Separation Axioms in Mated Fuzzy Topological space

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Abstract

We introduce separation axioms in mated fuzzy topological spaces and investigate some properties of them.

Keywords: Intuitionistic fuzzy set, Mated fuzzy topological space, Separation axioms in mated fuzzy topological spaces, MFR_m , $m = 0, 1, 2, 3, 4$.

الملخص:

في هذا البحث نحن عرفنا بديهيات الفصل في الفضاء التبولوجي الضبابي المزاوج , و برهنا بعض النتائج حول ذلك.

Introduction:

In 1986, Atanassov introduced intuitionistic fuzzy sets in a set X. Dogan Coker in 1995 constructed the basic concept, so-called"Intuitionistic Fuzzy points" and related object such that as "qusi-coincidence". After that in 1996, D. Coker and A. Demirci introduced a mated fuzzy topological space. In 2001, Eun. Pyo. Lee and Young. Bin Im introduced the concept of closure and interior defined by a mated fuzzy topological space. In 2002, T. Mondal and S. Smanta introduce a concept of intuitionistic gradation of openness of fuzzy subset of nonempty set X and an intuitionistic fuzzy topological spaces with respect to gradation of openness. They defined an intuitionistic fuzzy topological space (IFTS). In 2005, Won. K. Min and C. k. Park introduced the concept of closure and interior defined by intuitionistic gradation of openness. In 2006 , Yue and Fang considered the separation axioms T_0, T_1 *and* T_2 in an I-fuzzy topological space. Also in 2006, Yong Chan Kim and S. E. Abbas introduced separation axioms in intuitionistic fuzzy topological space. In this paper, we introduce separation axioms in mated fuzzy topological spaces and investigate some properties of them.

1.Mated Fuzzy Topological Space

Definition (1.1) Coker D. and Demirci M (1995)

Let $A=\{\langle x, \mu_A(x), \nu_A(x)\rangle : x\in X\}$ and $B=\{\langle x, \mu_B(x), \nu_B(x)\rangle : x\in X\}$ be two intuitionistic fuzzy sets in X. A is said to be quasi-coincident with B (written $A\ q\ B$) if there exists an element x *in* X such that $\mu_A(x) + \mu_B(x) > 1$ and $\nu_A(x) + \nu_B(x) < 1$.While if there is no such element $x \in X$ such that $\mu_A(x) + \mu_B(x) > 1$ and $\nu_A(x) + \nu_B(x) < 1$, *then it is denoted*

$(A\ \bar{q}\ B)$.

Definition (1. 2) Coker D. and Demirci M (1995)

Let $x_{(\alpha,\beta)} \in P_t(IF(X))$ *and* $A \in IF(X)$.We say that $x_{(\alpha,\beta)}$ quasi-coincident with A, denoted $x_{(\alpha,\beta)}qA$ if $\mu_A(x) + \alpha > 1$ and $\nu_A(x) + \beta < 1$.

Definition (1.3) Coker D. (1997)

An intuitionistic fuzzy topology on X is a family τ of intuitionistic fuzzy sets in X which satisfies the following properties:

(i) $\bar{0}, \bar{1} \in \tau$

(ii) If $A_1, A_2 \in \tau, then\ A_1 \cap A_2 \in \tau$

(iii) If $A_i \in \tau$ for all i, then $\cup A_i \in \tau$

The pair (X, τ) is called an intuitionistic fuzzy topological space.

Definition (1.4) Coker D. and Demirci M. (1996)

Let X be a nonempty set. A mated fuzzy topology (T, T^*) on X is two maps $T, T^*: IF(X)\rightarrow I$ which satisfy the following properties :

(i) $T(A) + T^*(A) \leq 1$

(ii) $T(\bar{0}) = T(\bar{1}) = 1$ *and* $T^*(\bar{0}) = T^*(\bar{1}) = 0$

(iii) $T(A\cap B) \geq T(A) \wedge T(B)$ *and* $T^*(A\cap B) \leq T^*(A) \vee T^*(B)$

(iv) $T(\cup A_i) \geq \bigwedge T(A_i)$ *and* $T^*(\cup A_i) \leq \bigvee T^*(A_i)$

The (X, T, T^*) is called mated fuzzy topological space. And we call T a gradation of openness and T^* is called a gradation of nonopenness.

Definition (1.5) Lee E. P. and Im Y. B. (2002)

Let A be an intuitionistic fuzzy set of a mated fuzzy topological space (X, T, T^*) and $r, s \in I, r+s \le 1$. Then A is called:

(i) an intuitionistic fuzzy (r, s)-open set if $T(A) \ge r$ and $T^*(A) \le s$.

(ii) an intuitionistic fuzzy (r, s)-closed set if $T(A^c) \ge r$ and $T^*(A^c) \le s$.

Definition (1.6) Lee E. P. and Im Y. B. (2002)

Let (X, T, T^*) be a mated fuzzy topological space. For each $(r, s) \in IXI$ and each $A \in IF(X)$. The fuzzy (r, s)-interior is defined by

$$MFI(A, r, s) = \cup\{B \in IF(X): B \subseteq A \text{ and } B \text{ is a fuzzy } (r, s)\text{-open set}\}.$$

And the fuzzy (r, s)-closure is defined by

$$MFC(A, r, s) = \cap\{B \in IF(X): A \subseteq B \text{ and } B \text{ is a fuzzy } (r, s)\text{-closed set}\}.$$

The operators $MFI: IF(X) \; X \; I \; X \; I \to IF(X)$ and $MFC: IF(X) \; X \; I \; X \; I \to IF(X)$ are called the fuzzy interior operator and fuzzy closure operator in (X, T, T^*), respectively.

Theorem (1.1) Lee E. P. and Im Y. B. (2002)

Let (X, T, T^*) be a mated fuzzy topological space and MFI the fuzzy interior operator in (X, T, T^*). Then for any $A, B \in IF(X)$ and $r, s \in I$, $r+s \le 1$.

(i) $MFI(\bar{0}, r, s)=\bar{0}$ and $MFI(\bar{1}, r, s)=\bar{1}$.

(ii) $MFI(A, r, s) \subseteq A$.

(iii) $MFI(A, r_2, s_2) \subseteq MFI(A, r_1, s_1)$ if $r_1 \le r_2$ and $s_1 \ge s_2$.

(iv) $MFI(A_1, r, s) \cap MFI(A_2, r, s) = MFI(A_1 \cap A_2, r, s)$.

(v) $MFI(MFI(A, r, s), r, s) = MFI(A, r, s)$.

Theorem (1.2) Lee E. P. and Im Y. B. (2002)

Let (X, T, T^*) be a mated fuzzy topological space and Let MFC the fuzzy closure operator in (X, T, T^*). Then for A , $B \in IF(X)$ and $r, s \in I$,$r+s \leq 1$

(i) $MFC(\overline{0}, r, s)=\overline{0}$ and $MFC(\overline{1}, r, s)=\overline{1}$

(ii) $A \subseteq MFC(A, r, s)$

(iii) $MFC(A, r_1, s_1) \subseteq MFC(A, r_2, s_2)$ if $r_1 \leq r_2$ and $s_1 \geq s_2$

(iv) $MFC(A, r, s) \cup MFC(B, r, s) = MFC(A \cup B, r, s)$

(v) $MFC(MFC(A, r, s), r, s) = MFC(A, r, s)$

Theorem (1.3) Lee E. P. and Im Y. B (2002)

For intuitionistic fuzzy set A of a mated fuzzy topological space (X, T, T^*) and $r, s \in I$, we have

(i) $MFI(A, r, s)^c = MFC(A^c, r, s)$.

(ii) $MFC(A, r, s)^c = MFI(A^c, r, s)$

Theorem (1.4) Coker D. and Demirci M. (1995)

If A and B are intuitionistic fuzzy sets in X, then :

(i) $A \bar{q} B$ if and only if $A \subseteq B^c$

(ii) $A q B$ if and only if $A \nsubseteq B^c$

Lemma (1. 5)

For $A, A_i, B \in IF(X)$ *and* $x_{(\alpha, \beta)} \in P_t(IF(X))$, we have:

(i) $A \subseteq B$ if and only if for $x_{(\alpha, \beta)} \in A$ then $x_{(\alpha, \beta)} \in B$.

(ii) $A \subseteq B$ if and only if for $x_{(\alpha, \beta)} q A$ then $x_{(\alpha, \beta)} q B$.

(iii) $x_{(\alpha, \beta)} q \cup A_i$ if and only if there exist $i \in \Gamma$ such that $x_{(\alpha, \beta)}\ q\ A_i$.

Proof:

(i)

Let $A \subseteq B$ and $x_{(\alpha, \beta)} \in A$, then $\mu_A(y) \leq \mu_B(y)$ and $\nu_A(y) \geq \nu_B(y)$. $\forall y \in X, \alpha \leq \mu_A(y)$ and $\beta \geq \nu_A(y)$.

Thus $\alpha \leq \mu_B(y)$ and $\beta \geq \nu_B(y)$. Hence $x_{(\alpha,\beta)} \in B$.

Let $x_{(\alpha,\beta)} \in A$, then $x_{(\alpha,\beta)} \in B$

This means $\alpha \leq \mu_A(x), \beta \geq \nu_A(x) \Longrightarrow \alpha \leq \mu_B(x), \beta \geq \nu_B(x)$

Thus $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$. $\forall x \in X$

Hence $A \subseteq B$.

(ii)

Let $x_{(\alpha,\beta)} qA$, then there exist $y \in X$ such that: $\alpha + \mu_A(y) > 1$ *and*

$\beta + \nu_A(y) < 1$. Since $A \subseteq B$, we have $\alpha + \mu_B(y) > 1$ and $\beta + \nu_B(y) < 1$. Hence $x_{(\alpha,\beta)} qB$.

Suppose that $x_{(\alpha,\beta)} qA$, we have $x_{(\alpha,\beta)} qB$.

Thus $A \subseteq B$.

(iii)

It is clear ■

<u>Lemma (1. 6)</u>

Let (X, T, T^*) be a mated fuzzy topological space, we have the following :

(i) $A\ q\ B$ if and only if $A\ q\ MFC(B, r, s)$, for all $T(A) \geq r$, $T^*(A) \leq s$,

(ii) $x_{(\alpha,\beta)}\ q\ MFC(A, r, s)$ if and only if $A\ q\ B$ for all $T(B) \geq r$, $T^*(B) \leq s$ with $x_{(\alpha,\beta)} \in B$.

Proof:

(i)

Since $A\ q\ B$ then there exist $x \in X$ such that: $\mu_A(x) + \mu_B(x) > 1$ and $\nu_A(x) + \nu_B(x) < 1$.

Since $B \subseteq MFC(B, r, s)$, put $MFC(B, r, s) = H$.Then $\mu_B(y) \leq \mu_H(y)$ and $\nu_B(y) \geq \nu_H(y)$ for all $y \in X$. Thus $\mu_A(x) + \mu_H(x) > 1$ and $\nu_A(x) + \nu_H(x) < 1$. Hence $A\ q\ MFC(B, r, s)$.

If there exists $A \in IF(X)$ with $T(A) \geq r$, $T^*(A) \leq s$ such that:

$A\ \bar{q}\ B$, then $B \subseteq A^c$

By definition of MFC, we have $MFC(B, r, s) \subseteq A^c$

Thus $A\ \bar{q}\ MFC(B, r, s)$, it is contradiction.

Hence $A\ q\ B$.

(ii)

Suppose that $x_{(\alpha, \beta)}\ q\ MFC(A, r, s)$

Since $x_{(\alpha, \beta)} \in B$, then $B\ q\ MFC(A, r, s)$

By (i), we have $B\ q\ A$ for all $T(B) \geq r$, $T^*(B) \leq s$.

Suppose that $x_{(\alpha, \beta)}\ \bar{q}\ MFC(A, r, s)$, then $x_{(\alpha, \beta)} \in MFI(A^c, r, s)$

Since $A \subseteq MFC(A, r, s)$, we have $A\ \bar{q}\ MFI(A^c, r, s)$, such that $T(MFI(A^c, r, s)) \geq r$ and

$T^*(MFI(A^c, r, s)) \leq s$, *and this contradict with* $A\ q\ B$ for all $T(B) \geq r$, $T^*(B) \leq s$ with $x_{(\alpha, \beta)} \in B$.

Hence $x_{(\alpha, \beta)}\ q\ MFC(A, r, s)$. ■

2.Separation Axioms in Mated Fuzzy Topological Space.

Definition (2. 1)

A mated fuzzy topological space (X, T, T^*) is said to be:

(i) MFR_0 if and only if for any intuitionistic fuzzy points $x_{(\alpha, \beta)}$, $y_{(\alpha_1,\beta_1)} \in P_t(IF(X))$ and

$x_{(\alpha, \beta)}\ \bar{q}\ MFC(y_{(\alpha_1,\beta_1)}\ r, s)$ implies $y_{(\alpha_1,\beta_1)}\bar{q}\ MFC(x_{(\alpha,\beta)}, r, s)$

(ii) MFR_1 if and only if for any $x_{(\alpha, \beta)}\ \bar{q}\ MFC(y_{(\alpha_1,\beta_1)}\ r, s)$, there exist $A_i \in IF(X)$ *with* $T(A_i) \geq r$, $T^*(A_i) \leq s$ *for* $i \in \{1, 2\}$ such that : $x_{(\alpha, \beta)} \in A_1\ y_{(\alpha_1,\beta_1)} \in A_2$ *and* $A_1 \bar{q} A_2$.

(iii) MFR_2 if and only if for any $x_{(\alpha, \beta)}\ \bar{q}\ A$ *with* $T(A^c) \geq r$, $T^*(A^c) \leq s$, there exist $A_i \in IF(X)$ *with* $T(A_i) \geq r$, $T^*(A_i) \leq s$ *for* $i \in \{1, 2\}$ such that : $x_{(\alpha, \beta)} \in A_1$, $A \subseteq A_2$ *and* $A_1\ \bar{q}\ A_2$.

(iv) MFR_3 if and only if for any $A_1 \bar{q} A_2$ with $T(A_i^c) \geq r$, $T^*(A_i^c) \leq s$ *for* $i \in \{1, 2\}$, there exist $B_i \in IF(X)$ *with* $T(B_i) \geq r$, $T^*(B_i) \leq s$ such that : $A_1 \subseteq B_1$, $A_2 \subseteq B_2$ *and* $B_1 \bar{q} B_2$.

(v) MFR_4 if and only if MFR_3 and MFR_0.

Theorem (2. 1)

Let (X, T, T^*) be a mated fuzzy topological space, then the following statements are equivalent:

(i) (X, T, T^*) is a MFR_0-space.

(ii) $MFC(x_{(\alpha,\beta)}, r, s) \subseteq B$ for all $T(B) \geq r$, $T^*(B) \leq s$ with $x_{(\alpha,\beta)} \in B$.

(iii) If $x_{(\alpha,\beta)}\, \bar{q}\, A$ with $T(A^c) \geq r$ *and* $T^*(A^c) \leq s$, then $\exists\, B \in IF(X)$ *wher* $T(B) \geq r$, $T^*(B) \leq s$ such that $x_{(\alpha,\beta)}\, \bar{q}\, B, A \subseteq B$.

(iv) $x_{(\alpha,\beta)}\, \bar{q}\, A$ with $T(A^c) \geq r$, $T^*(A^c) \leq s$, then $MFC(x_{(\alpha,\beta)}, r, s)\, \bar{q}\, A$.

(v)If $x_{(\alpha,\beta)}\, \bar{q}\, MFC(y_{(\alpha_1,\beta_1)}, r, s)$, then $MFC(x_{(\alpha,\beta)}, r, s)\, \bar{q}\, MFC(y_{(\alpha_1,\beta_1)}, r, s)$.

Proof:

If $y_{(\alpha_1,\beta_1)}\, q\, MFC(x_{(\alpha,\beta)}, r, s)$, by Lema 1. 6 (i), we show that only $y_{(\alpha_1,\beta_1)}\, q\, B$

for all $T(B) \geq r$, $T^*(B) \leq s$ with $x_{(\alpha,\beta)} \in B$.

From $y_{(\alpha_1,\beta_1)}\, q\, MFC(x_{(\alpha,\beta)}, r, s)$, *we get* $x_{(\alpha,\beta)}\, q\, MFC(y_{(\alpha_1,\beta_1)}, r, s)$.

Hence $y_{(\alpha_1,\beta_1)}\, qB$ for all $T(B) \geq r$, $T^*(B) \leq s$ with $x_{(\alpha,\beta)} \in B$ by Lemma 1. 6 (ii).

Let $x_{(\alpha,\beta)} \bar{q} A, T(A) \geq r, T^*(A^c) \leq s$

Since $MFC\big(y_{(\alpha_1,\beta_1)}, r, s\big) \subseteq MFC(A, r, s), for\ each\ y_{(\alpha_1,\beta_1)} \in A, and\ MFC(A, r, s) = A$

Then $MFC\big(y_{(\alpha_1,\beta_1)}, r, s\big) \subseteq A$. Thus $x_{(\alpha,\beta)} \bar{q}\ MFC\big(y_{(\alpha_1,\beta_1)}, r, s\big)$

Hence $x_{(\alpha,\beta)} \in (\ MFC(y_{(\alpha_1,\beta_1)}, r, s))^c$

By (ii) we have $MFC(x_{(\alpha,\beta)}, r, s) \subseteq (\ MFC(y_{(\alpha_1,\beta_1)}, r, s))^c \subseteq (y_{(\alpha_1,\beta_1)})^c$

Therefore $MFC(x_{(\alpha,\beta)}, r, s) \subseteq (y_{(\alpha_1,\beta_1)})^c$. Thus $y_{(\alpha_1,\beta_1)} \bar{q}\ MFC\big(x_{(\alpha,\beta)}\ , r, s\big)$

Hence (X, τ, τ^*) is a MFR_0-space.

By Lamma (1. 5)(ii), there exist $H \in IF(X), \tau(H) \geq r, \tau^*(H) \leq s$ such that: $x_{(\alpha,\beta)} \bar{q} H$

Let $B = \cup \{H : x_{(\alpha,\beta)} \bar{q} H\}$. By Lamma (1. 6)(ii), and definition (1. 4)

We have $x_{(\alpha,\beta)} \bar{q} B, A \subseteq B\ and\ , \tau(B) \geq r, \tau^*(B) \leq s$.

If $x_{(\alpha,\beta)}\ \bar{q}\ A$ with $\tau(A^c)\geq r,\ \tau^*(A^c)\leq s$. By (iii), there exists $B\in IF(X)$ such that $x_{(\alpha,\beta)}\ \bar{q}\ B$.

$A\subseteq B,\ \tau(B)\geq r,\ \tau^*(B)\leq$ s. Since $x_{(\alpha,\beta)}\ \bar{q}\ B$, we have $x_{(\alpha,\beta)}\subseteq B^c$.

Hence $MFC(x_{(\alpha,\beta)}, r, s)\ \bar{q}\ A$.

Let $x_{(\alpha,\beta)}\ \bar{q}\ MFC(y_{(\alpha_1,\beta_1)}, r, s)$, by (iv). Then $MFC(x_{(\alpha,\beta)}, r, s)\ \bar{q}\ MFC(y_{(\alpha_1,\beta_1)}, r, s)$.

Let $x_{(\alpha,\beta)}\ \bar{q}\ MFC(y_{(\alpha_1,\beta_1)}, r, s)$, by (v), we have $MFC(x_{(\alpha,\beta)}, r, s)\ \bar{q}\ MFC(y_{(\alpha_1,\beta_1)}, r, s)$.

Thus $y_{(\alpha_1,\beta_1)}\ \bar{q}\ MFC(x_{(\alpha,\beta)}, r, s)$. Hence (X, τ, τ^*) *is an* MFR_0. ■

Theorem (2. 2)

Let (X, T, T^*) be a mated fuzzy topological space, then:

$MFR_4 \Rightarrow MFR_2 \Rightarrow MFR_1 \Rightarrow MFR_0$

Proof:

$(MFR_4 \Rightarrow MFR_2)$

Let Let $x_{(\alpha,\beta)}\ \bar{q}\ A$ with $T(A^c)\geq r,\ T^*(A^c)\leq s$. Since (X, T, T^*) is a MFR_0-space, by theorem 2. 1

We have $MFC(x_{(\alpha,\beta)}, r, s)\ \bar{q}\ A$.

Since (X, T, T^*) *is a* MFR_3-space, there exist $A_i\in IF(X)$ *for* $i\in\{1, 2\}$such that:

$MFC(x_{(\alpha,\beta)}, r, s)\subseteq A_1,\ A\subseteq A_2$ *and* $A_1\ \bar{q}\ A_2,\ T(A_i)\geq r,\ T^*(A_i)\leq s$. Hence (X, T, T^*) *is a* MFR_2-space.

$(MFR_2 \Rightarrow MFR_1)$ It is trivial

$(MFR_1 \Rightarrow MFR_0)$

Let $x_{(\alpha,\beta)}\ \bar{q}\ MFC(y_{(\alpha_1,\beta_1)}, r, s)$ for intuitionistic fuzzy points $x_{(\alpha,\beta)}, y_{(\alpha_1,\beta_1)}$

Since (X, T, T^*) *is a* MFR_1-space, there exist $A_i\in IF(X)$ *for* $i\in\{1, 2\}$such that $x_{(\alpha,\beta)}\in A_1$, $y_{(\alpha_1,\beta_1)}\in A_2$ *and* $A_1\ \bar{q}\ A_2,\ T(A_i)\geq r,\ T^*(A_i)\leq s$.

Hence $x_{(\alpha,\beta)}\in A_1\subseteq A^c_2$, by definition of C it is follows $MFC(x_{(\alpha,\beta)}, r, s)\subseteq A^c_2$ *and* $A^c_2\subseteq (y_{(\alpha_1,\beta_1)})^c$. Thus $MFC(x_{(\alpha,\beta)}, r, s)\subseteq (y_{(\alpha_1,\beta_1)})^c$. Hence $y_{(\alpha_1,\beta_1)}\ \bar{q}\ MFC(x_{(\alpha,\beta)}, r, s)$.

Example(2. 1)

Let $X = \{x, y\}$ and (X, T, T^*) be a mated fuzzy topological space as follows:

$$T(G) = \begin{cases} 1 & \text{if } G = \bar{0} \text{ or } \bar{1} \\ \frac{1}{2} & \text{if } G = A \\ \frac{1}{2} & \text{if } G = B \\ 0 & \text{otherwise} \end{cases}, \quad T^*(G) = \begin{cases} 0 & \text{if } G = \bar{0} \text{ or } \bar{1} \\ \frac{1}{2} & \text{if } G = A \\ \frac{1}{2} & \text{if } G = B \\ 1 & \text{otherwise} \end{cases}$$

Where: $A = \{\langle x, 0.7, 0.3\rangle, \langle y, 0.2, 0.8\rangle\}$, $B = \{\langle x, 0.3, 0.7\rangle, \langle y, 0.8, 0.2\rangle\}$

We obtain the operator $MFC: IF(X) \ X \ I \ X \ I \rightarrow IF(X)$ as follows:

$$MFC(G, r, s) = \begin{cases} \bar{0} & \text{if } G = \bar{0},\ r, s \in I \\ A & \text{if } \bar{0} \neq G \subseteq A,\ 0 \leq r \leq \frac{1}{2},\ \frac{1}{2} \leq s < 1 \\ B & \text{if } \bar{0} \neq G \subseteq B,\ 0 \leq r \leq \frac{1}{2}, \\ \bar{1} & \text{otherwise} \end{cases}$$

(1) (X, T, T^*) is a MFR_0-space from because

For $0 \leq r \leq \frac{1}{2}$, $\frac{1}{2} \leq s$ $\quad x_{(\alpha, \beta)} \bar{q}\ MFC(y_{(\alpha_1, \beta_1)}, r, s) \Leftrightarrow y_{(\alpha_1, \beta_1)} \bar{q}\ MFC(x_{(\alpha, \beta)}, r, s)$.

Since $MFC(y_{(\alpha_1, \beta_1)}, r, s) = B$ *and* $MFC(x_{(\alpha, \beta)}, r, s) = A$ *where* $y_{(\alpha_1, \beta_1)} \in B$ *and* $x_{(\alpha, \beta)} \in A$.

(2) *Since* $A \bar{q} B$ *with* $T(A^c) \geq r$, $T^*(A^c) \leq s$ *for* $0 \leq r \leq \frac{1}{2}$, $\frac{1}{2} \leq s \leq 1$ *and* $A \subseteq A$, $B \subseteq B$ *and*

$T(A) \geq r$, $T^*(A) \leq s$, $T(B) \geq r$, $T^*(B) \leq s$. Hence (X, T, T^*) *is a* MFR_3-space

There for (X, T, T^*) *is a* MFR_4-space. Then (X, T, T^*) *is a* MFR_2 and MFR_1-space.

Example (2. 2)

Let $X = \{x, y\}$ *and* (X, T, T^*) be a mated fuzzy topological space as follows:

$$T(G) = \begin{cases} 1 & \text{if } G = \bar{0} \text{ or } \bar{1} \\ \frac{1}{2} & \text{if } G = A \\ \frac{1}{2} & \text{if } G = B \end{cases}, \quad T^*(G) = \begin{cases} 0 & \text{if } G = \bar{0} \text{ or } \bar{1} \\ \frac{1}{3} & \text{if } G = A \\ \frac{1}{2} & \text{if } G = B \end{cases}$$

0 *otherwise* 1 *otherwise*

Where: $A = \{\langle x, 0.7, 0.3\rangle, \langle y, 0.2, 0.8\rangle\}$, $B = \{\langle x, 0.3, 0.7\rangle, \langle y, 0.8, 0.2\rangle\}$

We obtain the operator $IFC: IF(X) \times I \times I \rightarrow IF(X)$ which is given by

$$MFC(G, r, s) = \begin{cases} \overline{0} & \text{if } G = \overline{0},\ r, s \in I \\ A & \text{if } \overline{0} \neq G \subseteq A,\ 0 \leq r \leq \frac{1}{2},\ \frac{1}{2} \leq s < 1 \\ B & \text{if } \overline{0} \neq G \subseteq B,\ 0 \leq r \leq \frac{1}{2},\ \frac{1}{2} \leq s < 1 \\ \overline{1} & \text{otherwise} \end{cases}$$

(X, T, T^*) *is not* MFR_0-space according to the following reasons:

For $0 \leq r \leq \frac{1}{2}$, $\frac{1}{3} \leq s < \frac{1}{2}$ any $x_{(\alpha, \beta)} \in A \Rightarrow MFC(x_{(\alpha, \beta)}, r, s) = \overline{1}$. And $y_{(\alpha_1, \beta_1)} \in B$

Therefore $MFC(y_{(\alpha_1,\beta_1)}, r, s) = B$. Thus $x_{(\alpha, \beta)}\, \overline{q}\, MFC(y_{(\alpha_1,\beta_1)}, r, s)$,

but $y_{(\alpha_1,\beta_1)}\, q\, MFC(x_{(\alpha, \beta)}, r, s)$. Since (X, T, T^*) *is not* MFR_0-space.Then (X, T, T^*) *is not* MFR_1-space, (X, T, T^*) *is not* MFR_2-space and (X, T, T^*) *is not* MFR_4-space

Example (2. 3)

Let $X = \{x, y, z\}$ *and* (X, T, T^*) be a mated fuzzy topological space as follows:

$$T(G) = \begin{cases} 1 & \text{if } G = \overline{0} \text{ or } \overline{1} \\ \frac{1}{4} & \text{if } G = A \\ \frac{1}{3} & \text{if } G = B \\ 0 & \text{otherwise} \end{cases}, \quad T^*(G) = \begin{cases} 0 & \text{if } G = \overline{0} \text{ or } \overline{1} \\ \frac{3}{4} & \text{if } G = A \\ \frac{2}{3} & \text{if } G = B \\ 1 & \text{otherwise} \end{cases}$$

Where: $A = \{\langle x, 1, 0\rangle, \langle y, 1, 0\rangle, \langle z, 0, 1\rangle\}$, $B = \{\langle x, 0, 1\rangle, \langle y, 0, 1\rangle, \langle z, 1, 0\rangle\}$

We can obtain the operator $MFC: IF(X) \times I \times I \rightarrow IF(X)$ as follows:

$$MFC(G, r, s) = \begin{cases} \overline{0} & \text{if } G = \overline{0},\ r, s \in I \\ A & \text{if } \overline{0} \neq G \subseteq A,\ 0 \leq r \leq \frac{1}{4},\ \frac{3}{4} \leq s < 1 \\ B & \text{if } \overline{0} \neq G \subseteq B,\ 0 \leq r \leq \frac{1}{3},\ \frac{2}{3} \leq s < 1 \end{cases}$$

$\overline{1}$ *otherwise*

(1) (X, T, T^*) is a MFR_0-space from the following: For $0 \leq r \leq \frac{1}{4}$, $\frac{3}{4} \leq s < 1$.

$x_{(\alpha,\beta)}\ \overline{q}\ MFC(z_{(\alpha_2,\beta_2)}, r, s) = B \Leftrightarrow z_{(\alpha_2,\beta_2)}\ \overline{q}\ MFC(x_{(\alpha,\beta)}, r, s) = A$

$y_{(\alpha_1,\beta_1)}\ \overline{q}\ MFC(z_{(\alpha_2,\beta_2)}, r, s) = B \Leftrightarrow z_{(\alpha_2,\beta_2)}\ \overline{q}\ MFC(y_{(\alpha_1,\beta_1)}, r, s) = A$

(2) Since $x_{(\alpha,\beta)}\ \overline{q}\ MFC(z_{(\alpha_2,\beta_2)}, r, s) = B$ but no there exist $G_1, G_2 \in IF(X)$, such that

(3) $x_{(\alpha,\beta)} \in G_1$, $z_{(\alpha_2,\beta_2)} \in G_2$ and $G_1\ \overline{q}\ G_2$. Then (X, T, T^*) is not MFR_1.

Example (2. 4)

Let $X=R$ be a set of real number. Define $T, T^*: IF(X) \to I$ as follows:

$$T(G) = \begin{cases} 1 & \text{if } G = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } G = A_a \\ 0 & \text{otherwise} \end{cases} \qquad T^*(G) = \begin{cases} 0 & \text{if } G = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } G = A_a \\ 1 & \text{otherwise} \end{cases}$$

Where $A_a = \{\langle x, \mu_{Aa}, \nu_{Aa}\rangle : x \in X\}$, *such that:*

$$\mu_{Aa}(x) = \begin{cases} 1 & \text{if } x \in (a, \infty) \\ 0 & \text{if } x \notin (a, \infty) \end{cases} \qquad \nu_{Aa}(x) = \begin{cases} 0 & \text{if } x \in (a, \infty) \\ 1 & \text{if } x \notin (a, \infty) \end{cases}$$

(1) (X, T, T^*) is MFR_1

(2) For $2_{(\alpha,\beta)}\ \overline{q}\ A_a^c$ with $T(A_a) \geq r$, $T^*(A_a) \leq s$, there exist an unique $\overline{1}$ such that $A_a^c \subseteq \overline{1}$, $T(\overline{1}) = 1$, $T^*(\overline{1}) = 0$. Hence (X, T, T^*) is not MFR_2.

Theorem (2. 3)

Let (X, T, T^*) be a mated fuzzy topological space, the following statements are equivalent:

(i) (X, T, T^*) is a MFR_2-space.

(ii) If $x_{(\alpha,\beta)} \in A$ with $T(A) \geq r$, $T^*(A) \leq s$, there exist $B_1 \in IF(X)$ with $T(B_1) \geq r$, $T^*(B_1) \leq s$ such that $x_{(\alpha,\beta)} \in B_1 \subseteq IFC(B_1, r, s) \subseteq A$.

(iii) If $x_{(\alpha, \beta)}\ \bar{q}\ A$ *with* $T(A^c) \geq r, T^*(A^c) \leq s$, there exist $B_i \in IF(X)$ with $T(B_i) \geq r, T^*(B_i) \leq s$ such that $x_{(\alpha, \beta)} \in B_1, A \subseteq B_2$ *and* $MFC(B_1, r, s)\ \bar{q}\ MFC(B_2, r, s)$.

Proof:

Let $x_{(\alpha, \beta)} \in A$ *with* $T(A) \geq r, T^*(A) \leq s$, then $x_{(\alpha, \beta)}\ \bar{q}\ A^c$ *with* $T(A) \geq r, T^*(A) \leq s$.

Since (X, T, T^*) *is an* MFR_2-space, there exist $B_i \in IF(X)$ *with* $T(B_i) \geq r, T^*(B_i) \leq s$

for $i \in [1, 2\}$ *such that* $x_{(\alpha, \beta)} \in B_1, A^c \subseteq B_2$ *and* $B_1\ \bar{q}\ B_2$.

It implies $x_{(\alpha, \beta)} \in B_1 \subseteq B_2{}^c \subseteq A$. *Since* $T(B_i) \geq r, T^*(B_i) \leq s$. Then $x_{(\alpha, \beta)} \in B_1 \subseteq MFC(B_1, r, s) \subseteq A$. Let $x_{(\alpha, \beta)}\ \bar{q}\ A$ *with* $T(A^c) \geq r, T^*(A^c) \leq s$. *Then* $x_{(\alpha, \beta)} \in A^c$.

By(ii), there exist $B \in IF(X)$ *with* $T(B) \geq r, T^*(B) \leq s$ *such that:* $x_{(\alpha, \beta)} \in B \subseteq MFC(B, r, s) \subseteq A^c$. Since $T(B) \geq r, T^*(B) \leq s$ *and* $x_{(\alpha, \beta)} \in B$.

By (ii), there exist $B_i \in IF(X)$ *with* $T(B_i) \geq r, T^*(B_i) \leq s$ *for* $i \in [1, 2\}$ *such that:*

$x_{(\alpha, \beta)} \in B_1 \subseteq MFC(B_1, r, s) \subseteq B \subseteq MFC(B, r, s) \subseteq A^c$. It implies $A \subseteq MFI(B^c, r, s) \subseteq B^c$

Put $B_2 = MFI(B^c, r, s)$. *Then* $T(B_2) \geq r, T^*(B_2) \leq s$, from definition of *MFI*.

Thus $MFC(B_2, r, s) \subseteq B^c \subseteq MFI(B_1{}^c, r, s)$. *Hence* $MFC(B_1, r, s)\ \bar{q}\ MFC(B_2, r, s)$.

Let $x_{(\alpha, \beta)}\ \bar{q}\ A$ *with* $T(A^c) \geq r, T^*(A^c) \leq s$.

By (iii), there exist $B_i \in IF(X)$ *with* $T(B_i) \geq r, T^*(B_i) \leq s$ *for* $i \in [1, 2\}$ *such that* $x_{(\alpha, \beta)} \in B_1$, $A \subseteq B_2$ *and* $MFC(B_1, r, s)\ \bar{q}\ MFC(B_2, r, s)$. *Since* $MFC(B_1, r, s)\ \bar{q}\ MFC(B_2, r, s)$.

Thus $B_1\ \bar{q}\ B_2$. Hence (X, T, T^*) *is* MFR_2. ■

<u>Theorem (2. 4)</u>

Let (X, T, T^*) be a mated fuzzy topological space, the following statements are equivalent:

(i) (X, T, T^*) *is a* MFR_3-space.

(ii) If $A \subseteq B$ for each $T(A^c) \geq r, T^*(A^c) \leq s$, *and* $T(B) \geq r, T^*(B) \leq s$, there exist $G \in IF(X)$ *with* $T(G) \geq r, T^*(G) \leq s$ *such that* $A \subseteq G \subseteq MFC(G, r, s) \subseteq B$.

(iii) If $A_1 \overline{q}\ A_2$ for each $T(A_i) \geq r$, $T^*(A_i) \leq s$, *with* $i \in \{1, 2\}$, there exist $B_i \in IF(X)$ with $T(B_i) \geq r$, $T^*(B_i) \leq s$ such that $A_i \subseteq B_i$ *and* $MFC(B_1, r, s)\ \overline{q}\ MFC(B_2, r, s)$.

Proof:

Let $A \subseteq B$, $T(A^c) \geq r$, $T^*(A^c) \leq s$, *and* $T(B) \geq r$, $T^*(B) \leq s$. *Since* $A \subseteq B$, *then* $A\ \overline{q}\ B^c$

Since (X, T, T^*) *is a* MFR_3-space, there exist $G_i \in IF(X)$ *with* $T(G_i) \geq r$, $T^*(G_i) \leq s$ *for* $i \in [1, 2\}$ *such that:* $A \subseteq G_1$, $B^c \subseteq G_2$ *and* $G_1\ \overline{q}\ G_2$. It implies $A \subseteq G_1 \subseteq G_2^c \subseteq B$, since $T(G_i) \geq r$, $T^*(G_i) \leq s$. *Then* $A \subseteq G_1 \subseteq IFC(G_1, r, s) \subseteq B$.

Let $A_1\ \overline{q}\ A_2$ *with* $T(A_i^c) \geq r$, $T^*(A_i^c) \leq s$, *with* $i \in \{1, 2\}$.

Then $A_1 \subseteq A_2^c$, by(ii), there exist $B \in IF(X)$ *with* $T(B) \geq r$, $T^*(B) \leq s$ *such that:* $A_1 \subseteq B \subseteq MFC(B, r, s) \subseteq A_2^c$. Since $T(B) \geq r$, $T^*(B) \leq s$ *and* $A_1 \subseteq B$.

By (ii), there exist $B_i \in IF(X)$ *with* $T(B_i) \geq r$, $T^*(B_i) \leq s$ *for* $i \in [1, 2\}$ such that *:* $A_1 \subseteq B_1 \subseteq MFC(B_1, r, s) \subseteq B \subseteq IFC(B, r, s) \subseteq A_2^c$. It implies $A \subseteq MFI(B^c, r, s) \subseteq B^c$,

put $B_2 = MFI(B^c, r, s)$. Then $T(B_2) \geq r$, $T^*(B_2) \leq s$, from definition of *MFI*.

Thus $MFC(B_2, r, s) \subseteq B^c \subseteq MFI(B_1^c, r, s)$. Hence $MFC(B_1, r, s)\ \overline{q}\ MFC(B_2, r, s)$.

Let $A_1\ \overline{q}\ A_2$ *with* $T(A_i^c) \geq r$, $T^*(A_i^c) \leq s$.

By (iii), there exist $B_i \in IF(X)$ *with* $T(B_i) \geq r$, $T^*(B_i) \leq s$ *for* $i \in [1, 2\}$ *such that* $A_i \subseteq B_i$ *and* $MFC(B_1, r, s)\ \overline{q}\ MFC(B_2, r, s)$. *Since* $MFC(B_1, r, s)\ \overline{q}\ MFC(B_2, r, s)$.

Thus $B_1\ \overline{q}\ B_2$. Hence (X, T, T^*) *is* MFR_3.

<u>Theorem (2. 5)</u>

A mated fuzzy topological space (X, T, T^*) *is a* MFR_1-space if and only if

$x_{(\alpha, \beta)}\ \overline{q}\ MFC(y_{(\alpha_1, \beta_1)}, r, s)$, there exist $B_i \in IF(X)$ *with* $T(A_i) \geq r$, $T^*(A_i) \leq s$ for $i \in [1, 2\}$ *such that :* $A_1\ \overline{q}\ A_2$ *and* $MFC(x_{(\alpha, \beta)}, r, s) \subseteq A_1$, $MFC(y_{(\alpha_1, \beta_1)}, r, s) \subseteq A_2$.

Proof:

Let $x_{(\alpha,\beta)}\,\overline{q}\,MFC(y_{(\alpha_1,\beta_1)}, r, s)$, there exist $A_i \in IF(X)$ *with* $T(A_i) \geq r$, $T^*(A_i) \leq s$ *for* $i \in [1, 2\}$ *such that :* $A_1\,\overline{q}\,A_2$ *and* $MFC(x_{(\alpha,\beta)}, r, s) \subseteq A_1$, $MFC(y_{(\alpha_1,\beta_1)}, r, s) \subseteq A_2$.

Since MFR_1 *implies* MFR_0, by theorem 2.1

$x_{(\alpha,\beta)}\,\overline{q}\,A_1^c$. Thus $MFC(x_{(\alpha,\beta)}, r, s)\,\overline{q}\,A_1^c$. Hence $MFC(x_{(\alpha,\beta)}, r, s) \subseteq A_1$.

And since $y_{(\alpha_1,\beta_1)}\,\overline{q}\,A_2^c$. Thus $MFC(y_{(\alpha_1,\beta_1)}, r, s)\,\overline{q}\,A_2^c$. Hence $MFC(y_{(\alpha_1,\beta_1)}, r, s) \subseteq A_2$.

Let $x_{(\alpha,\beta)}\,\overline{q}\,MFC(y_{(\alpha_1,\beta_1)}, r, s)$, there exist $A_i \in IF(X)$ *with* $T(A_i) \geq r$, $T^*(A_i) \leq s$ *for* $i \in [1, 2\}$ *such that :* $A_1\,\overline{q}\,A_2$ *and* $MFC(x_{(\alpha,\beta)}, r, s) \subseteq A_1$, $MFC(y_{(\alpha_1,\beta_1)}, r, s) \subseteq A_2$.

Since $x_{(\alpha,\beta)} \subseteq MFC(x_{(\alpha,\beta)}, r, s)$ *and* $y_{(\alpha_1,\beta_1)} \subseteq MFC(y_{(\alpha_1,\beta_1)}, r, s)$.

Thus $x_{(\alpha,\beta)} \in A_1$, $y_{(\alpha_1,\beta_1)} \in A_2$. Hence (x, T, T^*) *is* MFR_1. ■

3.A Homeomorphism on Mated Fuzzy Topological Spaces

Definition (3. 1)

Let f be a map from a mated fuzzy topological space (X, T, T^*) in to a mated fuzzy topological space (Y, U, U^*), then f is said to be as follows:

(i) *MF*-continuous if $T(f^{-1}(B)) \geq U(B)$ *and* $T^*(f^{-1}(B)) \leq U^*(B)$, for each $B \in IF(Y)$.

(ii) *MF*-open if $T(A) \leq U(f(A))$ *and* $T^*(A) \geq U^*(f(A))$, for each $A \in IF(X)$.

(iii) *MF*-closed if $T(A^c) \leq U(f(A^c))$ *and* $T^*(A^c) \geq U^*(f(A^c))$, for each $A \in IF(X)$.

(iv) *MF*-homeomorphism if and only if f is bijective and both f, f^{-1} are *MF*-continuous.

Theorem (3. 1)

Let (X, T, T^*) and (Y, U, U^*) are mated fuzzy topological spaces and $f:(X, T, T^*) \rightarrow (Y, U, U^*)$ a function . Then the following statements are equivalent, for each $A \in IF(X)$, $B \in IF(Y)$ and $r, s \in I$.

(i) f is *MF*-continuous.

(ii) $f(MFC(A, r, s)) \subseteq MFC(f(A), r, s)$

(iii) $MFC(f^{1}(B), r, s) \subseteq f^{1}(MFC(B, r, s))$

Proof:

Let f be $MF-continuous$

Then $T(f^{-1}(B))^c \geq U(B^c)$ and $T^*(f^{-1}(B))^c \leq U^*(B^C)$

$MFC(f(A), r, s)=\cap\{B\in IF(Y): f(A) \subseteq B, U(B^c) \geq r, U^*(B^c) \leq s\}$

$= \bigcap(B\in IF(Y), A \subseteq f^{-1}(B), T(f^{-1}(B^c)) \geq r, T^*(f^{-1}(B^c)) \leq s\}$

$\supseteq \bigcap\{f(f^{-1}B)\in IF(Y), A \subseteq f^{-1}(B), T(f^{-1}(B^c)) \geq r, T^*(f^{-1}B^c)) \leq s\}$

$\supseteq f(\bigcap\{f^{-1}(B)\in IF(X), A \subseteq f^{-1}B), T(f^{-1}(B^c)) \geq r, T^*(f^{-1}B^c)) \leq s\}$

$\supseteq f(MFC(A, r, s))$. Hence $f(MFC(A, r, s)) \subseteq MFC(f(A), r, s)$.

For all $B\in IF(Y)$, put $A=f^{-1}(B)$. By (ii), $f(MFC(f^{-1}B), r, s)) \subseteq MFC(f(f^{-1}(B), r, s) \subseteq$ $MFC(B, r, s)$. Thus $f(MFC(f^{-1}(B), r, s)) \subseteq MFC(B, r, s)$

Hence $MFC(f^{-1}(B), r, s) \subseteq f^{-1}(MFC(B, r, s))$.

Let $B^c \in IF(Y)$ and $U(B^c) = r, U^*(B^c) = s$.

Then $MFC(B, r, s) = B$ implies $f^{-1}(MFC(B, r, s)) = f^{-1}B)$.

From(iii), $MFC(f^{-1}(B), r, s) \subseteq f^{-1}(MFC(B, r, s))$

We have $MFC(f^{-1}(B), r, s) \subseteq f^{-1}(B)$, *thus* $MFC(f^{-1}B), r, s) = f^{-1}(B)$.

Hence $T(f^{-1}(B))^c \geq r$ implies $T(f^{-1}B^c)) \geq r$, and $T^*(f^{-1}(B))^c \leq s$ implies $T^*(f^{-1}(B^c)) \leq s$.

Then $T(f^{-1}(B^c)) \geq U(B^c)$ and $T^*(f^{-1} (B^c)) \leq U^*(B^c)$. ■

Theorem (3. 2)

Let f be a homeomorphism from a mated fuzzy topological space (X, T, T^*) in to a mated fuzzy topological space (Y, U, U^*), *then* (X, T, T^*) *is* MFR_0 if and only if (Y, U, U^*) *is* MFR_0.

Proof:

Suppose that (X, T, T^*) *is* MFR_0. Let $x_{(\alpha, \beta)}, y_{(\alpha_1,\beta_1)}$ be two intuitionistic fuzzy point in Y.

Since $x, y\in Y$ and f bijective, then there exist $a, b\in X$ such that $a=f^{-1}(x), b=f^{-1}(y)$. Thus

$a_{(\alpha,\beta)} = f^{-1}(x_{(\alpha,\beta)})$, $b_{(\alpha_1,\beta_1)} = f^{-1}(y_{(\alpha_1,\beta_1)})$.

Let $x_{(\alpha,\beta)}\ \bar{q}\ MFC(y_{(\alpha_1,\beta_1)}, r, s)$ *implies* $x_{(\alpha,\beta)} \subseteq MFC(y_{(\alpha_1,\beta_1)}, r, s)^c$

Thus $f^{-1}(x_{(\alpha,\beta)}) \subseteq f^{-1}(MFC(y_{(\alpha_1,\beta_1)}, r, s))^c$

Hence $a_{(\alpha,\beta)} \subseteq f^{-1}(MFC(y_{(\alpha_1,\beta_1)}, r, s)^c$, implies $a_{(\alpha,\beta)}\ \bar{q}\ f^{-1}(MFC(y_{(\alpha_1,\beta_1)}, r, s))$.

Since $MFC(f^{-1}(y_{(\alpha_1,\beta_1)}), r, s) \subseteq f^{-1}(MFC(y_{(\alpha_1,\beta_1)}), r, s))$.

Then $a_{(\alpha,\beta)}\ \bar{q}\ MFC(f^{-1}(y_{(\alpha_1,\beta_1)}), r, s)$. Implies $a_{(\alpha,\beta)}\ \bar{q}\ MFC(b_{(\alpha 1,\beta 1)}, r, s)$

Since (X, T, T^*) *is* MFR_0, *then* $b_{(\alpha_1,\beta_1)}\ \bar{q}\ MFC(a_{(\alpha,\beta)}, r, s)$

Therefore $f(b_{(\alpha_1,\beta_1)})\ \bar{q}\ f(MFC(a_{(\alpha,\beta)}, r, s))$. *Since* f^{-1}*continuous, then* $MFC(f(a_{(\alpha,\beta)}, r, s) \subseteq f(MFC((a_{(\alpha,\beta)}, r, s)$. $\Rightarrow f(b_{(\alpha_1,\beta_1)})\ \bar{q}\ MFC(f(a_{(\alpha,\beta)}, r, s)$. Thus $y_{(\alpha_1,\beta_1)}\ \bar{q}\ MFC((x_{(\alpha,\beta)}, r, s)$

Hence (Y, U, U^*) *is* MFR_0.

Suppose that (Y, U, U^*) *is* MFR_0. Let $a_{(\alpha,\beta)}$, $b_{(\alpha_1,\beta_1)}$ be two intuitionistic fuzzy point in *X*.

Such that $a_{(\alpha,\beta)}\ \bar{q}\ MFC(b_{(\alpha_1,\beta_1)}, r, s)$, then $a_{(\alpha,\beta)} \subseteq (MFC(b_{(\alpha_1,\beta_1)}, r, s)^c$.

Therefore $f(a_{(\alpha,\beta)}) \subseteq f((MFC(b_{(\alpha_1,\beta_1)}, r, s))^c$. *Thus* $f(a_{(\alpha,\beta)})\ \bar{q}\ f(MFC(b_{(\alpha_1,\beta_1)}, r, s))$

Since $f(MFC(b_{(\alpha_1,\beta_1)}, r, s)) \subseteq MFC(f(b_{(\alpha_1,\beta_1)}, r, s)$,

Therefore $f(a_{(\alpha,\beta)})\ \bar{q}\ MFC(f(b_{(\alpha_1,\beta_1)}, r, s)$. *Thus* $x_{(\alpha,\beta)}\ \bar{q}\ MFC(y_{(\alpha_1,\beta_1)}, r, s)$

Since (Y, U, U^*) *is* MFR_0. Then $y_{(\alpha_1,\beta_1)}\ \bar{q}\ MFC((x_{(\alpha,\beta)}, r, s)$

Therefore $y_{(\alpha_1,\beta_1)} \subseteq MFC(x_{(\alpha,\beta)}, r, s)^c$.*Thus* $f^{-1}(y_{(\alpha_1,\beta_1)}) \subseteq f^{-1}(MFC(x_{(\alpha,\beta)}, r, s))^c$

Hence $f^{-1}(y_{(\alpha_1,\beta_1)})\ \bar{q}\ f^{-1}(MFC(x_{(\alpha,\beta)}, r, s))$.

Since $b_{(\alpha_1,\beta_1)}\ \bar{q}\ MFC(a_{(\alpha,\beta)}, r, s)$. Then (X, T, T^*) *is* MFR_0. ■

Theorem (3. 3)

Let *f* be a homeomorphism from a mated fuzzy topological space (X, T, T^*) in to a mated fuzzy topological space (Y, U, U^*), *then* (X, T, T^*) *is* MFR_1 if and only if (Y, U, U^*) *is* MFR_1.

Proof:

Suppose that *(X, T, T^*) is MFR$_1$*.Let $x_{(\alpha,\beta)}$, $y_{(\alpha_1,\beta_1)}$ be two intuitionistic fuzzy point in *Y*.

Since *x, y*$\in$*Y* and *f* bijactive, then there exist *a, b*$\in$*X such that a=*$f^{-1}(x)$, *b=*$f^{-1}(y)$.

Thus $a_{(\alpha,\beta)} = f^{-1}(x_{(\alpha,\beta)})$, $b_{(\alpha_1,\beta_1)} = f^{-1}(y_{(\alpha_1,\beta_1)})$. Let $x_{(\alpha,\beta)}\ \overline{q}\ MFC(y_{(\alpha_1,\beta_1)}, r, s)$.

Therefore $f^{-1}(x_{(\alpha,\beta)})\ \overline{q}\ f^{-1}(MFC\ (y_{(\alpha_1,\beta_1)}, r, s))$..

Thus $f^{-1}(x_{(\alpha,\beta)})\ \overline{q}\ (MFC(f^{-1}(y_{(\alpha_1,\beta_1)}, r, s))$,

Hence $a_{(\alpha,\beta)}\ \overline{q}\ MFC(b_{(\alpha_1,\beta_1)}, r, s)$, since *(X, T, T^*) is MFR$_1$*, there exist $A_1, A_2 \in IF(X)$ *with* ($T(A_i) \geq r$, $T^*(A_i) \leq s$ *and* $i=\{1, 3\}$) *such that* $a_{(\alpha,\beta)} \in A_1$, $b_{(\alpha_1,\beta_1)} \in A_2$ *and* $A_1\ \overline{q}\ A_2$.

Since $A_i \in IF(X)$, then $f(A_i) \in IF(Y)$.

Since f^{-1}continuous *(f open)*, then $T(A_i) \leq U(f(A_i))$ *and* $T^*(A_i) \geq U^*(f(A_i))$.

Thus $U(f(A_i)) \geq r$, $U^*(f(A_i)) \leq s$ *and* $f(a_{(\alpha,\beta)}) \in f(A_1)$, $f(b_{(\alpha_1,\beta_1)}) \in f(A_2)$.

Since $A_1\ \overline{q}\ A_2 \Rightarrow f(A_1)\ \overline{q}\ f(A_2)$. Thus $x_{(\alpha,\beta)} \in f(A_1)$, $y_{(\alpha_1,\beta_1)} \in f(A_2)$ and $f(A_1)\ \overline{q}\ f(A_2)$. .

Hence (Y, U, U^*) is MFR$_1$.

Suppose that *(Y, U, U^*) is MFR$_1$*. Let $a_{(\alpha,\beta)}$, $b_{(\alpha_1,\beta_1)}$ be two intuitionistic fuzzy point in *X*.

Such that $a_{(\alpha,\beta)}\ \overline{q}\ MFC(b_{(\alpha_1,\beta_1)}, r, s)$, *then* $a_{(\alpha,\beta)} \subseteq (MFC(b_{(\alpha_1,\beta_1)}, r, s)^c$.

Therefore $f(a_{(\alpha,\beta)}) \subseteq f((MFC(b_{(\alpha_1,\beta_1)}, r, s))^c$.Thus $f(a_{(\alpha,\beta)})\ \overline{q}\ f(MFC(b_{(\alpha_1,\beta_1)}, r, s))$

Hence $x_{(\alpha,\beta)}\ \overline{q}\ MFC(f(b_{(\alpha_1,\beta_1)}, r, s)$,

Since f^{-1}*continuous*, then $MFC(f(b_{(\alpha_1,\beta_1)}, r, s) \subseteq f(MFC((b_{(\alpha 1,\beta 1)}, r, s)$

Therefore $x_{(\alpha,\beta)}\ \overline{q}\ MFC(f(b_{(\alpha_1,\beta_1)}, r, s)$. *Thus* $x_{(\alpha,\beta)}\ \overline{q}\ MFC(y_{(\alpha_1,\beta_1)},), r, s)$.

Since *(Y, U, U^*) is MFR$_1$, there exist* $B_1, B_2 \in IF(Y)$ *with* ($U(B_i) \geq r$, $U^*(B_i) \leq s$ *and* $i=\{1, 3\}$) *such that* $x_{(\alpha,\beta)} \in B_1$, $y_{(\alpha_1,\beta_1)}, \in B_2$ *and* $B_1\ \overline{q}\ B_2$.Since $B_i \in IF(Y)$, then $f^{-1}(B_i) \in IF(X)$

Since *f MF-continuous*, then $T(f^{-1}(B_i) \geq r$, $T^*(f^{-1}(B_i) \leq s$.

Since $B_1\ \overline{q}\ B_2$, then $f^{-1}(B_1)\ \overline{q}\ f^{-1}(B_2)$.

Thus $a_{(\alpha,\beta)} \in f^{-1}(B_1)$, $b_{(\alpha_1,\beta_1)} \in f^{-1}(B_1)$ *and* $f^{-1}(B_1)\ \overline{q}\ f^{-1}(B_2)$.

Hence (X, T, T^*) *is* MFR_1. ■

Theorem(3. 4)

Let f be a homeomorphism from a mated fuzzy topological space (X, T, T^*) in to a mated fuzzy topological space (Y, U, U^*), *then* (X, T, T^*) *is* MFR_2 if and only if (Y, U, U^*) *is* MFR_2.

Proof:

Suppose that (X, T, T^*) *is* MFR_2.

Let $x_{(\alpha,\beta)}$, be an intuitionistic fuzzy point in Y, and $x_{(\alpha,\beta)}\ \overline{q}\ B$ with $U(B^c) \geq r$, $U^*(B^c) \leq s$ *and* $B \in IF(Y)$, then $f^{1}(x_{(\alpha,\beta)})\ \overline{q}\ f^{1}(B)$.

Since $T(f^{-1}(B^c) \geq U(B^c)$, $T^*(f^{-1}(B^c) \leq U^*(B^c)$, then $T(f^{-1}(B^c) \geq r$, $T^*(f^{-1}B^c) \leq s$.

Since $x \in Y$ and f bijactive, then there exist $a \in X$ *such that* $a = f^{-1}(x)$.

Thus $a_{(\alpha,\beta)} = f^{-1}(x_{(\alpha,\beta)})$. Hence $a_{(\alpha,\beta)}\ \overline{q}\, f^{-1}(B)$.

Since (X, T, T^*) *is* MFR_2, there exist $A_1, A_2 \in IF(X)$ *with* $(T(A_i) \geq r$, $T^*(A_i) \leq s$ *and* $i=\{1, 2\})$ such that $a_{(\alpha,\beta)} \in A_1$, $f^{-1}(B) \subseteq A_2$ *and* $A_1\ \overline{q}\ A_2$. Since $A_i \in IF(X)$, then $f(A_i) \in IF(Y)$

Since f^{-1}*MF-continuous (f open), therefore* $T(A_i) \leq U(f(A_i))$ *and* $T^*(A_i) \geq U^*(f(A_i))$

Thus $U(f(A_i)) \geq r$, $U^*(f(A_i)) \leq s$. Since $a_{(\alpha,\beta)} \in A_1$, *then* $f(a_{(\alpha,\beta)}) \in f(A_1)$

Since $f^{1}(B) \subseteq A_2$, then $B \subseteq f(A_2)$. Since $A_1\ \overline{q}\ A_2$, then $f(A_1)\ \overline{q}\ f(A_2)$.Hence (Y, U, U^*) *is* MFR_2.

Suppose that (Y, U, U^*) *is* MFR_2.

Let $a_{(\alpha,\beta)}$, be an intuitionistic fuzzy point in X, and $a_{(\alpha,\beta)}\ \overline{q}\ A$ *with* $T(B^c) \geq r$, $T^*(B^c) \leq s$ *and* $A \in IF(Y)$. *Then* $f(a_{(\alpha,\beta)})\ \overline{q}\ f(A)$. Since f open , then $T(A) \leq U(f(A))$, $T^*(A) \geq U^*(f(A))$

Hence $U(f(A)) \geq r$, $U^*(f(A)) \leq s$. Since $a \in X$ and f bijactive, then there exist $x \in Y$ such that $x = f(a)$. Thus $x_{(\alpha,\beta)} = f(a_{(\alpha,\beta)})$. Hence $x_{(\alpha,\beta)}\ \overline{q}\ f(A)$, $x_{(\alpha,\beta)} \in P_{(\alpha,\beta)}(Y)$, $f(A) \in IF(Y)$.

Since (Y, U, U^*) *is* MFR_2, there exist $B_1, B_2 \in IF(Y)$ *with* $(U(A_i) \geq r$, $U^*(A_i) \leq s$ *and* $i=\{1, 2\})$ such that $x_{(\alpha,\beta)} \in B_1$, $f(A) \subseteq B_2$ *and* $B_1\ \overline{q}\ B_2$. Since $B_i \in IF(Y)$, then $f^{-1}(B_i) \in IF(X)$

Since f continuous, then $T(f^{-1}(B_i)) \geq U(B_i)$, $T^*(f^{-1}(B_i)) \leq U^*(B_i)$

Therefore $T(f^{-1}(B_i) \geq r$, $T^*(f^{-1}(B_i) \leq s$. Since $x_{(\alpha,\beta)}\ \overline{q}\ f(A)$, then $a_{(\alpha,\beta)}\ \overline{q}\ A$.

Since $x_{(\alpha,\beta)} \in B_1$, then $a_{(\alpha,\beta)} \in f^{1}(B_1)$. Since $f^{-1}A) \subseteq B_2$, then $A \subseteq f(B_2)$

Since $B_1\ \overline{q}\ B_2$, then $f(B_1)\ \overline{q}\ f(B_2)$.Hence (X, T, T^*) *is* MFR_2. ■

Theorem (3. 5)

Let f be a homeomorphism from a mated fuzzy topological space (X, T, T^*) in to a mated fuzzy topological space (Y, U, U^*), *then* (X, T, T^*) *is* MFR_3 if and only if (Y, U, U^*) *is* MFR_3.

Proof:

Suppose that (X, T, T^*) *is* MFR_3. Let $B_1\ \overline{q}\ B_2$ *with* $U(B_i^c) \geq r$, $U^*(B_i^c) \leq s$ *for* $i=1, 2$ such that $B_i \in IF(Y)$.

Since f continuous, then $f^{-1}(B_1)\ \overline{q}\ f^{-1}(B_2)$ *with* $T(f^{-1}(B_i^c) \geq U(B_i^c)$, $T^*(f^{-1}(B_i^c) \leq U^*(B_i^c)$.

Therefore $T(f^{-1}(B_i^c) \geq r$, $T^*(f^{-1}(B_i^c) \leq s$ such that $f^{1}(B_i) \in IF(X)$.

Since (X, T, T^*) *is* MFR_3, *there exist* $A_1, A_2 \in IF(X)$ *with* $(T(A_i) \geq r$, $T^*(A_i) \leq s$ *and* $i=\{1, 2\})$ such that $f^{-1}(B_1) \subseteq A_1$, $f^{-1}(B_2) \subseteq A_2$ *and* $A_1\ \overline{q}\ A_2$.

Since $f^{-1}(B_i) \subseteq A_i$, then $B_i \subseteq f(A_i)$ *and* $f(A_1)\ \overline{q}\ f(A_2)$. Since f open , then $T(A_i) \leq U(f(A_i)$ *and* $T^*(A_i) \geq U^*(f(A_i))$. Thus $U(f(A_i) \geq r$, $U^*(f(A_i)) \leq s$. Hence (Y, U, U^*) *is* MFR_3.

Suppose that (Y, U, U^*) *is* MFR_3. Let $A_1\ \overline{q}\ A_2$ *with* $T(A_i^c) \geq r$, $T^*(A_i^c) \leq s$ *for* $i=1, 2$ such that $A_i \in IF(Y)$. Since f open, then $f(A_1)\ \overline{q}\ f(A_2)$ *with* $T(A_i^c) \leq U(f(A_i^c))$, $T^*(A_i^c) \geq U^*(f(A_i^c))$, thus $U(f(A_i^c)) \geq r$, $U^*(f(A_i^c)) \leq s$ such that $f(A_i) \in IF(Y)$.

Since (Y, U, U^*) *is* MFR_3, there exist $B_1, B_2 \in IF(Y)$ *with* $(U(A_i) \geq r$, $U^*(A_i) \leq s$ *and* $i=\{1, 2\})$ such that $f(A_1) \subseteq B_1$, $f(A_2) \subseteq B_2$ *and* $B_1\ \overline{q}\ B_2$. Since f bijactive, then $A_i \subseteq f^{-1}(B_i)$ *and* $f^{-1}B_1)\ \overline{q}\ f^{-1}(B_2)$. Since f continuous, then $T(f^{-1}(B_i) \geq U(B_i)$, $T^*(f^{-1}(B_i)) \leq U^*(B_i)$

Thus $T(f^{-1}(B_i)) \geq r$, $T^*(f^{-1}(B_i)) \leq s$. Hence (X, T, T^*) *is* MFR_3. ■

Theorem (3. 6)

Let f be a homeomorphism from a mated fuzzy topological space (X, T, T^*) in to a mated fuzzy topological space (Y, U, U^*), *then* (X, T, T^*) *is* MFR_4 if and only if (Y, U, U^*) *is* MFR_4.

Proof:

(X, T, T^*) *is* $MFR_4 \Leftrightarrow (X, T, T^*)$ *is* MFR_3 *and* MFR_0

$\Leftrightarrow (Y, U, U^*)$ *is* MFR_3 *and* $MFR_0 \Leftrightarrow (Y, U, U^*)$ *is* MFR_4.

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