Lyapunov -Schmidt reduction in the analysis of bifurcation solutions of sixth order nonlinear differential equation

Mudhir A. Abdul Hussain
Ahmed K. Shanan
University of Basrah, College of Education, Department of Mathematics, Basrah, IRAQ

Abstract. In this paper we are interested in the study of bifurcation solutions of nonlinear sixth order differential equation with four parameters, by using Lyapunov -Schmidt method. The problem is reduced to the study of bifurcation solutions of a function of fourth degree in two variables.

الملخص:

في هذا البحث تمت دراسة الحلول التفرعيه للمعادلات التفاضلية غير الخطية من الدرجة السادسه وذات اربعة معاملات بأستخدام طريقة (Lyapunov –Schmidt) . المسأله الاساسيه تم تحويلها الى مسئلة دراسة الحلول التفرعيه لداله من الدرجه الرابعه وبمتغيرين .

Keywords: bifurcation, Lyapunov -Schmidt reduction, nonlinear differential equation

1. Introduction

It is known that many of the nonlinear problems in mathematics and physics can be written in the form of operator equation,

$$F(x,\lambda)=b\,,\ x\in O\subset X,\ b\in Y,\ \lambda\in R^n \qquad \dots(1.1)$$

in which F is a smooth Fredholm map of index zero, X, Y Banach spaces and O open subset of X. For these problems, the method of reduction to finite dimensional equation [8],

$$\Phi(\xi,\lambda)=\beta\ ,\ \xi\in\hat{M},\ \beta\in\hat{N}, \qquad \dots(1.2)$$

can be used, where $\hat{M}$ and $\hat{N}$ are smooth finite dimensional manifolds.

Passage from equation (1.1) into equation (1.2) (variant local scheme of Lyapunov – Schmidt) with the conditions, that equation (1.2) has all the topological and analytical properties of equation (1.1) (multiplicity, bifurcation diagram, etc) dealing with [3],[8],[11][12] .

Definition 1.1 Suppose that E and M are Banach spaces and $A:E\to M$ be a linear continuous operator. The operator A is called Fredholm operator, if

The kernel of A, Ker(A), is finite dimensional.

The range of A, Im(A), is closed in M.

The Cokernel of A, Coker(A), is finite dimensional.

The number

dim(Ker A)- dim(Coker A)

is called Fredholm index of the operator A .

Suppose that $F:\Omega \to M$ is a nonlinear Fredholm map of index zero. A smooth map $F:\Omega \to M$ has variational property, if there exist a functional $V:\Omega \to R$ such that $F = grad_H V$ or equivalently,

$$\frac{\partial V}{\partial x}(x,\lambda)h = \langle F(x,\lambda),h\rangle_H, \quad \forall\, x\in\Omega,\ h\in E.$$

where ($\langle\cdot,\cdot\rangle_H$ is the scalar product in Hilbert space H). In this case the solutions of equation $F(x,\lambda)=0$ are the critical points of functional V(x, λ). By using the method of finite dimensional reduction (Local method of Lyapunov-Schmidt) the problem,

$$V(x,\lambda) \to extr, \quad x\in E,\ \lambda\in R^n.$$

is reduce into an equivalent problem,

$$W(\xi,\lambda) \to extr, \quad \xi \in R^n.$$

the function $W(\xi,\lambda)$ is called Key function.

If $N = span\{e_1,...,e_n\}$ is a subspace of E, where $e_1,\dots, e_n$ are orthonormal set, then the key function $W(\xi,\lambda)$ can be defined in the form,

$$W(\xi,\lambda) = \inf_{x:\langle x,e_i\rangle=\xi_i\ \forall i} V(x,\lambda), \quad \xi = (\xi_1,...,\xi_n) .$$

The function W has all the topological and analytical properties of the functional V (multiplicity, bifurcation diagram, etc) [10]. The study of bifurcation solutions of functional V is equivalent to the study of bifurcation solutions of key function [10]. If F has variational property, then it is easy to check that,

$$\theta(\xi,\lambda) = grad\,W(\xi,\lambda).$$

Equation $\theta(\xi,\lambda)=0$ is called bifurcation equation.

Proposition 1.1 (Sapronov) [10]

Suppose that the map (1.2) is an identity map and assume that the condition

$$\langle \frac{\partial F}{\partial x}(x)h,\ h\rangle > 0,$$
$$\forall (x,h) \in E \times (E\setminus\{0\}),$$

is satisfied for this map. Then the marginal map

$$\varphi : \xi \mapsto \sum_{i=1}^{n} x_i e_i + h(\xi),$$

where h(ξ) given in (1.2) can be used to determine one-to-one correspondence between the critical points of the function W and the critical points of functional V.

Definition 1.2 The caustic Σ is defined to be the set of all λ for which the functional $V(\cdot,\lambda), \quad \lambda\in\Lambda\subset R^n$ have in $\Omega\subset R^n$ degenerate critical points**.**

i.e.

$$\sum = \{ \lambda : \frac{\partial V}{\partial x}(x,\lambda) = \frac{\partial^2 V}{\partial x^2}(x,V) = 0 \}.$$

Consider the following sixth order nonlinear ODE,

$$\alpha \frac{d^6 w}{d x^6} + \sigma \frac{d^4 w}{d x^4} + \beta \frac{d^2 w}{d x^2} + \varphi w + w^2 + w^3 = 0 \qquad \dots (1.3)$$

with the boundary conditions,

$$w(0) = w(\pi) = w''(0) = w''(\pi) = w''''(0)$$
$$= w''''(\pi) = 0. \qquad \dots(1.4)$$

An analogous problem of fourth order of equation (1.3) has been studied by Thompson J.M.T., Stewart H.B. [4] they showed numerically the existence of periodic solutions of equation (1.3) for some values of parameters. Bardin B., Furta S. [1] used the local method of Lyapunov-Schmidt and found the sufficient conditions of existence of periodic waves of equation (1.3), also they are introduced the solutions of equation (1.3) in the form of power series. Sapronov Yu.I ([2],[10],[[11],[12]) applied the local method of Lyapunov –Schmidt and found the bifurcation solutions of equation (1.3). Abdul Hussain M.A ([5],[6]) studied equation (1.3) with small perturbation when the nonlinear part has quadratic term and Mohammed M.J. [7] studied equation (1.3) in the variational case when the nonlinear part has quadratic term. In this paper we studied the bifurcation solutions of equation (1.3) near the critical point when the dimension of the null space is equal to four.

2. Reduction to the principal ports of key function.

Suppose that f : E → F is a nonlinear Fredholm operator of index zero from Banach space E to Banach space F defined by,

$$f(w, \lambda) = \alpha \frac{d^6 w}{d x^6} + \sigma \frac{d^4 w}{d x^4} + \beta \frac{d^2 w}{d x^2} + \varphi w + w^2 + w^3 \qquad (2.1)$$

where E = C^6([0, π],R) is the space of all continuous functions which have derivative of order at most six, F = C^o ([0, π],R) is the space of all continuous functions where w = w(x), x$\in$ [0, π] , $\lambda \in (\alpha, \sigma, \beta, \varphi)$. In this case the solutions of equation (1.3) is equivalent to the solutions of the operator equation,

$$f(w, \lambda) = 0. \qquad (2.2)$$

We note that the operator f has variational property that is; there exist a functional V such that $f(w,\lambda) = grad_H V(w,\lambda)$ or equivalently,

$$\frac{\partial V}{\partial w}(w,\lambda)h = \langle F(w,\lambda), h \rangle_H, \ \forall \ w \in \Omega, \ h \in E.$$

Where($\langle \cdot, \cdot \rangle_H$ is the scalar product in Hilbert space H) and

$$V(w,\lambda)=\int_0^{\pi}(\alpha\frac{(w''')^2}{2}-\sigma\frac{(w'')^2}{2}+\beta\frac{(w')^2}{2}+\varphi\frac{w^2}{2}+\frac{w^3}{3}$$
$$+\frac{w^4}{4})hdx=0.$$

In this case the solutions of equation (2.2) are the critical points of the functional $V(w,\lambda)$, where the critical points of the functional $V(w,\lambda)$ are the solutions of Euler-Lagrange equation,

$$\frac{\partial V}{\partial w}(w,\lambda)h=\int_0^{\pi}(\alpha\, w^{vi}+\sigma\, w^{iv}+\beta\, w''+\varphi\, w+w^2+w^3)dx.$$

and $\frac{\partial V}{\partial w}(w,\lambda)$ is the Frechet derivative of the functional $V(w,\lambda)$.

Thus, the study of equation (1.3) with the conditions (1.4) is equivalent to the study extremely problem,

$$V(w,\lambda)\to extr,\quad w\in E.$$

Analysis of bifurcation can be finding by using method of Lyapunov-Schmidt to reduce into finite dimensional space. By localized parameters,

$$\alpha=\alpha_1+\delta_1,\sigma=\sigma_1+\delta_2,\beta=\beta_1+\delta_3,\varphi=\varphi_1+\delta_4,$$

$\delta_1,\delta_2,\delta_3,\delta_4$ are small parameters.

The reduction lead to the function in four variables,

$$W(\xi,\delta)=\inf_{x:\langle x,e_i\rangle=\xi_i\ \ \forall i}V(x,\lambda),\quad \xi=(\xi_1,\xi_2,\xi_3,\xi_4),\ \delta$$
$$=(\delta_1,\delta_2,\delta_3,\delta_4)$$

It is well known that in the reduction of Lyapunov-Schmidt the function $W(\xi,m)$ is smooth. This function has all the topological and analytical properties of functional V [10]. In particular, for small δ there is one-to-one corresponding between the critical points of functional V and smooth function W, preserving the type of critical points (multiplicity, index Morse, etc.) [10]. By using the scheme of Lyapunov-Schmidt, the linearized equation corresponding to the equation (2.2) has the form:

$$\alpha\, h''''''+\sigma\, h''''+\beta\, h''+\varphi\, h=0,\ \ h\in E.$$
$$h(0)=h(\pi)=h''(0)=h''(\pi)=0.$$

The point $(\alpha,\sigma,\beta,\varphi)$ = (0,0,0,0) bifurcation point [10]. Localized parameters $\alpha,\sigma,\beta,\varphi$ as follows,

$$\alpha=0+\delta_1,\quad \sigma=0+\delta_2,$$
$$\beta=0+\delta_3,\quad \varphi=0+\delta_4$$

$\delta_1,\delta_2,\delta_3,\delta_4$ are small parameters**.**

Lead to the bifurcation along the modes e_1,e_2,e_3,e_4 where

$$e_1 = c_1 \sin(x), e_2 = c_2 \sin(2x), e_3 = c_3 \sin(3x), e_4 = c_4 \sin(4x),$$

$\|e_1\|_H = \|e_2\|_H = \|e_3\|_H = \|e_4\|_H = 1$ and

$$c_1 = c_2 = c_3 = c_4 = \sqrt{\frac{2}{\pi}} \cdot \text{let } N = \ker A = span\{e_1, e_2, e_3, e_4\}$$

where $A = f_w(0,\lambda) = \alpha \dfrac{d^6}{dx^6} + \sigma \dfrac{d^4}{dx^4} + \beta \dfrac{d^2}{dx^2} + \varphi$

then the space E can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$E = N \oplus \hat{E}, \qquad \hat{E} = N^{\perp} \cap E = \{v \in E : v \perp N\}.$$

Similarly, the space F decomposed in direct sum of two subspaces, N and orthogonal complement to N,

$$M = N \oplus \hat{F}, \qquad \hat{F} = N^{\perp} \cap F = \{v \in F : v \perp N\}.$$

There exists projections $p : E \to N$ and $I - p : E \to \hat{E}$ such that pw = u and (I − p)w = v, (I is the identity operator). Hence every vector $u \in E$ can be written in the form, $w = u + v, \quad u = \sum_{i=1}^{4} \xi_i \, e_i \in N, \quad N \perp v \in \hat{E}, \quad \xi_i = \langle w, e_i \rangle.$

Similarly, there exists projections $Q : M \to N$ *and* $(I - Q) : M \to \tilde{N}^{\perp}$ such that

$$f(w,\lambda) = Qf(w,\lambda) + (I - Q) f(w,\lambda)$$

Accordingly, equation (2.2) can be written in the form,

$$Qf\,(u + v, \lambda) = 0,$$
$$(I - Q) f\,(u + v, \lambda) = 0$$

By implicit function theorem, there exists a smooth map $\Phi : N \to \hat{E}$ (depending on λ), such that $\Phi(w,\lambda) = v$ and

$$W(\xi,\delta) = V(\Phi(w,\lambda),\delta),$$
$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4),\ \delta = (\delta_1, \delta_2, \delta_3, \delta_4)$$

and then key function can be written in the form,

$$W(\xi,\delta) = V(\xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4 + \Phi(\xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + \xi_4 e_4, \delta), \delta),$$
$$= \tilde{U}(\xi,\delta) + o(|\xi|^4) + O(|\xi|^4) O(\delta)$$

The function $\tilde{U}(\xi,\delta)$ can be found as follows, substitute the value of u in the above functional we have that,

$$\int_0^\pi \frac{(u''')^2}{2}\,dx = \frac{\xi_1^2}{2} + \frac{729\xi_3^2}{2} + 32\xi_2^2 + 2048\xi_4^2,$$

$$\int_0^\pi \frac{u^2}{2}\,dx = \frac{\xi_1^2}{2} + \frac{\xi_3^2}{2} + \frac{\xi_2^2}{2} + \frac{\xi_4^2}{2},$$

$$\int_0^\pi \frac{(u'')^2}{2}\,dx = \frac{\xi_1^2}{2} + \frac{81\xi_3^2}{2} + 8\xi_2^2 + 128\xi_4^2,$$

$$\int_0^\pi \frac{(u')^2}{2}\,dx = \frac{\xi_1^2}{2} + \frac{9\xi_3^2}{2} + 2\xi_2^2 + 8\xi_4^2,$$

$$\int_0^\pi \frac{u^3}{3}\,dx = \frac{8\sqrt{2}}{10395\pi^{\frac{3}{2}}}(2648\xi_4^2\xi_1 + 3696\,\xi_3\xi_4\xi_2$$
$$-1584\xi_1\xi_4\xi_2 + 385\,\xi_3^3 + 2673\,\xi_1\xi_3^2$$
$$+1980\,\xi_3\xi_2^2 + 1155\,\xi_1^3 + 1008\xi_3\xi_4^2$$
$$-693\xi_3\xi_1^2 + 2772\,\xi_1\xi_2^2),$$

$$\int_0^\pi \frac{u^4}{4}\,dx = \frac{1}{8\pi}(12\xi_1^2\xi_2^2 + 12\xi_1^2\xi_3^2 + 12\xi_1^2\xi_4^2$$
$$+3\,\xi_2^4 + 12\xi_2^2\xi_3^2 + 12\xi_3^2\xi_4^2 + 12\,\xi_1\xi_2^2\xi_3$$
$$-4\,\xi_1^3\xi_3 + 24\xi_1\xi_4\xi_3\xi_2 + 3\,\xi_4^4 + 3\xi_1^4$$
$$+3\xi_3^4 + 12\xi_2\xi_4\xi_3^2 + 12\xi_2^2\xi_1^2 - 12\,\xi_1^2\xi_4\xi_2).$$

and then

$$\widetilde{U}(\xi,\delta)=\frac{3}{8\pi}\xi_1^4+\frac{3}{8\pi}\xi_2^4+\frac{3}{8\pi}\xi_3^4+\frac{3}{8\pi}\xi_4^4\frac{1}{2}$$

$$(\alpha-\sigma+\beta+\varphi)\xi_1^{2}+(32\alpha-8\sigma+2\beta+\frac{1}{2}\varphi)\xi_2^2$$

$$+\frac{1}{2}(729\alpha-81\sigma+9\beta+\varphi)\xi_3^2+(2048\alpha-128\sigma$$

$$+8\beta+\frac{1}{2}\varphi)\xi_4^2+\frac{3}{2\pi}\xi_2^2\xi_3^2+\frac{3}{2\pi}\xi_4^2\xi_1^2$$

$$+\frac{3}{2\pi}\xi_1^2\xi_3^2+\frac{3}{2\pi}\xi_2^2\xi_4^2-\frac{1}{2\pi}\xi_1^3\xi_3+\frac{3}{2\pi}\xi_3^2\xi_4^2$$

$$+\frac{3}{2\pi}\xi_1^2\xi_2^2+\frac{3}{2\pi}\xi_1\xi_2^2\xi_3+\frac{3}{2\pi}\xi_4\xi_3^2\xi_2$$

$$-\frac{3}{2\pi}\xi_4\xi_1^2\xi_2+\frac{3}{\pi}\xi_1\xi_2\xi_4\xi_3-\frac{8\sqrt{2}}{15\pi^{\frac{3}{2}}}\xi_1^2\xi_3$$

$$+\frac{32\sqrt{2}}{15\pi^{\frac{3}{2}}}\xi_2^2\xi_1+\frac{32\sqrt{2}}{21\pi^{\frac{3}{2}}}\xi_2^2\xi_3+\frac{128\sqrt{2}}{165\pi^{\frac{3}{2}}}\xi_4^2\xi_3$$

$$+\frac{128\sqrt{2}}{63\pi^{\frac{3}{2}}}\xi_4^2\xi_1+\frac{72\sqrt{2}}{35\pi^{\frac{3}{2}}}\xi_3^2\xi_1+\frac{128\sqrt{2}}{45\pi^{\frac{3}{2}}}\xi_3\xi_4\xi_2$$

$$-\frac{128\sqrt{2}}{105\pi^{\frac{3}{2}}}\xi_1\xi_4\xi_2+\frac{8\sqrt{2}}{9\pi^{\frac{3}{2}}}\xi_1^3+\frac{8\sqrt{2}}{27\pi^{\frac{3}{2}}}\xi_3^3\quad ...(2.3)$$

The bifurcation equation of (2.3) is locally equivalent in the neighbourhood of point zero to the equation,

$$\widetilde{U}(\xi,\mu)=\frac{3}{8\pi}\xi_1^4+\frac{3}{8\pi}\xi_2^4+\frac{3}{8\pi}\xi_3^4+\frac{3}{8\pi}\xi_4^4+\mu_1\xi_1^2$$

$$+\mu_2\xi_2^2+\mu_3\xi_3^2+\mu_4\xi_4^2+\frac{3}{2\pi}\xi_2^2\xi_3^2+\frac{3}{2\pi}\xi_4^2\xi_1^2$$

$$+\frac{3}{2\pi}\xi_1^2\xi_3^2+\frac{3}{2\pi}\xi_2^2\xi_4^2-\frac{1}{2\pi}\xi_1^3\xi_3+\frac{3}{2\pi}\xi_3^2\xi_4^2$$

$$+\frac{3}{2\pi}\xi_1^2\xi_2^2+\frac{3}{2\pi}\xi_1\xi_2^2\xi_3+\frac{3}{2\pi}\xi_4\xi_3^2\xi-\frac{3}{2\pi}\xi_4\xi_1^2\xi$$

$$+\frac{3}{\pi}\xi_1\xi_2\xi_4\xi_3-\mu_5\xi_1^2\xi_3+\mu_6\xi_2^2\xi_1+\mu_7\xi_2^2\xi_3+\mu_8\xi_4^2\xi_3$$

$$+\mu_9\xi_4^2\xi_1+\mu_{10}\xi_3^2\xi_1+\mu_{11}\xi_3\xi_4\xi_2-\mu_{12}\xi_1\xi_4\xi_2$$

$$+\mu_{13}\xi_1^3+\mu_{14}\xi_3^3\quad ...(2.4)$$

Where $\mu=(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7,\mu_8,\mu_9,\mu_{10},\mu_{11},\mu_{12},\mu_{13},\mu_{14})$

and,

$\mu_1 = \frac{1}{2}(\alpha - \sigma + \beta + \varphi),\ \mu_2 = (32\alpha - 8\sigma + 2\beta + \frac{1}{2}\varphi),\ \mu_3 = \frac{1}{2}(729\alpha - 81\sigma + 9\beta + \varphi)$

$\mu_4 = (2048\alpha - 128\sigma + 8\beta + \frac{1}{2}\varphi),\ \mu_5 = \frac{8\sqrt{2}}{15\pi^{\frac{3}{2}}},\ \mu_6 = \frac{32\sqrt{2}}{15\pi^{\frac{3}{2}}},\ \mu_7 = \frac{32\sqrt{2}}{21\pi^{\frac{3}{2}}},\ \mu_8 = \frac{128\sqrt{2}}{165\pi^{\frac{3}{2}}},$

$\mu_9 = \frac{128\sqrt{2}}{63\pi^{\frac{3}{2}}},\ \mu_{10} = \frac{72\sqrt{2}}{35\pi^{\frac{3}{2}}},\ \mu_{11} = \frac{128\sqrt{2}}{45\pi^{\frac{3}{2}}},\ \mu_{12} = \frac{128\sqrt{2}}{105\pi^{\frac{3}{2}}},\ \mu_{13} = \frac{8\sqrt{2}}{9\pi^{\frac{3}{2}}},\ \mu_{14} = \frac{8\sqrt{2}}{27\pi^{\frac{3}{2}}},$

3. Bifurcation Analysis.

By changing variables $\xi_i = \sqrt[4]{\frac{8\pi}{3}}\ \eta_i,\ \ i = 1,\ldots,4$ we have equation (2.4) is equivalent to the following equation,

$$U(\hat{\eta},\hat{\alpha}) = \eta_1^4 + \eta_2^4 + \eta_3^4 + \eta_4^4 + \alpha_1\eta_1^2 + \alpha_2\eta_2^2 + \alpha_3\eta_3^2 + \alpha_4\eta_4^2$$
$$+ 4\eta_2^2\eta_3^2 + 4\eta_4^2\eta_1^2 + 4\eta_1^2\eta_3^2 + 4\eta_2^2\eta_4^2 - \frac{4}{3}\eta_1^3\eta_3 + 4\eta_3^2\eta_4^2$$
$$+ 4\eta_1^2\eta_2^2 + 4\eta_1\eta_2^2\eta_3 + 4\eta_4\eta_3^2\eta_2 - 4\eta_4\eta_1^2\eta_2 + 8\eta_1\eta_2\eta_4\eta_3$$
$$- \alpha_5\eta_1^2\eta_3 + \alpha_6\eta_2^2\eta_1 + \alpha_7\eta_2^2\eta_3 + \alpha_8\eta_4^2\eta_3 + \alpha_9\eta_4^2\eta_1 + \alpha_{10}\eta_3^2\eta_1$$
$$+ \alpha_{11}\eta_3\eta_4\eta_2 - \alpha_{12}\eta_1\eta_4\eta_2 + \alpha_{13}\eta_1^3 + \alpha_{14}\eta_3^3 + \ldots \quad \ldots.(3.1)$$

where, $\hat{\eta} = (\eta_1, \eta_2, \eta_3, \eta_4)$,

and $\hat{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14})$.

In the complex variables,

$$z_1 = \eta_1 + i\eta_2, \qquad z_2 = \eta_3 + i\eta_4, \qquad i^2 = -1$$

The function (3.1) has the following form ,

$$
\begin{aligned}
U(z,\hat{\alpha}) &= -\frac{1}{2} z_1^2 z_2^2 + \frac{7}{4}|z_1|^4 + \frac{1}{2} z_2^2 \bar{z}_1^2 + \frac{3}{4}|z_2|^4 \\
&-\frac{1}{4}(\alpha_5 - \alpha_7)|z_1|^2 z_2 - \frac{1}{4}(\alpha_5 - \alpha_9) z_1 |z_2|^2 \\
&+\frac{1}{4}(\alpha_{10} + \alpha_7)|z_1|^2 \bar{z}_2 + \frac{1}{4}(\alpha_{10} + \alpha_9)|z_2|^2 \bar{z}_1 \\
&+4|z_1|^2|z_2|^2 + \frac{1}{8}(\alpha_{12} - \alpha_5 - \alpha_7) z_1^2 z_2 - \frac{1}{8} z_2^4 \\
&-\frac{1}{8}(\alpha_{12} + \alpha_5 + \alpha_7)\bar{z}_1^2 z_2 - \frac{1}{8}(\alpha_{11} - \alpha_{10} + \alpha_9) z_2^2 z_1 \\
&+\frac{1}{8}(\alpha_{11} + \alpha_{10} - \alpha_9)\bar{z}_2^2 z_1 + \frac{1}{4}\bar{z}_1 \bar{z}_2 |z_1|^2 \\
&-\frac{1}{8}(\alpha_{11} - \alpha_{10} + \alpha_9)\bar{z}_2^2 \bar{z}_1 - \frac{1}{8}(\alpha_{12} + \alpha_5 + \alpha_7) z_1^2 \bar{z}_2 \\
&+\frac{1}{8}(\alpha_{11} + \alpha_8 + 3\alpha_{13}) z_2 |z_2|^2 - \frac{1}{8}\bar{z}_2^4 \\
&+\frac{1}{8}(\alpha_{12} - \alpha_5 - \alpha_7)\bar{z}_1^2 \bar{z}_2 + \frac{1}{8}(\alpha_{10} - \alpha_9) z_2^2 \bar{z}_1 + \frac{1}{4} z_1 z_2 |z_1|^2 \\
&-\frac{1}{4} z_1 \bar{z}_2 |z_1|^2 - \frac{1}{4}\bar{z}_1 z_2 |z_1|^2 - \frac{1}{8}\bar{z}_1^4 \\
&-\frac{7}{12} z_1^3 \bar{z}_2 - \frac{7}{12}\bar{z}_1^3 z_2 - \frac{1}{12}\bar{z}_1^3 \bar{z}_2 - \frac{1}{4} z_2^3 z_1 + \frac{1}{4}\bar{z}_2^3 z_1 \\
&+\frac{1}{4} z_2^3 \bar{z}_1 - \frac{1}{4}\bar{z}_2^3 \bar{z}_1 - \frac{1}{8} z_1^4
\end{aligned}
$$

$$-\frac{1}{12}z_1^3 z_2+\frac{1}{4}\bar{z}_2 z_1|z_2|^2-\frac{1}{4}\bar{z}_2\bar{z}_1|z_2|^2-\frac{1}{4}z_2 z_1|z_2|^2$$
$$+\frac{1}{4}z_2\bar{z}_1|z_2|^2+\frac{1}{8}(\alpha_6+3\alpha_{14})z_1|z_1|^2$$
$$+\frac{1}{8}(\alpha_8+3\alpha_{13})\bar{z}_2|z_2|^2+\frac{1}{2}(\alpha_1+\alpha_2)|z_1|^2-\frac{1}{8}\alpha_6 z_1^3$$
$$-\frac{1}{8}\alpha_6\bar{z}_1^3-\frac{1}{8}\alpha_8 z_2^3-\frac{1}{8}\alpha_8\bar{z}_2^3+\frac{1}{4}\alpha_1 z_1^2$$
$$+\frac{1}{4}\alpha_1\bar{z}_1^2+\frac{1}{8}\alpha_{13}\bar{z}_2^3+\frac{1}{2}\alpha_4|z_2|^2+\frac{1}{8}\alpha_{14}z_1^3+\frac{1}{8}\alpha_{14}\bar{z}_1^3$$
$$-\frac{1}{4}\alpha_2 z_1^2-\frac{1}{4}\alpha_2\bar{z}_1^2+\frac{1}{8}(\alpha_6+3\alpha_{14})\bar{z}_1|z_1|^2$$
$$-\frac{1}{4}\alpha_4 z_2^2-\frac{1}{4}\alpha_4\bar{z}_2^2-\frac{1}{8}i\alpha_3 z_2^3+\frac{1}{8}i\alpha_3\bar{z}_2^3-\frac{1}{8}i\alpha_3 z_2^2\bar{z}_2$$
$$+\frac{1}{8}i\alpha_3\bar{z}_2^2 z_2+\ldots \qquad \ldots(3.2)$$

where,

$|z_1|^2=\eta_1^2+\eta_2^2$, $|z_2|^2=\eta_2^2+\eta_4^2$, $\bar{z}_1,\bar{z}_2$ are the conjugates of z_1 and z_2 respectively,.

To study the behavior of the function (3.2) near the critical point .It is convenient to consider this function in polar coordinate $z_1=r_1e^{i\theta_1}$, $z_2=r_2e^{i\theta_2}$, . The real part of function (3.2) has the following form,

$$\hat{U}(\hat{r},\hat{\beta})=U(\hat{r},\hat{\beta})+O(|\hat{r}|^4)$$

where

$$U(\hat{r},\hat{\beta})=\beta_1 r_1^4+\beta_2 r_2^4+\beta_3 r_1^3+\beta_4 r_2^3+\beta_5 r_2^2+\beta_6 r_1^2$$
$$+\beta_7 r_1^2 r_2^2+\beta_8 r_1^3 r_2+\beta_9 r_2^3 r_1+\beta_{10} r_1^2 r_2$$
$$+\beta_{11} r_2^2 r_1 \qquad \ldots.(3.3)$$

where $\hat{r}=(r_1,r_2)$, $\hat{\beta}=(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\beta_7,\beta_8,\beta_9,\beta_{10},\beta_{11})$,

$$\beta_1 = \cos(\theta_1)^4 + \sin(\theta_1)^4 + 4\cos(\theta_1)^2 \sin(\theta_1)^2,$$
$$\beta_2 = \cos(\theta_2)^4 + \sin(\theta_2)^4 + 4\cos(\theta_2)^2 \sin(\theta_2)^2,$$
$$\beta_3 = \alpha_6 \cos(\theta_1)\sin(\theta_1)^2 + \alpha_{14} \cos(\theta_1)^3,$$
$$\beta_4 = \alpha_6 \cos(\theta_2)\sin(\theta_2)^2 + \alpha_{13} \cos(\theta_2)^3,$$
$$\beta_5 = \alpha_4 \cos(\theta_2)^2,$$
$$\beta_6 = \alpha_2 \sin(\theta_1)^2 + \alpha_1 \cos(\theta_1)^2,$$
$$\beta_9 = 4\sin(\theta_2)\cos(\theta_2)^2 \sin(\theta_1),$$

$$\beta_7 = 4\cos(\theta_1)^2 \sin(\theta_2)^2 + 4\cos(\theta_2)^2 \sin(\theta_1)^2 + 4\cos(\theta_2)^2 \cos(\theta_1)^2 + 4\sin(\theta_1)^2 \sin(\theta_2)^2 + 8\cos(\theta_2)\sin(\theta_2)\cos(\theta_1)\sin(\theta_1),$$
$$\beta_8 = 8\cos(\theta_2)\cos(\theta_1)\sin(\theta_1)^2 - 4\sin(\theta_2)\cos(\theta_1)^2 \sin(\theta_1) - \frac{4}{3}\cos(\theta_2)\cos(\theta_1)^3,$$
$$\beta_{10} = \alpha_7 \cos(\theta_2)\sin(\theta_1)^2 - \alpha_5 \cos(\theta_2)\cos(\theta_1)^2 - \alpha_5 \sin(\theta_2)\cos(\theta_1)\sin(\theta_1),$$
$$\beta_{11} = \alpha_{10} \cos(\theta_1)\cos(\theta_2)^2 + \alpha_9 \cos(\theta_1)\sin(\theta_2)^2 + \alpha_{11} \sin(\theta_2)\cos(\theta_2)\sin(\theta_1)$$

By changing variables $r_1 = \frac{1}{\sqrt[4]{\beta_1}} h_1,\ r_2 = \frac{1}{\sqrt[4]{\beta_2}} h_2$, we have equation (3.3) which is equivalent to the following equation,

$$U(\hat{h},\hat{\gamma}) = h_1^4 + h_2^4 + \gamma_1 h_1^3 + \gamma_2 h_2^3 + \gamma_3 h_2^2 + \gamma_4 h_1^2 + \gamma_5 h_1^2 h_2^2 + \gamma_6 h_1^3 h_2 + \gamma_7 h_2^3 h_1 + \gamma_8 h_1^2 h_2 + \gamma_9 h_2^2 h_1 \quad \text{....(3.4)}$$

where $\hat{h} = (h_1, h_2),\ \hat{\gamma} = (\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5,\gamma_6,\gamma_7,\gamma_8,\gamma_9)$,

$$\gamma_1 = \frac{\beta_3}{\beta_1^{3/4}},\ \gamma_2 = \frac{\beta_4}{\beta_2^{3/4}},\ \gamma_3 = \frac{\beta_5}{\sqrt{\beta_2}},\ \gamma_4 = \frac{\beta_6}{\sqrt{\beta_1}},\ \gamma_5 = \frac{\beta_7}{\sqrt{\beta_1}\sqrt{\beta_2}},$$

$$\gamma_6 = \frac{\beta_8}{\beta_1^{3/4}\beta_2^{1/4}},\ \gamma_7 = \frac{\beta_9}{\beta_2^{3/4}\beta_1^{1/4}},\ \gamma_8 = \frac{\beta_{10}}{\sqrt{\beta_1}\ \beta_2^{1/4}},\ \gamma_9 = \frac{\beta_{11}}{\sqrt{\beta_2}\ \beta_1^{1/4}}.$$

The elements $h_1^3, h_2^3, h_1^3 h_2, h_2^3 h_1$ belongs to the tangent space generated by the first derivatives $\frac{\partial U}{\partial h_1}, \frac{\partial U}{\partial h_2}, U_0(\hat{h}) = h_1^4 + h_2^4$ then from the theory of germs we have that the function (3.4) is equivalent to the following function

$$U(\hat{h}, \hat{f}) = h_1^4 + h_2^4 + f_3 h_2^2 + f_4 h_1^2 + f_5 h_1^2 h_2^2$$
$$+ f_8 h_1^2 h_2 + f_9 h_2^2 h_1 \qquad ....(3.5)$$

The geometrical form of bifurcations of critical points and the first asymptotic of branches of bifurcating points for the function $W(\xi, \delta)$ are completely determined by its principal part $U(\hat{\xi}, \hat{\delta})$.the geometric description of the caustic and the bifurcation of critical points of the function $U(\hat{h}, \hat{\gamma})$ will be study in another paper.

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