Approximation Results For q-Szãsz Mirakjan type operators

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Abstract

In this paper, we introduce and study a new type q-Szãsz Mirakjan operators $L_n(f, q_n; x)$ based on the q-integers in the case $q_n \in (0,1)$, where $q_n \to 1$ as $n \to \infty$. First, we prove that these operators are converging to the function being approximated. Then, we define the m-th order q-moments and find a recurrence relation for these q-moments. Finally, we prove Voronovskaja-type asymptotic formulas for the operators $L_n(f, q_n; x)$.

Keywords: q-integers; q- Szãsz Mirakjann operators; linear positive operators; sequence of positive numbers; Voronovskaja-type asymptotic formulas

الملخص:

في هذا البحث, نقدم وندرس صيغة جديدة هي $L_n(f, q_n; x)$ من المؤثرات من النمط q-Szãsz المتضمنة q-integers في الحالة $q_n \in (0,1)$, حيث $q_n \to 1$ عندما $n \to \infty$. في البداية برهنا هذه المؤثرات تتقارب الى الدالة المستخدمة في التقريب ثم عرفنا العزم m من النمط q وأوجدنا الصيغة التكرارية لها. وأخيرا أثبتنا صيغة فورونو فسكي للتقارب (Voronovskaja-type asymptotic formula) للمؤثرات $L_n(f, q_n; x)$.

1. Introduction and preliminaries

The approximation of functions by using linear positive operators introduced via q-Calculus is currently under intensive research. Aral in [1] proposed and studies a generalization of Szãsz Mirakjan operators [8], (1950) involving on the q-integers. In [2], (2006), [6], (2010), and [7], (2010) the reader can be finding a study of modifications for q-Szãsz Mirakjan operators. In this paper, we first introduce a new generalization of the operators defined by Mahmudov [8] and study the weighted function properties of the generalize q-Szãsz Mirakjan operators with the help of the Korovkin type approximation theorem. We also gives the quantitative Voronovskaja-type asymptotic formula. Throughout the paper we employ the standard notations of q-calculus, see [3], [4].

Let $q_n \in (0,1)$ and for any $n \in N^0 = \{0,1, \dots\}$ the q-integer $[n]$ is defined by

$$[n]:=\begin{cases}\dfrac{q_n^n-1}{q_n-1}=1+q_n+q_n^2+\cdots+q_n^{n-1}, & if\ q_n\neq 1\\ n & ,if\ q_n=1\end{cases}$$

and the q-factorial $[n]!$ by

$$[n]!:=\begin{cases}1 & if\ \ n=0,\\ [n]\times[n-1]\times\ldots\times[1] & if\ \ n\geq 1.\end{cases}$$

For the integers n,k, the q-binomial, or the Gaussian coefficients are defined by

$$\begin{bmatrix}n\\k\end{bmatrix}=\begin{cases}\dfrac{[n]!}{[k]!\ [n-k]!}, & n\geq k\geq 0,\\ 0, & \text{otherwise.}\end{cases}$$

The q-derivative of a function $f\colon R\to R$ denoted by $D_q f$ is defined by

$$D_q f(x)=\frac{d_q f(x)}{d_q x}=\frac{f(qx)-f(x)}{(q-1)x}\ ,\qquad x\neq 0, q\in R^+/\{1\}.$$

Note that

$$\lim_{q\to 1} D_q f(x)=\frac{df(x)}{dx}\ \text{and}\ \lim_{n\to\infty}[n]=\frac{1}{1-q}.$$

The q-derivative of a function f, denoted by $D_q f$, the q-derivative of sum defined as

$$D_q\big(u(x)+v(x)\big)=D_q(u(x))+D_q(v(x)),$$

and the formula for the q-derivative of a product of two functions is defined by

$$D_q\big(u(x)\,v(x)\big)$$
$$=D_q(u(x))\,v(x)+u(qx)\,D_q(v(x)).$$

Also, it is known that $D_q x^n=\dfrac{(qx)^n-x^n}{(q-1)x}=\dfrac{q^n-1}{q-1}x^{n-1}=[n]x^{n-1}$.

The q-analuge of $(t-x)^m$ is defined as:

$$(t-x)_q^m=\begin{cases}1 & if\ \ m=0,\\ (t-x)(t-qx)\ldots(t-q^{m-1}x) & if\ \ m\geq 1.\end{cases}$$

The $-$ exponential function is defined as e_{q_n}

$$=\sum_{k=0}^{\infty}\frac{x^k}{[k]!}\ \text{ such that}$$

$D_{q_n}\left(e_{q_n}(ax)\right)=ae_{q_n}(ax)$ and $D_{1/q_n}\left(e_{q_n}(x)\right)=e_{q_n}(q_n^{-1}x)$.

Let $C[0,\infty)$ denote the class of all continuous function on interval $[0,\infty)$. For $h>0$ and $f\in C_h[0,\infty)=\{f\in C[0,\infty)\colon |f(t)|\leq Ce_{q_n}^{ht}\ \ for\ some\ C>0\}$.

Definition 1. Let $0 < q_n < 1$, $n \in N$ and $q_n \to 1 \;\; as \;\; n \to \infty$. For $f \in C_h[0,\infty)$, we define the operators as

$$L_n(f, q_n; x) = \sum_{k=0}^{\infty} S_{n,k+p}(q_n; x)\, f\left(\frac{[k+p+\alpha]}{[n+\beta]}\right) \qquad (1.1)$$

where, $$S_{n,k+p}(q_n; x) = \frac{1}{q_n^{k+p(k+p-1)/2}} \frac{[n]^{k+p} x^{k+p}}{[k+p]!} e_{q_n}\left(-[n] q_n^{-(k+p)} x\right).$$

Clearly whenever $\alpha = \beta = p = 0$, then $L_n(f, q_n; x)$ becomes the operators in [7].

2. Some properties of the weight function $S_{n,k+p}(q_n; x)$.

The next lemmas exposition some importance properties of the weight functions $S_{n,k+p}(q_n; x)$.

Lemma 1. For $x \in [0,\infty)$ and $m \in N^0$, we have

(i) $$\sum_{k=0}^{\infty} S_{n,k+p}(q_n; x) = 1;$$

(ii) $$x D_{q_n} \; S_{n,k+p}(q_n; x) = ([k+p] - [n]x)\, S_{n,k+p}(q_n; x); \qquad (2.1)$$

(iii) $$L_n(t^{m+1}, q_n; x) = \frac{x}{[n+\beta]} D_{q_n} L_n(t^m, q_n; x) + \frac{(x[n] + q_n[\alpha])}{[n+\beta]} L_n(t^m, q_n; x);$$

(iv) Suppose that

$$\varphi_{n,m}(q_n; x) = \sum_{k=0}^{\infty} [k+P]^m \; S_{n,k+p}(q_n; x),$$

then

$$\varphi_{n,m+1}(q_n; x) = x D_{q_n} \varphi_{n,m}(q_n; x) + [n]x\, \varphi_{n,m}(q_n; x); \qquad (2.2)$$

and

$$\varphi_{n,m}(q_n; x) = ([n]x)^m + \left(\sum_{j=0}^{m-1} (m-1-j) q_n^j\right) ([n]x)^{m-1} + \; T.L.P.(x), \qquad (2.3)$$

where $T.L.P.(x)$ are the terms in lower power of x;

(v) $$\sum_{k=0}^{\infty} [k+p] \; S_{n,k+p}(q_n; x) = [n]x;$$

(vi) $$\sum_{k=0}^{\infty} [k+p]^2 \;\; S_{n,k+p}(q_n; x) = ([n]x)^2 + [n]x;$$

$(vii)\ \sum_{k=0}^{\infty}[k+p]^3\ \ S_{n,k+p}(q_n;x)=([n]x)^3+(2+q_n)([n]x)^2+[n]x;$

$(I)\ \sum_{k=0}^{\infty}[k+p]^4\ \ S_{n,k+p}(q_n;x)$

$$=([n]x)^4+(3+2q_n+q_n^2)([n]x)^3+(3+3q_n+q_n^2)([n]x)^2+[n]x.$$

Proof. (i) For a fixed $x\in[0,\infty)$, using the q-Taylor's theorem [4], (2000) for the function $f(t)$, we obtain

$$f(t)=\sum_{k=0}^{\infty}\frac{(t-x)_{1/q_n}^{k+p}}{[k+p]!_{1/q_n}}\ D_{1/q}^{k+p}\ f(x).$$

Also we have $(-x)_{1/q_n}^{k+p}\ =(-1)^{k+p}\ x^{k+p}\ q_n^{k+p(k+p-1)/2}$,

and $D_{1/q_n}^{k+p}\ e_{q_n}(-[n]x)=(-1)^{k+p}\ \ q_n^{k+p(k+p-1)/2}\ e_{q_n}\left(-[n]q_n^{-(k+P)}x\right)$.

Hence we get

$$1=e_{q_n}(0)=f(0)=\sum_{k=0}^{\infty}\frac{(-x)_{1/q_n}^{k+p}}{[k+p]!_{1/q_n}}\ D_{1/q_n}^{k}\ f(x)=\sum_{k=0}^{\infty}\frac{1}{q_n^{k+p(k+p-1)/2}}\ \frac{[n]^{k+p}}{[k+p]!}\times$$

$$e_{q_n}\left(-[n]q_n^{-(k+p)}x\right)=\sum_{k=0}^{\infty}S_{n,k+p}(q_n;x)\ .$$

$$(ii)\ xD_{q_n}\ \ S_{n,k+p}(q_n;x)=[k+p]\frac{1}{q_n^{k+p(k+p-1)/2}}\ \frac{[n]^{k+p}x^{k+p}}{[k+p]!}\times$$

$$e_{q_n}\left(-[n]q_n^{-(k+p)}x\right)$$

$$-[n]x\frac{1}{q_n^{k+p(k+p-1)/2}}\ \frac{[n]^{k+p}x^{k+p}}{[k+p]!}\times$$

$$e_{q_n}\left(-[n]q_n^{-(k+p)}x\right)$$

$$=([k+p]-[n]x)\ S_{n,k+p}(q_n\ ;x).$$

$$(iii)\ xD_{q_n}\ L_n(t^m,q_n;x)=\sum_{k=0}^{\infty}\left(\frac{[k+p+\alpha]}{[n+\beta]}\right)^m([k+p]-[n]x)\ S_{n,k+p}(q_n\ ;x)$$

$$=[n+\beta]\sum_{k=0}^{\infty}\left(\frac{[k+p+\alpha]}{[n+\beta]}\right)^m\left(\frac{[k+p+\alpha]}{[n+\beta]}-x-\frac{q_n([\alpha]-x[\beta])}{[n+\beta]}\right)S_{n,k+p}(q_n\ ;x)$$

$$= [n+\beta]\sum_{k=0}^{\infty}\left(\frac{[k+p+\alpha]}{[n+\beta]}\right)^{m+1} S_{n,k+p}(q_n\ ;x)$$

$$-\left(x[n]+q_n[\alpha]\right)\sum_{k=0}^{\infty}\left(\frac{[k+p+\alpha]}{[n+\beta]}\right)^{m} S_{n,k+p}(q_n\ ;x)$$

$$L_n(t^{m+1},q_n;x)=\frac{x}{[n+\beta]}D_{q_n}\,L_n(t^m,q_n;x)+\frac{(x[n]+q_n[\alpha])}{[n+\beta]}L_n(t^m,q_n;x).$$

$$(iv)\ xD_{q_n}\varphi_{n,m}(q_n\ ;x)=\sum_{k=0}^{\infty}[k+p]^m\ ([k+p]-[n]x)S_{n,k+p}(q_n\ ;x)$$

$$xD_{q_n}\varphi_{n,m}(q_n\ ;x)=\sum_{k=0}^{\infty}[k+p]^{m+1}\ S_{n,k+p}(q_n\ ;x)$$

$$-[n]x\sum_{k=0}^{\infty}[k+p]^m\ S_{n,k+p}(q_n\ ;x)$$

$$\varphi_{n,m+1}(q_n\ ;x)$$

$$= xD_{q_n}\varphi_{n,m}(q_n\ ;x)+[n]x\ \varphi_{n,m}(q_n\ ;x).$$

To prove (2.3) using the induction on m, then from (2.2) we get:

$\varphi_{n,1}(q_n\ ;x)=[n]x,$

$\varphi_{n,2}(q_n\ ;x)=([n]x)^2+[n]x,$ and

$$\varphi_{n,3}(q_n\ ;x)=([n]x)^3+(2+q_n)([n]x)^2+[n]x.$$

Suppose that the relation is true for m, then we must prove it for $m+1$.

By using (2.2) we have:

$$\varphi_{n,m+1}(q_n;x)=x\left\{[m][n]^m\ x^{m-1}+\left(\sum_{j=0}^{m-1}(m-1-j)q_n^j\right)[m-1][n]^{m-1}\ x^{m-2}\right.$$

$$\left.+T.L.P.(x)\right\}$$

$$+[n]x\left\{[n]^m\ x^m+\left(\sum_{j=0}^{m-1}(m-1-j)q_n^j\right)[n]^{m-1}\ x^{m-1}\right\}$$

$$+T.L.P.(x)$$

$$=[n]^{m+1}\ x^{m+1}+\left(\sum_{j=0}^{m-1}(m-1-j)q_n^j+[m]\right)[n]^m\ x^m+T.L.P.(x)$$

$= [n]^{m+1}\, x^{m+1} + (m + (m-1)q_n + (m-2)q_n^2 + \cdots + 2q_n^{m-2} + q_n^{m-1})[n]^m\, x^m$
$+T.L.P.(x)$

$= [n]^{m+1}\, x^{m+1} + \left(\sum_{j=0}^{m}(m-j)q_n^j\right)\times$

$[n]^m\, x^m + T.L.P.(x).$

To prove $(v), (vi)$ and (vii), put $m = 0, 1, 2$ respectively in (2.2), then we get the proof.

Now , to proof (I) put $m = 3$ in (2.2) then we get

$$\varphi_{n,4}(q_n; x) = \sum_{k=0}^{\infty}[k+p]^4\ \ S_{n,k+p}(q_n; x) = x([3][n]^3x^2 + [2](2+q_n)[n]^2\, x + [n])$$

$$+[n]x([n]^3x^3 + (2+q_n)[n]^2x^2 + [n]x)$$

$$= [3][n]^3x^3 + [2](2+q_n)[n]^2\, x^2 + [n]x + [n]^4x^4 + (2+q_n)[n]^3x^3 + [n]^2x^2$$

$$= [n]^4x^4 + \big([3] + (2+q_n)\big)[n]^3x^3 + ([2](2+q_n) + 1)[n]^2\, x^2 + [n]x$$

$$= [n]^4x^4 + (3 + 2q_n + q_n^2)[n]^3x^3 + (3 + 3q_n + q_n^2)[n]^2\, x^2 + [n]x. \qquad \blacksquare$$

Lemma 2. For $x \in [0,\infty)$, the following conditions are holds

$(i)\ L_n(1, q_n; x) = 1;$

$(ii)\ L_n(t, q_n; x) = \dfrac{[n]x + q_n^n[\alpha]}{[n+\beta]}\ \to x\ as\ n \to \infty;$

$(iii)\ L_n(t^2, q_n; x) = \dfrac{([n]x)^2 + [n]x + 2q_n^n[\alpha][n]x + q_n^{2n}[\alpha]^2}{[n+\beta]^2}\ \to x^2\ as\ n \to \infty;$

$(iv)\ L_n(t^3, q_n; x)$

$$= \frac{1}{[n+\beta]^3}\big(([n]x)^3 + ([n]x)^2(2 + q_n + 3q_n^n[\alpha])$$

$$+ [n]x(1 + 3q_n^n[\alpha] + 3q_n^{2n}[\alpha]^2) + q_n^{3n}[\alpha]^3\big)\ \to x^3\ as\ n \to \infty.$$

Proof. By applying Lemma 1

(i) The proof easily.

$$(ii)\ L_n(t, q_n; x) = \frac{1}{[n+\beta]}\sum_{k=0}^{\infty}[k+p+\alpha]\ \ S_{n,k+p}(q_n; x)$$

$$= \frac{1}{[n+\beta]}\left(\sum_{k=0}^{\infty}[k+p]\ \ S_{n,k+p}(q_n;x) + q^n[\alpha]\sum_{k=0}^{\infty}\ \ S_{n,k+p}(q_n;x)\right)$$

$$= \frac{[n]x + q_n^n[\alpha]}{[n+\beta]} \to x\ as\ n \to \infty.$$

$$(iii)\ L_n(t^2, q_n; x) = \frac{1}{[n+\beta]^2}\sum_{k=0}^{\infty}[k+p+\alpha]^2\ \ S_{n,k+p}(q_n;x)$$

$$= \frac{1}{[n+\beta]^2}\left(\sum_{k=0}^{\infty}\ \ S_{n,k+p}(q_n;x)\ ([k+p]^2 + 2q_n^n[\alpha][k+p] + q_n^{2n}[\alpha]^2)\right)$$

$$= \frac{1}{[n+\beta]^2}(([n]x)^2 + [n]x + 2q_n^n[\alpha][n]x + q_n^{2n}[\alpha]^2)$$

$$= \frac{([n]x)^2 + [n]x + 2q_n^n[\alpha][n]x + q_n^{2n}[\alpha]^2}{[n+\beta]^2} \to x^2\ as\ n \to \infty.$$

$$(iv)\ L_n(t^3, q_n; x) = \frac{1}{[n+\beta]^3}\sum_{k=0}^{\infty}[k+p+\alpha]^3\ \ S_{n,k+p}(q_n;x)$$

$$= \frac{1}{[n+\beta]^3}\left(\sum_{k=0}^{\infty}\ \ S_{n,k+p}(q_n;x)\ ([k+p]^3 + 3q_n^n[\alpha][k+p]^2 + 3q_n^{2n}[\alpha]^2[k+p] + q_n^{3n}[\alpha]^3)\right)$$

$$= \frac{1}{[n+\beta]^3} \times (([n]x)^3 + ([n]x)^2(2 + q_n + 3q_n^n[\alpha]) + [n]x(1 + 3q_n^n[\alpha] + 3q_n^{2n}[\alpha]^2) + q_n^{3n}[\alpha]^3) \to x^3\ as\ n \to \infty.$$ ■

Theorem 1. Suppose that $f \in C_h[0,\infty)$, for some $h > 0$ and f exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} L_n(f, q_n; x) = f(x). \qquad (2.4)$$

Further, if f exist and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then (2.4) holds uniformly on $[a,b]$.

Proof. By using Lemma 2, the proof of above theorem is holds. ■

3. The q-moment of the operators $L_n(f, q_n; x)$.

In this section we defined the m-th order q-moment for the operators $L_n(f, q_n; x)$ which is denoted by $E_{n,m}(x)$.

Definition 2. For $m \in N^0$, the *m*-th order q- moment $E_{n,m}(q_n;x)$ for the operator $L_n(f,q_n;x)$ is defined as:

$$E_{n,m}(q_n;x) = L_n\left((t-x)_{q_n}^m, q_n;x\right) = \sum_{k=0}^{\infty} S_{n,k+p}(q_n;x)\left(\frac{[k+p+\alpha]}{[n+\beta]} - x\right)_q^m.$$

Lemma 3. For the function $E_{n,m}(q;x)$, we have:

(i) $E_{n,0}(q_n;x) = 1;$

(ii) $E_{n,1}(q_n;x) = \dfrac{q_n^n([\alpha]-x[\beta])}{[n+\beta]};$

(iii) $E_{n,2}(q_n;x)$

$$= \frac{1}{[n+\beta]^2}$$

$$\times\left([n]x(1+q_n^n[\alpha](1-q) - xq_n^n[\beta][2]) + q_n^{2n}(1-x)([\alpha]-q_n x[\alpha])\right)$$

Proof. By direct computation, we can easily prove (i).

$$(ii) E_{n,1}(q_n;x) = L_n(t,q_n;x) - xL_n(1,q_n;x) = \frac{[n]x + q_n^n[\alpha]}{[n+\beta]} - x = \frac{q_n^n([\alpha]-x[\beta])}{[n+\beta]}.$$

$$(iii)\ E_{n,2}(q_n;x) = L_n\left((t-x)_{q_n}^2, q_n;x\right)$$

$$= L_n(t^2,q_n;x) - [2]xL_n(t,q_n;x) + q_n\, x^2 L_n(1,q_n;x)$$

$$= \frac{([n]x)^2 + [n]x + 2q_n^n[\alpha][n]x + q_n^{2n}[\alpha]^2}{[n+\beta]^2} - [2]x\left(\frac{[n]x + q_n^n[\alpha]}{[n+\beta]}\right) + q_n\, x^2$$

$$= \frac{1}{[n+\beta]^2} \times \left([n]x(1+q_n^n[\alpha](1-q_n) - xq_n^n[\beta][2])\right.$$

$$\left. + q_n^{2n}(1-x)([\alpha]-q_n x[\alpha])\right). \qquad ■$$

4. The recurrence relation for the *q*-moments of the operators $L_n(f,q_n;x)$.

In this section we give some lemmas which help us to prove Voronovskaja-type asymptotic formula for the operators $L_n(f,q_n;x)$.

Lemma 4. For $m \in N^0$;

$[n+\beta]E_{n,m+1}(q_n;x)$

$$= xD_{q_n}E_{n,m}(q_n;x) + q_n([\alpha]-x[\beta])E_{n,m}(q_n;x) + [m]xE_{n,m-1}(q_n;q_n x)$$

$$+ [n+\beta]x(1-q_n^m)E_{n,m}(q_n;x). \qquad (4.1)$$

Further more

a. $E_{n,m}(q_n;x)$ is a polynomial in x of degree $\leq m$, whenever n is sufficiently large.

b. For every $x \in [0,\infty)$, $E_{n,m}(q_n;x) = O\left([n]^{-\left[\frac{m+1}{2}\right]}\right)$,

where $[c]$ is an integer part of $c \geq 0$.

Proof.

$$E_{n,m}(q_n;x) = L_n\left((t-x)_{q_n}^m, q_n;x\right) = \sum_{k=0}^{\infty} S_{n,k+p}(q_n;x)\left(\frac{[k+p+\alpha]}{[n+\beta]} - x\right)_{q_n}^m.$$

$$D_{q_n}E_{n,m}(q_n;x)$$
$$= \sum_{k=0}^{\infty}\left(\frac{[k+p+\alpha]}{[n+\beta]} - x\right)_{q_n}^m D_{q_n} S_{n,k+p}(q_n;x)$$
$$- [m]\sum_{k=0}^{\infty}\left(\frac{[k+p+\alpha]}{[n+\beta]} - q_n x\right)_{q_n}^{m-1} S_{n,k+p}(q_n;q_n x)$$

$$xD_{q_n}E_{n,m}(q_n;x)$$
$$= \sum_{k=0}^{\infty}\left(\frac{[k+p+\alpha]}{[n+\beta]} - x\right)_{q_n}^m ([k+p]-[n]x)\, S_{n,k+p}(q_n;x)$$
$$- [m]xE_{n,m-1}(q_n;q_n x)$$

$$= [n+\beta]\sum_{k=0}^{\infty} S_{n,k+p}(q_n;x)\left(\frac{[k+p+\alpha]}{[n+\beta]} - x\right)_{q_n}^m \left(\frac{[k+p+\alpha]}{[n+\beta]} - xq_n^m + xq_n^m - x\right.$$
$$\left. - \frac{q_n([\alpha]-x[\beta])}{[n+\beta]}\right) - [m]xE_{n,m-1}(q_n;q_n x)$$

$$= [n+\beta]E_{n,m+1}(q_n;x) - [n+\beta]x(1-q_n^m)E_{n,m}(q_n;x) - q_n([\alpha]-x[\beta])E_{n,m}(q_n;x)$$
$$- [m]xE_{n,m-1}(q_n;q_n x)$$

$$[n+\beta]E_{n,m+1}(q_n;x)$$
$$= xD_{q_n}E_{n,m}(q_n;x) + q_n([\alpha]-x[\beta])E_{n,m}(q_n;x) + [m]xE_{n,m-1}(q_n;q_n x)$$
$$+ [n+\beta]x(1-q_n^m)E_{n,m}(q_n;x).$$

Now, the consequence (a) can be prove easily by using (4.1) and the induction on m. So the details are omitted.

Now, the proof of (b) is doing by using the induction on m. So, we get:

The consequence, is true for $m = 1$.

Now, let this consequence be true for m, we have:

$$[n+\beta]E_{n,m+1}(q_n;x)=O\left([n]^{-\left[\frac{m+1}{2}\right]}\right)+O\left([n]^{-\left[\frac{m+1}{2}\right]}\right)+O\left([n]^{-\left[\frac{m}{2}\right]}\right)$$

$$=O\left([n]^{-\left[\frac{m+1}{2}\right]}\right)+O\left([n]^{-\left[\frac{m}{2}\right]}\right)$$

$$=\begin{cases}O\left([n]^{-\left[\frac{m+1}{2}\right]}\right) & if\ m\ is\ odd\ ;\\ O\left([n]^{-\left[\frac{m}{2}\right]}\right) & if\ m\ is\ even\ .\end{cases}$$

Then$E_{n,m+1}(q_n;x)=\begin{cases}O\left([n]^{-\left[\frac{m+3}{2}\right]}\right) & if\ m\ is\ odd\\ O\left([n]^{-\left[\frac{m+2}{2}\right]}\right) & if\ m\ is\ even\end{cases}$

$E_{n,m+1}(q_n;x)=O\left([n]^{-\left[\frac{m+1}{2}\right]}\right)$, so the result is true for $m+1$.

Hence, $E_{n,m+1}(q_n;x)=O\left([n]^{-\left[\frac{m+1}{2}\right]}\right)$ for every $x\in[0,\infty)$. ■

Lemma 5. Let δ and α be any two positive real numbers and $[a,b]\subset(0,\infty)$. Then, for any $s>0$ we have:

$$\sup_{x\in[a,b]}\left|\sum_{\left|\frac{[k+p+\alpha]}{[n+\beta]}-x\right|\geq\delta}S_{n,k+p}(q_n;x)e_{q_n}^{ht}\right|=O([n]^{-s}).$$

Making use of q-Taylor's expansion, Schwarz inequality for summation and Lemma 4 (b), the proof of this Lemma easily follows, hence the details are omitted.

5. Voronovaskaja-type asymptotic formula for $L_n(f,q;x)$.

The next theorem is a Voronovaskaja-type asymptotic formula for the operators $L_n(f,q;x)$.

Theorem 2. Let $f\in C_h[0,\infty)$ for some $\alpha>0$. If D_qf, D_q^2f are exist at a point $x\in(0,\infty)$, then for sufficiently large n we have:

$$\lim_{n\to\infty}[n](L_n(f(t),q_n;x)-f(x))=q_n([\alpha]-x[\beta])D_{q_n}f(x)+\left(\frac{x}{[2]}\right)D_{q_n}^2\ f(x) \qquad (5.1)$$

Proof. By using q-Taylor's expansion [4] of f, we get:

$$f(t)=f(x)+\frac{D_{q_n}}{[1]!}f(x)\ (t-x)_{q_n}+\frac{D_{q_n}^2}{[2]!}\ f(x)(t-x)_{q_n}^2+r(t;x)(t-x)_{q_n}^2,$$

where $r(t;x)\to 0\ as\ t\to x$.

Then

$$L_n(f(t),q_n;x)=f(x)\ L_n(1,q_n;x)+\frac{D_{q_n}}{[1]!}f(x)\ L_n\left((t-x)_{q_n},q_n;x\right)$$

$$+\frac{D_{q_n}^2}{[2]!}\,f(x)\,L_n\big((t-x)_{q_n}^2,q_n;x\big)+L_n\big(r(t;x)(t-x)_{q_n}^2,q_n;x\big)$$

$$=f(x)+\frac{D_{q_n}}{[1]!}f(x)E_{n,1}(q_n;x)+\frac{D_{q_n}^2}{[2]!}\,f(x)\,E_{n,2}(q_n;x)$$

$$+\sum_{k=0}^{\infty}S_{n,k+p}(q_n;x)\left(\frac{[k+p+\alpha]}{[n+\beta]}-x\right)_{q_n}^2 r\left(\frac{[k+p+\alpha]}{[n+\beta]};x\right). \qquad (5.2)$$

Let

$$\rho=\sum_{k=0}^{\infty}S_{n,k+p}(q_n;x)\left(\frac{[k+p+\alpha]}{[n+\beta]}-x\right)_{q_n}^2 r\left(\frac{[k+p+\alpha]}{[n+\beta]};x\right).$$

Since $r(t;x)\to 0\ as\ t\to x$, for a given $\varepsilon>0$ there exist $\delta>0$ such that $\left|r\left(\frac{[k+p+\alpha]}{[n+\beta]};x\right)\right|<\varepsilon$, whenever $0<\left|\frac{[k+p+\alpha]}{[n+\beta]}-x\right|<\delta$, for $\left|\frac{[k+p+\alpha]}{[n+\beta]}-x\right|\geq\delta$, there exists a constant $C>0$ such that $\left|\left(\frac{[k+p+\alpha]}{[n+\beta]}-x\right)_{q_n}^2 r\left(\frac{[k+p+\alpha]}{[n+\beta]};x\right)\right|\leq Ce_{q_n}^{h\left(\frac{[k+p+\alpha]}{[n+\beta]}\right)}$.

Hence,

$$|\rho|=\left|\sum_{\left|\frac{[k+p+\alpha]}{[n+\beta]}-x\right|<\delta}S_{n,k+p}(q_n;x)\left(\frac{[k+p+\alpha]}{[n+\beta]}-x\right)_{q_n}^2 r\left(\frac{[k+p+\alpha]}{[n+\beta]};x\right)\right|$$

$$+\left|C\sum_{\left|\frac{[k+p+\alpha]}{[n+\beta]}-x\right|\geq\delta}S_{n,k+p}(q_n;x)\,e_{q_n}^{ht}\right|:=\lambda_1+\lambda_2.$$

Now, by using Schwarz inequality for summation, we get:

$$\lambda_1\leq\varepsilon\left(\sum_{k=0}^{\infty}S_{n,k+p}(q_n;x)\left(\frac{[k+p+\alpha]}{[n+\beta]}-x\right)_{q_n}^4\right)^{1/2}\left(\sum_{k=0}^{\infty}S_{n,k+p}(q_n;x)\right)^{1/2}.$$

Now, by using Lemma 4, and since ε is arbitrary, hence:

$$\lambda_1\leq\varepsilon\big(E_{n,4}(q_n;x)\big)^{1/2}=\varepsilon O([n]^{-2})=o(1)\ \ as\ n\to\infty.$$

In view of Lemma 5, we have:

$$\lambda_1=C\ O([n]^{-s})=O([n]^{-s}).$$

Hence: $|\rho|=o(1)\ \ as\ n\to\infty$.

Then

$$\lim_{n\to\infty} = \sum_{k=0}^{\infty} S_{n,k+p}(q_n;x)\left(\frac{[k+p+\alpha]}{[n+\beta]}-x\right)_{q_n}^2 r\left(\frac{[k+p+\alpha]}{[n+\beta]};x\right)\to 0 \quad as\, n\to\infty.$$

By applying this result in (5.2), we get:

$$\lim_{n\to\infty}[n](L_n(f(t),q_n;x)-f(x)) = q_n([\alpha]-x[\beta])D_{q_n}f(x)+\left(\frac{x}{[2]}\right)D_{q_n}^2\, f(x)$$

To prove the uniformly assertion, it is sufficient to remark that $\delta(\varepsilon)$ in the above proof can be chosen to be independent of $x\in[a,b]$, and also the other estimates hold uniformly in $x\in[a,b]$. The proof is completed. ■

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