Lie Symmetries to Ordinary Differential Equations (ODEs)

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Abstract

In this paper, we have provided some examples motivating the use of Lie symmetries. Also, we have discussed the properties of one- parameter groups of transformations (Lie groups) and introduced the ideas of invariants of the group of transformations. In the case of ordinary differential equations use of Lie symmetries resulted in a separable ODE when the governing DE was first order and a reduction of order when the governing DE was of order greater than one.

Keywords: *Lie symmetries, the group of transformations*, the *infinitesimals,ordinarydifferential equations, canonical coordinates.*

الملخص:

في هذا البحث،جهزنا بعضا الأمثلة لاستخدام تناظرات لي . كذلك، ناقشنا خصائص مجموعات التحويلات ذات البارامیتر الواحد وتسمى (مجموعات لي). كما ناقشنا حل المعادلات التفاضلية الاعتيادية باستخدام تناظرات لي نتيجة لطريقة فصل المتغيرات وذلك عندما تكون المعادلة التفاضلية الاعتيادية من الرتبة الاولى او اختزال رتبتها اذا كانت اكبر من واحد.

1.1 Introduction

In this section, we discuss the construction of a one-parameter group of transformations which leaves a given ordinary differential equation (ODE) unchanged. We shall find that if an ODE is invariant under a one-parameter group of transformations then use of an invariant of the group results in a simplification of the ODE. If the differential equation is of first order then the equation will become a separable differential equation. For a higher order differential equation, the use of an invariant leads to the reduction in the order of the equation by one. We firstly illustrate finding infinitesimals for an ODE from first principles and then quote a general result for a general first –order ODE.

1.2 (one- parameter transformation groups)

Definition: In the (x,y) plane, the transformation

$$x_1 = f(x,y,\epsilon) y_1 = g(x,y,\epsilon) \qquad (1.1)$$

is a **one- parameter group of transformations** if the following properties hold:

(i) (identity) the value $\epsilon = 0$ characterises the identity transformation,

$$x = f(x,y,0), \qquad y = g(x,y,0)$$

(ii) (inverse) the parameter $-\epsilon$ characterises the inverse transformation,

$$x = f(x_1,y_1,-\epsilon), \qquad y = g(x_1,y_1,-\epsilon)$$

(iii) (closure) if $x_2 = f(x_1,y_1,\delta)$, $y_2 = g(x_1,y_1,\delta)$ then the product of the two transformation is also a member of the set of the transformation (1.1) and moreover is characterised by the parameter $\epsilon+\delta$, that is $x_2=f(x,y,\epsilon+\delta)$, $y_2=g(x,y,\epsilon+\delta)$, see [1],[5].

Example 1.1

Show that the transformation

$$x_1=\frac{x}{1+\epsilon x}, \qquad y_1=(1+\epsilon x)^2 y \qquad (1.2)$$

does indeed form a one-parameter group of transformations as defined above.

1) Firstly when $\epsilon=0$, $x_1=x$, $y_1=y$, so (i) is satisfied.

2) On rearranging (1.2) we obtain

$$(1+\epsilon x)x_1=x$$

$$\Rightarrow x(\epsilon x_1-1)+x_1=0$$

$$\Rightarrow x=\frac{x_1}{1-\epsilon x_1}$$

And $$y=\frac{y_1}{(1+\epsilon x)^2}$$

$$\Rightarrow y=\frac{y_1}{(1+\epsilon\frac{x_1}{1-\epsilon x_1})^2}$$

$$\Rightarrow y=y_1(1-\epsilon x_1)^2$$

So, $x=\frac{x_1}{1-\epsilon x_1}$, $y=y_1(1-\epsilon x_1)^2$

so that $-\epsilon$ characterizes the inverse and (ii) is satisfied .

3) we see that if

$x_2 = \frac{x_1}{1+\delta x_1}$, $y_2 = (1+\delta x_1)^2 y_1$ then we have

$$x_2 = \frac{\frac{x}{1+\epsilon x}}{1+\delta(\frac{x}{1+\epsilon x})}$$

$\Rightarrow x_2 = \frac{x}{1+(\epsilon+\delta)x}$

and

$$y_2 = (1+\delta.\frac{x}{1+\epsilon x})^2.(1+\epsilon x)^2 y$$

$\Rightarrow y_2 = \left[(1+\delta.\frac{x}{1+\epsilon x}).(1+\epsilon x)\right]^2 y$

$\Rightarrow y_2 = [1+\epsilon x+\delta x]^2 y$

$\Rightarrow y_2 = (1+(\epsilon+\delta)x)^2 y,$

so that x_2 and y_2 are members of the group of transformations, characterised by $\epsilon+\delta$. Therefore (iii) is satisfied.

Remark:

The transformations (1.1) are called point transformations because the transformed values only depend on the dependent and independent variables $x \; and \; y$ (and not derivatives of variables) and the parameter ϵ.The functions $f(x,y,\epsilon) \; and \; g(x,y,\epsilon)$ are referred to as *the global form* of the group,[7]. For small values of ϵ, we can expand $f \; and \; g$ and since

$$x_1 = x, \qquad y_1 = y \qquad at \;\; \epsilon = 0$$

we have

$$x_1 = x + \epsilon(\frac{df}{d\epsilon})_{\epsilon=0} + O(\epsilon^2),\ y_1 = y + \epsilon(\frac{dg}{d\epsilon})_{\epsilon=0} + O(\epsilon^2) \quad (1.3)$$

where $O(\epsilon^2)$ indicates terms of order ϵ^2 and higher.Defining $X(x,y)$ *and* $Y(x,y)$ by

$$X(x,y) = (\frac{df}{d\epsilon})_{\epsilon=0} \quad and \quad Y(x,y) = (\frac{dg}{d\epsilon})_{\epsilon=0} \quad (1.4)$$

then we obtain

$$x_1 = x + \epsilon X(x,y) + O(\epsilon^2),\ y_1 = y + \epsilon Y(x,y) + O(\epsilon^2)(1.5)$$

and (1.5) is referred to as ***the infinitesimal form*** of the group. $X(x,y), Y(x,y)$ are often referred to as "***the infinitesimal***", see [1],[5].

1.3 Invariants of a group

Definition:

In [2],[4], A differential function $F(x,y)$ is called an ***invariant function*** of a group G where$x^* = f(x,y,\epsilon),\quad y^* = g(x,y,\epsilon)$,

if $\qquad F(x^*,y^*) = F(x,y)$,(1.6)

Definition:

The ***infinitesimal generator*** of a one-parameter Lie group of transformations where

$$\bar{x} = x + \epsilon X(x,y) + O(\epsilon^2),\quad \bar{y} = y + \epsilon Y(x,y) + O(\epsilon^2)(1.7)$$

is given by $\qquad \Gamma = X(x,y)\frac{\partial}{\partial x} + Y(x,y)\frac{\partial}{\partial y}. \qquad (1.8)$

see [4],[6].

Example (1.2)

Consider the standard first order equation (Bernoulli's equation)[3],

$\frac{dy}{dx}+p(x)y=q(x)y^n \quad n\neq 1$(1.9)

We wish to find the *infinitesimals* for a group of transformations of the form

$$x_1=f(x,\epsilon), \qquad y_1=g(x,\epsilon)y$$

which leaves (1.9) invariant. Hence, consider

$\frac{dy_1}{dx_1}+p(x_1)y_1=q(x_1)y_1^n$ (1.10)

and we rewrite this in terms of x and y.

Now, $\frac{dy_1}{dx_1}=\frac{d}{dx}(g(x,\epsilon)y)\frac{dx}{dx_1}$

$$\Rightarrow\frac{dy_1}{dx_1}=\left[y\, g_x(x,\epsilon)+g(x,\epsilon)\frac{dy}{dx}\right]\left(\frac{1}{f_x(x,\epsilon)}\right) \qquad (1.11)$$

Now, we will substitute (1.11) in (1.10) to get

$$\left[y\, g_x(x,\epsilon)+g(x,\epsilon)\frac{dy}{dx}\right]\left(\frac{1}{f_x(x,\epsilon)}\right)+p(f)g(x,\epsilon)y=q(f)g(x,\epsilon)^n y^n$$

$$\Rightarrow\frac{dy}{dx}+y\left[\frac{g_x(x,\epsilon)}{g(x,\epsilon)}+f_x(x,\epsilon)p(f)\right]=q(f)g(x,\epsilon)^{n-1}f_x(x,\epsilon)y^n.$$

Equating this with our DE $\frac{dy}{dx}+p(x)y=q(x)y^n$

we may deduce

$$p(x)=\frac{g_x(x,\epsilon)}{g(x,\epsilon)}+f_x(x,\epsilon)p(f) \qquad (1.12)$$

$$q(x) = q(f)g(x,\epsilon)^{n-1} f_x(x,\epsilon)\cdot \qquad (1.13)$$

On substituting the expansions

$f(x,\epsilon) = x + \epsilon X(x) + O(\epsilon^2)$, $g(x,\epsilon) = 1 + \epsilon Y(x) + O(\epsilon^2)$ ....(a)

$p(f) = p(x) + \epsilon X(x)p'(x) + O(\epsilon^2)$, ...(b)

$q(f) = q(x) + \epsilon X(x)q'(x) + O(\epsilon^2)$, ...(c)

into (1.12) we obtain on equating terms of order ϵ,

$$Y' + X'p + Xp' = 0\ . \qquad (1.14)$$

On substituting the expansions (a),(b) and(c) into (1.13) we obtain

$\Longrightarrow$ $q(x) =$

$$[q(x) + \epsilon X(x)q'(x) + O(\epsilon^2)][1 + \epsilon X'(x)]\left(1 + \epsilon Y(x) + O(\epsilon^2)\right)^{n-1}$$

$\Longrightarrow$ $q(x)[1 + \epsilon Y(x) + O(\epsilon^2)] = [q(x) + \epsilon X(x)q'(x) + O(\epsilon^2) + \epsilon X'(x)q(x) + \epsilon^2 X'(x)X(x)q'(x) + \epsilon X'(x)O(\epsilon^2)][1 + (\epsilon Y(x) + O(\epsilon^2))]^n$

Since $(1+X)^n = 1 + nX + \frac{n(n-1)}{2!}X^2 + \frac{n(n-1)(n-2)}{3!}X^3 + \cdots$

$\Longrightarrow$ $q(x) + \epsilon q(x)Y(x) + q(x)O(\epsilon^2) =$

$$[q(x) + \epsilon X(x)q'(x) + O(\epsilon^2) + \epsilon X'(x)q(x) + \epsilon^2 X'(x)X(x)q'(x) + \epsilon X'(x)O(\epsilon^2)]\{1 + n[\epsilon Y(x) + O(\epsilon^2)] + \frac{n(n-1)}{2!}[\epsilon Y(x) + O(\epsilon^2)]^2 + \cdots\}$$

$\Longrightarrow$ $q(x) + \epsilon q(x)Y(x) + q(x)O(\epsilon^2) = \{q(x) + \epsilon X(x)q'(x) + O(\epsilon^2) + \epsilon X'(x)q(x) + \epsilon^2 X'(x)X(x)q'(x) + \epsilon X'(x)O(\epsilon^2) + \epsilon nY(x)q(x) + \epsilon^2 nY(x)X(x)q'(x) +$

i.e. a separable equation and so has a solution of the form

$t = \eta(u) + c,$

for certain functionsφ and η . For higher order DEs, the differential equation can be treated as " t $absent$ "and so the substitution $p = \frac{dt}{du}$

reduces the order by one.

In order to find canonical coordinates we note that when we change variables from (x, y) to (u, t), the differential operator (1.29) is transformed to

$$\Gamma = \Gamma(t)\frac{\partial}{\partial t} + \Gamma(u)\frac{\partial}{\partial u},$$

where $\Gamma(t)$ and $\Gamma(u)$ denote the effect of Γ on the functions $t(x, y)$ and $u(x, y)$.

As we require the operator to become $\Gamma = \frac{\partial}{\partial t}$,

the equations $\Gamma(t) = 1,\ \Gamma(u) = 0$ must be satisfied.

Solving these equations we can find the canonical coordinates.

That is, we solve

$X\frac{\partial t}{\partial x} + Y\frac{\partial t}{\partial y} = 1, X\frac{\partial u}{\partial x} + Y\frac{\partial u}{\partial y} = 0.$

1.5 The case study

Consider the first-order ODE

$$\frac{dy}{dx} = f(x)y^n + g(x)y^m \qquad (1.31)$$

for real constants m, n .

This equation includes the Riccati equation.

$y' = a(x)y^2 + b(x)y + c(x),$

(when $n = 2,\ m = 1, c(x) = 0$),

the Bernoulli equation

$$y' + \varphi(x)y = \psi(x)y^k, \text{(when } n = 1\text{)}$$

and Abel's equation of the first kind (when n, m are integers) .

We want to solve (1.31) using symmetries when

$$f(x) \neq 0 \quad and \quad g(x) \neq 0; \quad n \neq m \cdot$$

To find the infinitesimals for the equation $\frac{dy}{dx} = F(x, y)$ where

$F(x, y) = f(x)y^n + g(x)y^m$ (1.32)

we use the criteria (1.21) with $X = X(x),\ Y = Y(y)$and find

$$y^n[Xf' + X'f] + [X'g + Xg']y^m + y^{n-1}[nYf] + y^{m-1}[mYg] - y^n[Y'f] - y^m[Y'g] = 0 \quad (1.33)$$

We need to consider the cases where

1.$y^n, y^m, y^{n-1}Y, y^{m-1}Y, y^n\acute{Y}$ are linearly independent.

2.$y^n, y^m, y^{n-1}Y, y^{m-1}Y, y^n\acute{Y}$ may be linearly dependent.

In the case where $y^n, y^m, y^{n-1}Y, y^{m-1}Y, y^n\acute{Y}$ are linearly independent, each of the coefficients in (1.33) must be identically zero. This then leads to

and$\Gamma v = 1$,

$$\Rightarrow \left(\frac{\alpha(1-n)}{f} \int f dx\right) v_x + \alpha y v_y = 1 \cdot$$

By solving these equations, we obtain

$$u = y^{n-1}\left(\int f\, dx\right), \qquad \text{and} \qquad v = \frac{1}{\alpha}\ln y.$$

and our ODE then becomes under the new variables

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \alpha\left[(n-1)u + \frac{1}{1 + A\left(\frac{1-m}{1-n}\right)u^{\frac{n-m}{1-n}}}\right]$$

which is separable.

As an example, let $f(x) = x^2$ and $n = 2,\ m = 3$

Since $g = A\left(\frac{1-m}{1-n}\right)\left(\int f\, dx\right)^{\frac{n-m}{1-n}} f(x)$

then, $g = \bar{\bar{\alpha}}x^5$ *where* $\bar{\bar{\alpha}}$ *is constant.*

Hence, the differential equation we are solving is

$$\acute{y} = x^2y^2 + \bar{\bar{\alpha}}x^5\, y^3$$

Canonical coordinates then are

$u = y^{n-1}\left(\int f\, dx\right) = \frac{x^3 y}{3}$ and $v = \ln y$

(using $\bar{\bar{\alpha}} = 1$).

Substituting the new variables into the DE, we get the separable DE

$$\frac{du}{dv} = \frac{u(1 + \bar{\bar{\alpha}}u) + 1}{1 + \bar{\bar{\alpha}}u}\cdot$$

$$f = g = 0 \quad or \quad X = Y = 0 \cdot$$

In the situation where $y^n, y^m, y^{n-1}Y, y^{m-1}Y, y^n\acute{Y}$ may be linearly dependent, we find the following non-trivial cases:

<u>Case (1):</u>$Y = \alpha y$ with $m, n, f, g \neq 0$ and $m \neq n$

To find the infinitesimal X for equation (1.31) from equation (1.33) we obtain

where α is an arbitrary constant.

Hence,

$$g = A\left(\frac{1-m}{1-n}\right)\left(\int f\,dx\right)^{\frac{n-m}{1-n}} f(x)$$

This corresponds to the ODE (1.31)

$$\frac{dy}{dx} = f(x)y^n + \left[A\left(\frac{1-m}{1-n}\right)\left(\int f\,dx\right)^{\frac{n-m}{1-n}} f(x)\right]y^m$$

and the *infinitesimal generator* (1.29), is

$$\Gamma = \left(\frac{\alpha(1-n)}{f}\int f dx\right)\frac{\partial}{\partial x} + (\alpha y)\frac{\partial}{\partial y}$$

Now, to find the corresponding invariant solutions we first find the canonical coordinates:

Solving, $\Gamma(u) = 0$

$\Rightarrow\left(\frac{\alpha(1-n)}{f}\int f dx\right)u_x + \alpha y u_y = 0$

Solving this DE we get, for $\bar{\bar{\alpha}} > \frac{1}{4}$

$$\frac{1}{2}\ln|\bar{\bar{\alpha}}u^2 + u + 1| + \frac{1}{\sqrt{4\bar{\bar{\alpha}} - 1}}\tan^{-1}\left(\frac{2\bar{\bar{\alpha}}u + 1}{\sqrt{4\bar{\bar{\alpha}} - 1}}\right) = v + k$$

where k is an arbitrary constant. Then, we substitute in u, v to get the solution in terms of the original variables x, y.

<u>Case (2):</u>$Y = 0$

In this case $X = \frac{\alpha}{f(x)}$ and requires $g(x) = \gamma f(x)$,

so the DE is $\frac{dy}{dx} = f(x)(y^n + \gamma y^m) \cdot$

The canonical coordinates are $u = y,\ v = \int f(x)dx$

and under these coordinates, the DE reduces to

$\frac{du}{dv} = u^n + \gamma u^m.$

<u>Case (3) :</u>$Y = \gamma y^n$ with $m = 1$

In this case $X = \frac{\gamma(n-1)}{f(x)}\int g(x)dx$

and require

$f(x) = \frac{\gamma(n-1)}{\epsilon} g(x)\int g(x)dx.$

so the governing DE is

$$\frac{dy}{dx} = \left(\frac{\gamma(n-1)}{\epsilon}g(x)\int g(x)dx\right)y^n + g(x)y.$$

The canonical coordinates are found to be

$$u = [\int g(x)dx] - \frac{\epsilon}{\gamma(1-n)} y^{1-n}, \qquad v = \frac{y^{1-n}}{\gamma(1-n)} + \beta$$

and under these coordinates the DE reduces to

$$\frac{du}{dv} = \frac{\epsilon}{(n-1)u} - \epsilon.$$

As an example, with $n = 2$, $g = x^2$ $and\ \gamma = \epsilon = 1$, our DE i

$$y' = \left(x^2 \int x^2\right) y^2 + x^2 y$$

i.e.
$$y' = \frac{x^5}{3} y^2 + x^2 y$$

[**Note:** This equation could be solved as a Bernoulli equation

$$\frac{dy}{dx} - x^2 y = \frac{x^5}{3} y^2 \quad \text{with } v = y^{1-2} = y^{-1} \cdot]$$

Using the *canonical coordinatesu* $= \frac{x^3}{3} + y^{-1}$ and $v = -y^{-1}$

and substituting into the DE, we get $\frac{du}{dv} = \frac{1}{u} - 1$, a separable DE.

Solving this separable DE we get$u + \ln(1-u) = -v + k$; (where k is an arbitrary constant). So that in terms of the original variables:

$$\left(\frac{x^3}{3} + \frac{1}{y}\right) + \ln\left(1 - (\frac{x^3}{3} + \frac{1}{y})\right) = \frac{1}{y} + k \cdot$$

1.6 CONCLUSION

Symmetries are most useful in the study of differential equations and they are useful in different ways. Firstly, the use of symmetry transformation can be used to solve ordinary differential equations by leading to a variable-absent equation and thus simplifying the problem. This is achieved by the use of canonical coordinates.. Secondly, a symmetry of a differential equation transforms solutions into other solutions and thus symmetries can be used to generate new solutions from old ones.

In this paper, we investigated the application of Lie symmetries to the equation

$$\frac{dy}{dx} = f(x)y^n + g(x)y^m$$

where $f, g \neq 0$ and $n \neq m$·

We found special cases for the coefficient functions that allowed either a full general solution of the ODE or at least a transformation that lead to a separable equation.

References

[1] Bluman, G. W. and Kumei, S., *Symmetries and Differential Equations*, Springer-Verlag, 1989.

[2] Edwards, M.P., Lectures in:*One Parameter Group-MATH902*, School of Mathematics and Applied Statistics, University of Wollongong, 2010.

[3] Fushchych, W.I., Shtelen, V.M., and Serov, NI. , *Symmetry analysis and Exact Solutions of Equations of Nonlinear Mathematical Physics*,(Dordrecht: Kluwer), 1993.

[4] Goard, J.M.,"*New Solutions to the bond-Pricing equation via Lie's Classical Method",Journal of Mathematical and Computer Modelling,* 2000 vol.32, pp.299-313.

[5] Hill, J.M.,*Differential Equations and Group Methods for Scientists and Engineers*,CRC Press, Inc,1992.

[6] Hill, J.M.,Avagliano,A., Edwards, M.P., "*Some Exact Results for Nonlinear Diffusion with absorption" .IMA Journal of Applied Mathematics*, 1992,vol 48, pp.283-304.

[7] Lie, S., "*Über die Integration durch bestimmte Intergart van einer Klasse linearer partieller Differentialgleichungen",Arch Math*, 1881, vol6, pp. 328-368.