On Some Concepts of Metric in S^*-Orlicz Spaces

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Abstract
The main purpose of this paper is to study some concepts of metric in S^*-Orlicz spaces and we give some definitions that is related to it, where $S^* = S^*[0,1]$ is the ring of all real measurable functions on [0,1].

Keywords: metric space, S^*-Orlicz Spaces, Banach space

الملخص :

الهدف الرئيسي للبحث هو دراسة بعض المفاهيم المتريه لفضاء S^*-Orlicz وقد اعطينا بعض التعاريف المتعلقه بهذا الفضاء في حالة كون $S^* = S^*[0,1]$ والتي تمثل حلقه لكل الدوال القياسيه الحقيقه في [0,1].

1.Introduction and Preliminaries

The notion of the Orlicz space is generalized to spaces of the Banach space of valued functions. A well-known generalization is based on N-functions of a real variable.

A metric space need not have any kind of algebraic structure defined on it. In many applications, however, the metric space is a linear space with a metric derived linear spaces.

We shall denote by L_F the S^*-Orlicz class, $C_\infty(Q(\nabla))$ the set of all continuous functions on the Stone compactum $Q(\nabla)$, P the Lebesgue measure and $L_1(m)$ the set of all integrable by the measure m elements from $C_\infty(Q(\nabla)$.

Definition 1.1: [12]

A pair $(X, \leq)$ consisting of a real vector space X and a partial order $\leq$ defined on X is called a vector lattice if the following conditions are satisfied for all $x, y, z \in X$ and all real numbers $\alpha > 0$.

1. If $x \leq y$ then $x + z \leq y + z$.
2. If $x \leq y$ then $\alpha x \leq \alpha y$.
3. X is a lattice.

Remark 1.2:[11]

For a vector lattice X and $x \in X$, we make use of the following notation :
The positive part x_+ and the negative part x_- of X are given respectively by $x_+ = x \vee 0$, $x_- = (-x) \vee 0$.
The modulus $|x|$ of X is defined to be $|x| = (-x) \vee x$. It is obvious that $-x_- = x \wedge 0$ and for any $x \in X$ we have
$x = x_+ - x_-$, $x_+ \wedge x_- = 0$, $|x| = x_+ + x_-$.
The positive cone of a vector lattice is denoted X_+ , that is , $X_+ = \{x \in X : 0 \leq x\}$.

Example 1.3:[13]

The most obvious example of a vector lattice is the reals with all usual operations. The usual or standard order on $\mathbb{R}^n$ is that in which$(x_1, x_2, \dots, x_n) \leq (y_1, y_2, \dots, y_n)$ means that $x_k \leq y_k$ for $k = 1,2, \dots, n$. This order makes $\mathbb{R}^n$ into a vector lattice in which $(x_k) \vee (y_k) = (x_k \vee y_k)$ and $(x_k) \wedge (y_k) = (x_k \wedge y_k)$. Hence $(x_k)^+ = (x_k{}^+)$, $(x_k)^- = (x_k{}^-)$ and $|(x_k)| = (|x_k|)$.

Definition 1.4: [11]

An element from X_+ is called a Freudenthal unit and denoted by $\hat{1}$, if it follows from $x \in X, x \wedge \hat{1} = 0$, that $x = 0$. If $x_\alpha \xrightarrow{(o)} x$ and $\{x_\alpha\}$ is increasing(decreasing) then, we write $x_\alpha \uparrow x$, (respectively, $x_\alpha \downarrow x$).[5]

Remark 1.5: [11]

If a vector lattice X has a Freudenthal unit, then we will consider that this unit is chosen and fixed. This unit will be exactly denoted by $\hat{1}$.

Definition 1.6: [14]

The function $F(u): [0, \infty) \to [0, \infty)$ is called an N-function
if it has the following properties:
1. F is even, continuous, convex ;
2. $F(0) = 0$ and $F(u) > 0$ for all $u = 0$;
3. $\lim_{u \to 0} \frac{F(u)}{u} = 0$ and $\lim_{u \to \infty} \frac{F(u)}{u} = \infty$.

It is well-known that $F(u)$ is an N-function ,if and only if, $F(u) = \int_0^{|u|} f(t)dt$, where $f(t)$ is the right derivative of $F(u)$ satisfies:
1. $f(t)$ is the right-continuous and non- decreasing;
2. $f(t) > 0$ whenever $t > 0$; (3) $f(0) = 0$ and $\lim_{t \to \infty} f(t) = \infty$.

For an N-function F define $G(v) = \sup\{u|v| - F(u): u \geq 0\}$. Then G is an N-function and it is called the complement of F. [1]

If F and G are two in mutually complementary N-function then $uv \leq F(u) + G(v)$ $\forall\, u, v \in R$ (Young's Inequality).[2]

Definition 1.7:[7]

We say that X be a bimodule over $S^* = [0,1]$, i.e. X is abelian group with respect to addition operation $(+)$ and right and left multiplication by element from S^* are defined on X having the properties:
1. $\lambda(x + y) = \lambda x + \lambda y$, $(x + y)\lambda = x\lambda + y\lambda$
2. $(\lambda + \mu)x = \lambda x + \mu x$, $x(\lambda + \mu) = x\lambda + x\mu$
3. $\lambda(\mu x) = (\lambda\mu)x$, $(x\lambda)\mu = x(\lambda\mu)$
4. $\hat{1} \cdot x = x \cdot \hat{1}$, for all $x, y \in X, \lambda, \mu \in S^*$.

Remark 1.8:[7]

A bimodule X over S^* is called a normal S^*-module if :
1. For all $x \in X, \lambda \in S^*$, then $\lambda x = x\lambda$
2. For any $e \in \nabla(S^*)$, $e \neq 0$, there exists $x \in X$ such that $xe \neq 0$

3. For any decomposition of the identity $\{e_i\} \subset \nabla(S^*)$ and for any $\{x_i\} \subset X$ there exists $x \in X$ such that $x\, e_i = x_i\, e_i$, $i = 1,2,\dots,n$
4. For any $x \in X$ and any sequence $\{e_n\}$ of mutually disjoint elements from $\nabla(S^*)$ it follows the equalities $e_n x = 0, n = 1,2,\dots$ that

$$\left(\sup_{n\geq 1} e_n\right) x = 0.$$

It is clear that the condition 4 implies a validity of the analogous property for increasing sequences of idempotent from S^*.

Definition 1.9:[11]

A normal S^*-module is called an S^*-vector lattice if X is simultaneously lattice, i.e. an ordered set in which for any two elements $x, y \in X$ there exists their supremum $x \vee y$, infimum $x \wedge y$ and, in addition, the following algebraic operations and order agreement conditions are fulfilled :

1. for any $z \in X$ it follows from $x \leq y$ that $x + z \leq y + z$;
2. if $x \geq 0$, $\lambda \in S^*$, $\lambda x \geq 0$.

It is evident that any S^*-vector lattice X is a vector lattice in a usual sense (it is sufficient to consider X as a vector space over the field $\{\alpha \cdot \hat{1} : \alpha \in R\} \approx R$).

S^* itself consider as a bimodul over S^* is a simplest example of S^*-vector lattice.

2. Basic Concepts

In this section, we start recalling the usual definitions of S^*-valued metric and S^*-Orlicz spaces.

Definition 2.1:[6]

A mapping $\|\cdot\|: X \longrightarrow S^*$ from a normal S^*-module X into S^* is called an **S^*-norm** if

1. $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
2. $\|\lambda x\| = |\lambda|\|x\|$ for any $x \in X$, $\lambda \in S^*$.
3. $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in X$.

Definition 2.2: [11]

We say that the S^*-vector lattice X with an S^*-norm $\|\cdot\|$ is a **normed S^*-vector lattice**, if $|x| \leq |y|$, $\forall\, x, y \in X$, then $\|x\| \leq \|y\|$.

Definition 2.3: [4]

A mapping $\rho: X \times X \to S^*$ is called a **metric** on a set X with values in S^* if

1. $\rho(x, y) \geq 0$ for any $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$.
2. $\rho(x, y) = \rho(y, x)$ for any $x, y \in X$.
3. $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for any $x, y, z \in X$.

Definition 2.4:

An S^*-vector lattice X with an S^*-metric ρ is called a **metric S^*-vector lattice**, if it follows from $|x - y| \leq |z - w|$, $x, y, z, w \in X$, then $\rho(x, y) \leq \rho(z, w)$.

Now, we define the **S^*-Orlicz spaces** L^*_F. Let G be the complementary N-function to the N-function F.

Set

$L_F^* = \{x \in C_\infty(Q(\nabla)) : \lambda^{-1}x \in L_F \text{ for some number } \lambda = \lambda(x) > 0\}$.

We shall denote μ by the integral constructed by the measure m.

3.The Main Results

In this section, we investigate the important results concerning with the S^*-valued metric in S^*-Orlicz spaces.

Firstly, we need the following information.

Proposition 3.1:[8]

For every $x \in L_F^*$, we have

$$\bigvee_{y \in A(G)} |\mu(xy)| < \infty,$$

where $A(G) = \{y \in L_G : \mu(G(y)) \leq \hat{1}\}$.

This leads to define the following norm on L_F^* which is called the Orlicz norm:

$$\|x\|_F = \bigvee_{y \in A(G)} |\mu(xy)|,$$

for every $x \in L_F^*$.

Proposition 3.2:[9]

If $x \in L_F^*$, then

$$\|x\|_F = \||x|\|_F = \bigvee_{y \in A(G)} \mu(|xy|).$$

Remark 3.3:[10]

$\|x\|_F$ is a S^*-norm on L_F^*. In addition $|x| \leq |z|, x, z \in L_F$, implies,that $\|x\|_F \leq \|z\|_F$. Thus, $(L_F^*, \|\cdot\|_F)$ is a normed S^*-vector lattice.

Proposition 3.4:

Let L_F^* be S^*-Orlicz space, L_F be S^*-Orlicz class and $\|\cdot\|$ be an S^*-norm, then $\|x\|_F$ is a S^*-valued metric on L_F^*. In addition $|x-y| \leq |z-w|,\ x, y, z, w \in L_F$, implies, that $\rho(x, y) \leq \rho(z, w)$. Furthermore, (L_F^*, ρ) is a metric S^*-vector lattice.

Proof:

1. for any $x, y \in L_F^*$, then

$$\rho(x, y) = \|x-y\|_F = \bigvee_{z \in A(G)} |\mu(x-y)z| \geq 0.$$

and for any $x, y \in L_F^*$,

$$\rho(x, y) = 0 \Leftrightarrow \|x-y\|_F = 0 \Leftrightarrow \||x-y|\|_F = 0 \quad \Leftrightarrow \bigvee_{z \in A(G)} [\mu|(x-y)z|] = 0$$

$$\Leftrightarrow \mu[|xz-yz|] = 0 \Leftrightarrow xz-yz = 0 \Leftrightarrow x = y.$$

2. For any $x, y \in L_F^*$, then

$$\rho(x, y) = \|x-y\|_F = \bigvee_{z \in A(G)} |\mu(x-y)z|$$

$$= \bigvee_{z\in A(G)} \mu[|(y-x)z|] = \|y-x\|_F = \rho(y,x).$$

3. $\rho(x,y) = \|x-y\|_F = \|x-z+z-y\|_F$

$$= \bigvee_{w\in A(G)} \mu[|(x-z+z-y)w|]$$

$$= \bigvee_{w\in A(G)} \mu[|(x-z)w+(z-y)w|]$$

$$\le \bigvee_{w\in A(G)} \mu[|(x-z)w|] \quad + \bigvee_{w\in A(G)} \mu[|(z-y)w|]$$

$$= \|x-z\|_F + \|z-y\|_F = \rho(x,z) + \rho(z,y).$$

Thus, $\|x\|_F$ is a S^*-valued metric on L_F^*.

Now, let $x, y, z, w \in L_F$ and $|x-y| \le |z-w|$. Then

$$\rho(x,y) = \|x-y\|_F$$

$$= \bigvee_{r\in A(G)} \mu[|(x-y)r|]$$

$$= \bigvee_{r\in A(G)} \mu[|x-y||r|] \le \bigvee_{r\in A(G)} \mu[|z-w||r|]$$

$$= \|z-w\|_F = \rho(z,w)$$

Therefore, (L_F^*, ρ) is a metric S^*-vector lattice.

Remark 3.5:[10]

If $x \in L_F^*$ and $\|x\|_F \le \hat{1}$, then $x \in L_F$ and $\mu\big(F(x)\big) \le \|x\|_F$.

Proposition 3.6:

If $x, y \in L_F^*$ and $\rho(x,y) \le \hat{1}$, then $x, y \in L_F$ and $\mu\big(F(x,y)\big) \le \rho(x,y)$.

Proof:

Clearly $x, y \ge 0$. Choose a sequence of simple elements $z_n \ge 0$ such that $z_n = (x_{n,}y_n)$ and $(x_{n,}y_n) \uparrow (x,y)$.

Then $(x_{n,}y_n) \in L^*{}_F$ and $\rho\big(x_{n,}y_n\big) \le \hat{1}$ (see Remark 1.5.1 [9] and Proposition 3.3)).

Let

$$\big(x_{n,}y_n\big) = z_n = \sum_{i=1}^{k(n)} \lambda_i{}^{(n)}\, e_i{}^{(n)} \quad \text{and} \quad w_n = \sum_{i=1}^{k(n)} f(\lambda_i{}^{(n)})\, e_i{}^{(n)},$$

where f(t) is the right-hand derivative of the N-function F. By (Lemma 1.6.1 [3]), we have $\mu(G(w_n)) \le \hat{1}$.

By Young's Inequality , we get

$$(x_n, y_n)w_n = F(x_n, y_n) + G(w_n)$$

From this we have

$$\mu\big(F(x_n, y_n)\big) \le \mu\big(F(x_n, y_n)\big) + \mu\big(G(w_n)\big) = \mu((x_n, y_n)w_n) \le \rho(x_n, y_n) \le \rho(x,y).$$

Since $(x_{n,}y_n) \uparrow (x,y)$, then $F(x_{n,}y_n) \uparrow F(x,y)$ (see Lemma 1.5.1 [5]).

Since $\mu\big(F(x_n, y_n)\big) \le \rho(x,y)$, it follows from (Levi's Theorem [4]), that $F(x,y) \in L_1(m)$, i.e. $x, y \in L_F$ and $\mu\big(F(x,y)\big) \le \rho(x,y)$.

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