolvable arrangement and its Sp has exponent vector $d = (1, \ldots, 1)$.

As given in (3.2), the symetric group $A_{n-1} = \mathbf{S}_n$ acts on$\mathbb{R}^n$ by sending $e_i - e_j$ to its negative $e_j - e_i$, $1 \le i < j \le n$. The semi direct product of A_{n-1} and the group $(\frac{\mathbb{Z}}{2\mathbb{Z}})^n$ produce a reflection group $B_n = (\frac{\mathbb{Z}}{2\mathbb{Z}})^n \propto A_{n-1}$, of order $2^n n!$. In general, $A_{\mathbb{C}}(B_n)$ is an essential supersolvable arrangement with supersolvable partition $\Pi = (\Pi_1, \dots, \Pi_n)$, where;

$$\Pi_j = \{H_{e_j}\} \cup \{H_{e_i - e_{j+1}} | 1 \le i \le j\} = \{x_j = 0, x_1 = x_j, \dots, x_{j-1} = x_j\} \dots (1);$$

and $|\Pi_j| = j$, for $j = 1, \dots, n$, i.e. $A_{\mathbb{C}}(B_n)$ has d-vector $d = (1,2,..,n)$. However, we can applying theorem (1.3), in order to compute its G-complex, broken circuit complex, as well as its minimal free resolution can be computed. Via the SpΠ we have$NBC_{\trianglelefteq}(M_{A(B_n)}) \equiv S_\Pi(A(B_n))$, i.e. $NBC_{\trianglelefteq}(M_{A(B_n)}) \equiv S_\Pi(A(B_n))|_1 * \cdots * S_\Pi(A(B_n))|_n$is factored completely and;

$$f_{k-1}^\Delta = \chi_{M_{A(B_n)}} \sum_{i_1=1}^{n-k} \sum_{i_2=i_1+1}^{n-k+1} \dots \sum_{i_k=i_{k-1}+1}^{n} d_{i_1} d_{i_2} \dots d_{i_k}, \text{ for } 1 \le k \le n;$$

where$f^\Delta = (f_0^\Delta, f_1^\Delta, \dots, f_{n-1}^\Delta)$ be the f-vector of $NBC_{\trianglelefteq}(M_{A(B_n)})$ and$\chi_{A(B_n)}(t) = (t) =$ $f_{-1}^\Delta t^n - f_0^\Delta t^{n-1} + \cdots + (-1)^n f_{n-1}^\Delta$, where $f_{-1}^\Delta = 1$.$h_n = \beta_{rk(M_{A(B_n)})}(A_\Delta) = f_{n-1}^\Delta = n!$is the type of the Cohen-Macaulay ring A_Δ and it has a minimal free resolution,

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow A_\Delta \longrightarrow 0,$$

where for $0 \le k \le n$, $rk(M_k) = \beta_k(A_\Delta) = |NBC_{\trianglelefteq}^k(M)| = f_{k-1}^\Delta$.

For $n = 3$, the Matriod$M_{A(B_3)} = (A(B_3), \Delta = \bigcup_{k=0}^{2} \Delta_k)$ of the complexification of$A(B_3)$ is given as;

$$\Delta_0 = \{[1],[2],[3],[4],[5],[6]\}, \Delta_1 = \{[i,j],\ 1 \le i < j \le 9\} \text{and};$$

Δ_2 $= \{[1,2,3],[1,2,5],[1,2,6],[1,3,4],[1,3,6],[1,4,5],[1,4,6],[1,5,6],[2,3,4],[2,3,5],[2,4,5],[2,4,6],$ $[2,5,6],[3,4,5],[3,4,6],[3,5,6]\}$. That is the f-vector of Δ is $f = (6,15,16)$ and by easy Maple calculation (see appendix (2)), we have $H(\Delta,0) = 1 H(\Delta,1) = 6 H(\Delta,2) = 21$, $H(\Delta,3) = 52$, $H(\Delta,4) = 99$,… and the h-vector of Δ is $h = (1,3,6,6)$. the rank of$M_{A(B_3)}$ is$rk(M_{A(B_3)}) = rk(A(B_3)) = \dim(\Delta) + 1 = 3$. $A_{\mathbb{C}}(B_3)$has Sp,$\Pi = (\{H_1^{\mathbb{C}}\}, \{H_2^{\mathbb{C}}, H_3^{\mathbb{C}}\}, \{H_4^{\mathbb{C}}, H_5^{\mathbb{C}}, H_6^{\mathbb{C}}\})$. As illsutration of theorem (1.3) theBC -complex of $A(B_3)$ is;

$$NBC(M_{A(B_3)}) \equiv S_\Pi(A(B_3))|_1 * S_\Pi(A(B_3))|_2 * S_\Pi(A(B_3))|_3$$

$$=\{[1]\} * \{[2],[3]\} * \{[4],[5],[6]\}.$$

The f-vector of $NBC(M_{A(B_3)})$ is $f^\Delta = (1,6,11,6)$ and;

$$\chi_{M_{A(B_3)}}(t) = t^3 - 6t^2 + 11t - 6.$$

The type of the Cohen-Macaulay ring

$$A_\Delta = \sum_{m=0}^{\infty} A_m^{rk(A_m)} = A_0^1 \oplus A_1^6 \oplus A_2^{21} \oplus A_3^{53} \oplus A_4^{99} \oplus \dots,$$

Is$h_3 = \beta_3(A_\Delta) = f_2{}^\Delta = 6$ and it has a minimal free resolution;

$$0 \to M_3^6 \to M_2^{11} \to M_1^6 \to M_0^1 \to A_\Delta \to 0;$$

and its homological dimension is $hd_{A_\Delta} = n - dimA_\Delta = 6 - 3 = 3$. The reduced broken circuit $\overline{NBC_{\trianglelefteq}}(M_{A(B_3)})$of $M_{A(B_3)}$is;

$$\overline{NBC_{\trianglelefteq}}(M_{A(B_3)}) \equiv S_\Pi\big(A(B_3)\big)|_2 * S_\Pi\big(A(B_3)\big)|_3 = \{[2],[3]\} * \{[4],[5],[6]\}.$$

and as an application of the theorems (1.1) and (1.2), we have;

$$\widetilde{H}_d(\Delta) \cong \begin{cases} \mathbb{Z}^6 & \text{if } d = 2 \\ 0 & \text{if } d \neq 2 \end{cases} \text{and;} \widetilde{H}_1\big(\overline{NBC}\big(M_{A(B_3)}\big); \mathbb{Z}\big) \cong \mathbb{Z}^2;$$

where, $\beta(M) = (-1)^3\chi(\overline{NBC_{\trianglelefteq}}(M_{A(B_3)}))$ and $\chi\left(\overline{NBC_{\trianglelefteq}}\big(M_{A(B_3)}\big)\right) = -2$.

(2.4) The Reflection Matroidof $A(C_n), n \geq 3$:

Let $V = \mathbb{R}^n$ and let $e_1, e_2, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. The reflections S_{2e_i} that sending an $2e_i$ to its negative $-2e_i$ and fixing all other $2e_j$, $1 \leq i \neq j \leq n$ and the reflections $S_{e_i - e_j}$ sending $e_i - e_j$ to its negative $e_j - e_i$, $1 \leq i < j \leq n$, define The reflection group C_n. So, its Coxeter arrangement $A(C_n)$ has a complexification $A_{\mathbb{C}}(C_n)$ defined as $Q(A_{\mathbb{C}}(C_n))$= $2^n Q(A_{\mathbb{C}}(B_n))$. Therefore, $A_{\mathbb{C}}(C_n)$ and $A_{\mathbb{C}}(B_n)$ have the same lattice. Thus they have isomorphic Matroids, i.e.$M_{A(C_3)} = (A(C_3), \Delta = \bigcup_{k=0}^{2} \Delta_k) \cong M_{A(B_3)} = (A(B_3), \Delta = \bigcup_{k=0}^{2} \Delta_k)$.

(2.5)The Reflection Matroid of A (D_n), $n = 4$:

A subgroup D_n of index 2 of the reflection group of type B_n is also a reflection that generated by reflections $S_{e_i+e_j}$ that sending $e_i + e_j$ in to $-(e_i + e_j)$, where $i \neq j$ and fixing pointwise the orthogonal hypeplane $H_{e_i+e_j}$when $i = j$. The Coxeter arrangement $A(D_n)$ has a complexification $A_{\mathbb{C}}(D_n)$ that contains $n(n-1)$ complex hyperplanes defined by the following defining polynomial;

$$Q\big(A_{\mathbb{C}}(D_n)\big) = \textstyle\prod_{1 \leq i < j \leq n}(x_i \pm x_j).$$

Now for $n = 4$ the defining polynomial of $A(D_4)$ is;

$$Q\big(A(D_4)\big) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$$

$$(x_1 + x_2)(x_1 + x_3)(x_1 + x_4)(x_2 + x_3)(x_2 + x_4)(x_3 + x_4).$$

The matroid$M_{A(D_4)}$= $(A(D_4)\ ,\Delta)$ has$f$-vector of Δ is, $f = (12,66,204,315)$ and by an easy Maple calculation (see appendix(2)), we have $H(\Delta,0) = 1$, $H(\Delta,1) = 12$, $H(\Delta,2) = 78$, $H(\Delta,3) = 348$, $H(\Delta,4) = 1137, \ldots$ and the$h$-vector of Δ is $h = (1,8,36,104,66)$. The NBC-basis of $A(D_4)$ is defined by $NBC\big(M_{A(D_4)}\big) = \bigcup_{k=0}^{3} NBC_k\ (M_{A(D_4)})$ with$f$-vector of $NBC_{\trianglelefteq}\big(M_{A(D_4)}\big)$ is $f^\Delta = (1,12,50,60,21)and\chi_{M_{A(D_4)}}(t) = t^4 - 12t^3 + 50t^2 - 60t + 21$.The type of the Cohen-Macaulay ring

$$\mathrm{A}_\Delta = \textstyle\sum_{m=0}^{\infty} \mathrm{A}_m^{rk(\mathrm{A}_m)} = \mathrm{A}_0^1 \oplus \mathrm{A}_1^{12} \oplus \mathrm{A}_2^{78} \oplus \mathrm{A}_3^{348} \oplus \mathrm{A}_4^{1137} \oplus \ldots,$$

is $h_4 = \beta_4(\mathrm{A}_\Delta) = f_3^\Delta = 21$ and it has a minimal free resolution;

$$0 \to M_4^{21} \to M_3^{60} \to M_2^{50} \to M_1^{12} \to M_0^{1} \to A_\Delta \to 0;$$

is completely determines by f^Δ, and the homological dimension of A_Δ is $hd_{A_\Delta} = n - \dim A_\Delta = 8$. The f-vector of the reduced broken circuit $\overline{NBC}_{\trianglelefteq}(M_{A(D_4)})$of $M_{A(D_4)}$, will be $f_{\overline{NBC}_{\trianglelefteq}(M_{A(D_4)})} = (11,39,21)$. Finally, fromapplication of the theorems (1.1) and (1.2), we have;

$$\tilde{H}_d(\Delta_G) \cong \begin{cases} \mathbb{Z}^{166} & \text{if } d = 3 \\ 0 & \text{if } d \neq 3 \end{cases} \text{and;} \tilde{H}_2(\overline{NBC}(M_{A(D_4)}); \mathbb{Z}) \cong \mathbb{Z}^8;$$

where, $\beta(M) = (-1)^4 \chi(\overline{NBC}_{\trianglelefteq}(M_{A(D_4)}))$ and $\chi\left(\overline{NBC}_{\trianglelefteq}(M_{A(D_4)})\right) = -8$.

(3) The Complex Reflection Matroids

This section consists four parts.Each one of them, study the reflection Matroid for one of the complex reflection arrangements$A(G_{24})$,$A(G_{25})$,$A(G_{26})$and$A(G_{27})$. Moreover,they provide the detailed expositions for the constructions of theirCohen-Macaulay rings and their minimal free resolutions.

(3.1): The Complex Reflection Matroid$A(G_{24})$:

The complex reflection group $G_{24} \subset U(\mathrm{C}^3) \cong (3, \mathrm{C}) \subset GL(3, \mathrm{C})$ of order 336, is not the complexification of real group [23].The corresponding reflection arrangement $A(G_{24})$ has (21) hyperplanes and its defining polynomialis given by:

$$A(G_{24}) = x_1 x_2 x_3 \prod_{i,j,k=1,2,3} (x_i \mp x_j)(\beta x_i \mp x_j \mp x_k);$$

whereβ is the root of the equation $(t^2 - t + 2) = 0$, i.e. $\beta = \frac{1}{2}(1 - i\sqrt{7})$ and $\bar{\beta}$=$\frac{-1}{2}(1 + i\sqrt{7})$. We will order the hyperplanes$H_i = Ker(\alpha_{H_i})$, $1 \leq i \leq 21$, as shown in the following table:

$H_1: x_1 = 0$	$H_2: x_2 = 0$	$H_3: x_3 = 0$
$H_4: x_1 + x_2 = 0$	$H_5: x_1 + x_3 = 0$	$H_6: x_2 + x_3 = 0$
$H_7: x_1 - x_2 =0$	$H_8: x_1 - x_3 = 0$	$H_9: x_2 - x_3 = 0$
$H_{10}: \beta x_1 + x_2 + x_3 = 0$	$H_{11}: \beta x_1 - x_2 + x_3 = 0$	$H_{12}: \beta x_1 + x_2 - x_3 = 0$
$H_{13}: \beta x_1 - x_2 - x_3 = 0$	$H_{14}: \beta x_2 + x_1 + x_3 = 0$	$H_{15}: \beta x_2 - x_1 + x_3 = 0$
$H_{16}: \beta x_2 + x_1 - x_3 = 0$	$H_{17}: \beta x_2 - x_1 - x_3 = 0$	$H_{18}: \beta x_3 + x_1 + x_2 = 0$
$H_{19}: \beta x_3 - x_1 + x_3 = 0$	$H_{20}: \beta x_3 + x_1 - x_2 = 0$	$H_{21}: \beta x_3 + x_1 - x_2 = 0$

Table (3.1): The hyperplane of $A(G_{24})$

The matroid$M_{A(G_{24})}$= ($A(G_{24})$,Δ) on $A(G_{24})$has$f$-vector of Δ is $f = (21,210,1162)$, and by easy Maple calculation (see appendix (2)), we have $H(\Delta,0) = 1$, $H(\Delta,1) = 21$, $H(\Delta,2) = 231$, $H(\Delta,3) = 1603$, $H(\Delta,4) = 4137$,… and the h-vector of Δ is $h = (1,18,171,972)$.The rank of $rk(M_{A(G_{24})}) = rk(A(G_{24})) = \dim(\Delta) + 1 = 3$. The broken circuit complex of

$A(G_{24})$have a structure $NBC(M_{A(G_{24})}) = \cup_{k=0}^{2} NBC_k\,(M_{A(G_{24})})$ with$f$ −vectoris$f^{\Delta} = (1,21,119,99)$ and;

$$\chi_{M_{A(G_{24})}}(t) = t^3 - 21t^2 + 119t - 99.$$

The type of the Cohen-Macaulay ring$A_{\Delta} = \sum_{m=0}^{\infty} A_m^{rk(A_m)} = A_0^1 \oplus A_1^{21} \oplus A_2^{231} \oplus A_3^{1603} \oplus A_4^{4137} \oplus ...$,is $h_3 = \beta_3(A_{\Delta}) = f_2{}^{\Delta} = 99$ and it has a minimal free resolution;

$$0 \to M_3^{99} \to M_2^{119} \to M_1^{21} \to M_0^1 \to A_{\Delta} \to 0;$$

and the homological dimension of A_{Δ} is $hd_{A_{\Delta}} = n - dimA_{\Delta} = 21 - 3 = 18$. The reduced broken circuit $\overline{NBC_{\trianglelefteq}}(M_{A(G_{24})})$ of $M_{A(G_{24})}$has f-vector, $f_{\overline{NBC_{\trianglelefteq}}(M_{A(G_{24})})} = (20,99)$. Thus,from application of the theorems (1.1) and (1.2), we have;

$$\widetilde{H}_d(\Delta) \cong \begin{cases} \mathbb{Z}^{972} & \text{if } d = 2 \\ 0 & \text{if } d \neq 2 \end{cases} \text{and;} \widetilde{H}_1(\overline{NBC}(M_{A(G_{24})}); \mathbb{Z}) \cong \mathbb{Z}^{80};$$

where, $\beta(M) = (-1)^3 \chi(\overline{NBC_{\trianglelefteq}}(M_{A(G_{24})}))$ and$\chi\left(\overline{NBC_{\trianglelefteq}}(M_{A(G_{24})})\right) = -80$.

(3.2): The Complex Reflection Matroidof $A(G_{25})$:

The defining polynomial of $A(\,G_{25})$ is;

$$Q(A(G_{25})) = xyz \prod_{0 \leq i,j \leq 2} (x + \omega^i y + \omega^j z), \text{(see [17]);}$$

where $\omega = e^{\frac{2\pi i}{3}}$. We will order the hyperplanes $H_i = Ker(\alpha_{H_i})$, $1 \leq i \leq 21$, as shown in the following table,see AL- ALeyawee(2005):

$H_1\colon\ x = 0$	$H_2\colon\ y = 0$	$H_3\colon\ z = 0$
$H_4\colon\ x + y + z = 0$	$H_5\colon\ x + y + \omega z = 0$	$H_6\colon\ x + y + \omega^2 z = 0$
$H_7\colon\ x + \omega y + z = 0$	$H_8\colon\ x + \omega y + \omega z = 0$	$H_9\colon\ x + \omega y + \omega^2 z = 0$
$H_{10}\colon\ x + \omega^2 y + z = 0$	$H_{11}\colon\ x + \omega^2 y + \omega z = 0$	$H_{12}\colon\ x + \omega^2 y + \omega^2 z = 0$

Table (3.2): The hyperplanesof $A(G_{25})$

The Matriod$M_{A(G_{25})} = (A(G_{25}), \Delta)hasf$-vector of$\Delta$ is $f = (12,66,184)$.By easy Maple calculation (see appendix (2)), we have $H(\Delta,0) = 1$, $H(\Delta,1) = 12$, $H(\Delta,2) = 78$, $H(\Delta,3) = 328$, $H(\Delta,4) = 762$, … and the h-vector of Δ is $h = (1,9,45,129)$. The broken circuit complex of $A(G_{25})$ was computedby using appendix (1) as $NBC(M_{A(G_{25})}) = \cup_{k=0}^{2} NBC_k(M_{A(G_{25})})$ with$f$-vector of $NBC(M)$ is $f^{\Delta} = (1,12,39,28)$and $\chi_M(t) = t^3 - 12t^2 + 39t - 28$. The type of the Cohen-Macaulay ring $A_{\Delta} = \sum_{m=0}^{\infty} A_m^{rk(A_m)} = A_0^1 \oplus A_1^{12} \oplus A_2^{78} \oplus A_3^{328} \oplus A_4^{762} \oplus ...$ is $h_3 = \beta_3(A_{\Delta}) = f_2{}^{\Delta} = 28$ and it has a minimal free resolution;

$$0 \to M_3^{28} \to M_2^{39} \to M_1^{12} \to M_0^1 \to A_{\Delta} \to 0;$$

with homological dimension of A_{Δ} is $hd_{A_{\Delta}} = n - \dim A_{\Delta} = 12 - 3 = 9$. We computed the reduced broken circuit $\overline{NBC_{\trianglelefteq}}(M_{A(G_{25})})$of $M_{A(G_{25})}$,and it has f-vector, $f_{\overline{NBC_{\trianglelefteq}}(M_{A(G_{25})})} = (1,11,28)$. However;

$$\widetilde{H}_{\Delta}(\Delta) \cong \begin{cases} \mathbb{Z}^{129} & \text{if } d = 2 \\ 0 & \text{if } d \neq 2 \end{cases} \text{and } \widetilde{H}_1\left(\overline{NBC}\left(M_{A(G_{25})}\right); \mathbb{Z}\right) \cong \mathbb{Z}^{18};$$

where, $\beta\left(M_{A(G_{25})}\right) = (-1)^3\chi(\overline{NBC}_{\trianglelefteq}\left(M_{A(G_{25})}\right))$ and $\chi\left(\overline{NBC}_{\trianglelefteq}\left(M_{A(G_{25})}\right)\right) = -18$.

(3.3) The Complex Reflection Matroidof $A(G_{26})$:

The defining polynomial of $A(\,G_{26})$ is;

$Q(A(G_{26})) = (xyz\prod_{0\leq i,j\leq 2}(x + \omega^i y + \omega^j z))\,(x^3 - y^3)(x^3 - z^3)(y^3 - z^3)$, (see [17]);

where $\omega = e^{\frac{2\pi i}{3}}$. The hyperplanes of $A(G_{26})$ will ordered as given in table (3.3):

H_1: $x = 0$	H_2: $y = 0$	H_3: $z = 0$
H_4: $x + y + z = 0$	H_5: $x + y + \omega z = 0$	H_6: $x + y + \omega^2 z = 0$
H_7: $x + \omega y + z = 0$	H_8: $x + \omega y + \omega z = 0$	H_9: $x + \omega y + \omega^2 z = 0$
H_{10}: $x + \omega^2 y + z = 0$	H_{11}: $x + \omega^2 y + \omega z = 0$	H_{12}: $x + \omega^2 y + \omega^2 z = 0$
H_{13}: $x - y = 0$	H_{14}:$x - \omega y = 0$	H_{15}: $x - \omega^2\, y = 0$
H_{16}:$x - z = 0$	H_{17}: $x - \omega z = 0$	H_{18}:$x - \omega^2 z = 0$
H_{19}: $y - z = 0$	H_{20}: $y - \omega z = 0$	H_{21}:$y - \omega^2 z = 0$

Table (3.3): The hyperplane arrangement of G_{26}

The matroid$M_{A(G_{26})}$= $(A(G_{26})$,Δ) on $A(G_{26})$ has$f$-vector of Δ is $f = (21,210,1174)$, and by easy Maple calculation (see appendix (2)), we have $H(\Delta,0) = 1$, $H(\Delta,1) = 21$, $H(\Delta,2) = 231$, $H(\Delta,3) = 1615$, $H(\Delta,4) = 4173$,… and the h-vector of Δ is $h = (1,18,171,984)$. By applying appendix one, the broken circuit complex of $A(G_{26})$ is determined as, $NBC\left(M_{A(G_{26})}\right) = \bigcup_{k=0}^{2} NBC_k\,(M_{A(G_{26})})$ and the f-vector of $NBC(M_{A(G_{26})})$ is$f^{\Delta} = (1,21,111,91)$and $\chi_{M_{A(G_{26})}}(t) = t^3 - 21t^2 + 111t - 91$.The type of the Cohen-Macaulay ring;$A_\Delta = \sum_{m=0}^{\infty} A_m^{rk(A_m)} = A_0^1 \oplus A_1^{21} \oplus A_2^{231} \oplus A_3^{1615} \oplus A_4^{4173} \oplus ...$;

is$h_3 = \beta_3(A_\Delta) = f_2{}^{\Delta} = 87$ and it has a minimal free resolution;

$$0 \to M_3^{91} \to M_2^{111} \to M_1^{21} \to M_0^{1} \to A_\Delta \to 0;$$

and the homological dimension of A_Δ is $hd_{A_\Delta} = n - dimA_\Delta = 21 - 3 = 18$.

The reduced broken circuit $\overline{NBC}_{\trianglelefteq}(M_{A(G_{26})})$has f-vector, $f_{\overline{NBC}_{\trianglelefteq}(M_{A(G_{26})})} = (20,91)$,and as an application of the theorems (1.1) and (1.2), we have;

$$\widetilde{H}_d(\Delta) \cong \begin{cases} \mathbb{Z}^{984} & \text{if } d = 2 \\ 0 & \text{if } d \neq 2 \end{cases} \text{and } \widetilde{H}_1\left(\overline{NBC}\left(M_{A(G_{26})}\right); \mathbb{Z}\right) \cong \mathbb{Z}^{72};$$

where, $\beta\left(M_{A(G_{26})}\right) = (-1)^3\chi(\overline{NBC}_{\trianglelefteq}\left(M_{A(G_{26})}\right))$ and $\chi\left(\overline{NBC}_{\trianglelefteq}\left(M_{A(G_{26})}\right)\right) = -72$

(3.4) The Complex Reflection Group$A(G_{27})$:

All the reflections of G_{27} are of order 2, but they are not the complexification of a real reflections.The corresponding reflection arrangement has "45" hyperplane and defined by:

$$Q\big(A(G_{27})\big) = \prod_{1\le i,j,k\le 3}\big(x_i \pm \omega x_j\big)\big(x_i \pm \gamma x_j + \pm\gamma^2 x_k\big)\big(x_i \pm \omega\gamma^2 x_j \pm \omega^2\gamma x_k\big)$$

$$(x_i \pm (1-\omega^2\gamma)x_j \pm \omega x_k);$$

where(i,j,k) is a cyclic permutation of (1 2 3), $\omega = e^{\frac{2\pi i}{3}}$ and γ is the root of the equation $t^2 + t - 1 = 0$, i.e. $\gamma = \frac{1}{2}(-1+\sqrt{5})$, (see Shephard and Todd (1954)). Therefore, the hyperplanes of $A(G_{27})$ will orderd as given in table (3.4):

$H_1: x = 0$	$H_2 : y = 0$	$H_3 : z = 0$
$H_4 : y + \omega x = 0$	$H_5 : y - \omega x = 0$	$H_6 : z + \omega y = 0$
$H_7 : z - \omega y = 0$	$H_8 : x + \omega z = 0$	$H_9 : x - \omega z = 0$
$H_{10} : x + \gamma\, y + \gamma^2\, y = 0$	$H_{11} : x - \gamma y - \gamma^2 y = 0$	$H_{12} : x + \gamma y - \gamma^2 z = 0$
$H_{13} : x - \gamma\, y + \gamma^2 z = 0$	$H_{14} : y + \gamma z + \gamma^2 x = 0$	$H_{15} : y - \gamma z - \gamma^2 x = 0$
$H_{16} : y + \gamma z - \gamma^2 x = 0$	$H_{17} : y - \gamma z + \gamma^2 x = 0$	$H_{18} : z + \gamma x + \gamma^2 y = 0$
$H_{19} : z - \gamma x - \gamma^2 y = 0$	$H_{20} : z + \gamma x - \gamma^2 y = 0$	$H_{21} : z - \gamma x + \gamma^2 y = 0$
$H_{22} : x + (1-\omega^2\gamma)y + \gamma z = 0$	$H_{23} : x - (1-\omega^2\gamma)y - \omega z = 0$	$H_{24} : x + (1-\omega^2\gamma)y - \omega z = 0$
$H_{25} : x - (1-\omega^2\gamma)y + \omega z = 0$	$H_{26} : y + (1-\omega^2\gamma)z + \omega x = 0$	$H_{27} : y - (1-\omega^2\gamma)z - \omega = 0$
$H_{28} : y + (1-\omega^2\gamma)z - \omega = 0$	$H_{29} : y - (1-\omega^2\gamma)z + \omega x = 0$	$H_{30} : z + (1-\omega^2\gamma)x + \omega y = 0$
$H_{31} : z - (1-\omega^2\gamma)x - \omega y = 0$	$H_{32} = z + (1-\omega^2\gamma)x - \omega = 0$	$H_{33} : z - (1-\omega^2\gamma)x + \omega y = 0$
$H_{34} : x + \omega\gamma^2 y + \omega^2\gamma z = 0$	$H_{35} : x - \omega\gamma^2 y - \omega^2\gamma z = 0$	$H_{36} : x + \omega\gamma^2 y - \omega^2\gamma z = 0$
$H_{37} : x - \omega\gamma^2 y + \omega^2\gamma z = 0$	$H_{38} : y + \omega\gamma^2 z + \omega^2\gamma x = 0$	$H_{39} : y - \omega\gamma^2 z - \omega^2\gamma x = 0$
$H_{40} : y + \omega\gamma^2 z - \omega^2\gamma x = 0$	$H_{41} : y - \omega\gamma^2 z + \omega^2\gamma x = 0$	$H_{42} : z + \omega\gamma^2 x + \omega^2\gamma y = 0$
$H_{43} : z - \omega\gamma^2 x - \omega^2\gamma y = 0$	$H_{44} : z + \omega\gamma^2 x1 - \omega^2\gamma y = 0$	$H_{45} : z - \omega\gamma^2 x + \omega^2\gamma y = 0$

Table (3.4) The hyperplane arrangement of $A(G_{27})$.

The matroid$M_{A(G_{27})}$= $(A(G_{27})$,Δ) on $A(G_{27})$has f-vector of Δ is $f = (45,990,13290)$, and by Maple calculation in appendix (2), we have $H(\Delta,0) = 1$, $H(\Delta,1) = 45$, $H(\Delta,2) = 1035$, $H(\Delta,3) = 15668$, $H(\Delta,4) = 42885$,… and the h-vector of Δ is$h = (1,42,903,12344,0)$. The broken circuit complex of $M_{A(G_{27})}$is given as,$NBC\big(M_{A(G_{27})}\big) = \cup_{k=0}^{2} NBC_k\,(M_{A(G_{27})})$, with$f$-vector, $f^\Delta = (1,45,519,475)$ and $\chi_M(t) = t^3 - 45t^2 + 519t - 475$. The type of the Cohen-Macaulay ring A_Δ

$$A_\Delta = \sum_{m=0}^{\infty} A_m^{rk(A_m)} = A_0^1 \oplus A_1^{45} \oplus A_2^{1035} \oplus A_3^{15668} \oplus A_4^{42885} \oplus \ldots;$$

is$h_3 = \beta_3(A_\Delta) = f_2{}^\Delta = 475$ and it has the following minimal free resolution;

$$0 \to M_3^{475} \to M_2^{519} \to M_1^{45} \to M_0^1 \to A_\Delta \to 0;$$

and the homological dimension of A_Δ is $hd_{A_\Delta} = n - dimA_\Delta = 45 - 3 = 42$. The reduced broken circuit $\overline{NBC_{\trianglelefteq}}(M_{A(G_{27})})$ of $M_{A(G_{27})}$has f-vector, $f_{\overline{NBC_{\trianglelefteq}}(M_{A(G_{27})})} = (1,44,475)$and we have;

$$\widetilde{H}_d(\Delta) \cong \begin{cases} \mathbb{Z}^{12344} & \text{if } d = 2 \\ 0 & \text{if } d \neq 2 \end{cases} \text{and;} \widetilde{H}_1\left(\overline{NBC}\left(M_{A(G_{27})}\right); \mathbb{Z}\right) \cong \mathbb{Z}^{432};$$

where, $\beta\left(M_{A(G_{27})}\right) = (-1)^3\chi(\overline{NBC_{\trianglelefteq}}\left(M_{A(G_{27})}\right))$ and $\chi\left(\overline{NBC_{\trianglelefteq}}\left(M_{A(G_{27})}\right)\right) = -432.$

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Appendix (1): In the following Maple program we compute the intersection between any two hyperplanes and among three hyperplanes by using the instruction "solve" in order to construct the whole intersection lattice of an arrangement of rank three. For general reference in Maple programs, we refer the reader to Wright (2002).

```
>restart:
h(1):=(x1,x2,x3)->x1:
h(2):=(x1,x2,x3)->x2:
h(3):=(x1,x2,x3)->x3:
.
.
.
h(|A|):=(x1,x2,x3)->…:
k:=0:
for i from 1 to 20 do
for j from i+1 to 21 do
k:=k+1:
Ai(k):=i;
Aj(k):=j;
d(k):=solve({h(i)(x1,x2,x3),h(j)(x1,x2,x3)},[x1,x2,x3]);
od;
od;
for z from 1 to k do
A:= vector [row] ([Ai(z),Aj(z)]);
d(z);
od;
k:=0:
for i from 1 to 19 do
for j from i+1 to 20 do
for m from j+1 to 21 do
k:=k+1:
Ai(k):=i;
Aj(k):=j;
Am(k):=m;
d(k):=solve({h(i)(x1,x2,x3),h(j)(x1,x2,x3),h(m)(x1,x2,x3)},[x,x2,x]);
od;
od;`
od;
for z from 1 to k do
A:= vector [row] ([Ai(z),Aj(z),Am(z)]);
d(z);
od;
```

Appendix (2):In the following Maple program, we give the values of the f-vector of any simplicial complex Δin order to calculate the Hilbert function $H(\Delta, n)$ and the h-vector of Δ.

```
> restart:
with(PolynomialTools):
delta:=l: m:=100:
A:=([ [f_0, ..., f_{l+1}]]):
for i from 1 to delta+1 do
f[i-1]:=A[i];
od;
H(Delta,0):=1;
for n from 1 to m do
H(Delta,n):=sum(f[k]*(binomial(n-1,k)),k=0..delta);
od;
eq:=(1-x)^(delta+1):
eq[1]:=expand(eq):
eq[2]:=eq[1]*sum(H(Delta,k)*x^k,k=0..m):
eq[3]:=expand(eq[2]):
e:=CoefficientList(eq[3],x):
for i from  1 to 10 do
h[i-1]:=e[i];
od;
```

Appendix (3):In the following program, we give the values of the d-vector of any supersolvable arrangement in order to compute the f-vector to its broken circuit complex as an application of theorem(1.3) in [3] :

```
> restart:
with(PolynomialTools):
l:=rk(A):
d:=[ [d_1, ..., d_l]];
q[1]:=(t-1):
g[1]:=(t+1):
for i from 2 to l do
q[i]:=q[i-1]*(t-d[i]):
g[i]:=g[i-1]*(t+d[i]):
od:
chi(A,t)=expand(q[l]);
eq[1]:=expand(g[l]):
e:=CoefficientList(eq[1],t):
for j from -1 to l do
f[j]^(Delta)=e[l-j]:
od;
```

Presses inter to make the programs work.