Some Generalization of Coquasi-Dedekind Modules

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Abstract

Let R be a commutative ring with unity and M is a unitary R-module. In this paper we introduce the concepts of weakly Coquasi-Dedekind modules and p –Coquasi Dedekind module as a generalization of the concepts of Coquasi- Dedekind module, and gives some of their basic properties, characterizations and examples. Another hand we study the relationships of these concepts with some classes of modules.

Keywords: Coquasi-Dedekind modules, p –Coquasi Dedekind module, unitary R-module

بعض تعميمات المقاسات الشبه الديدكانية المضادة

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الخلاصة

لتكن R حلقة ابدالية بمحايد ولتكن M مقاسا احاديا على R . في هذا البحث قدمنا مفهومين للمقاسات شبه ديدكننية المضادة وهما المقاسات شبه الديدكنية المضادة الضعيفة والمقاسات شبه الديدكانية المضادة من النمط P - كأعمام للمقاسات شبه الديدكانية المضادة وأعطينا بعض الخواص الاساسية والتشخيصات والامثلة. من جهة أخرى درسنا علاقة هذين المفهومين مع بعض اصناف المقاسات الاخرى .

Introduction

Let R be a commutative ring with identity. We call a proper submodule N of M Coquasi-invertible if $\mathrm{Hom}(M, N) = 0$, and the R-module M is called Coquasi-Dedekind if every proper submodule of M is Coquasi-invertible [6]. And an R-module M is called coprime, if for every $r \in R$ either $rM = M$ or $rM = 0$ [4]. A proper submodule N of M is called quasi essential submodule in M if $N \cap Q \neq 0$ for each non-zero quasi-prime submodule Q of M [2], where a proper submodule Q of an R-module M is called quasi-prime if , $r_1 r_2 m \in Q, m \in M, r_1, r_2 \in R$, then either $r_1 m \in Q$ or $r_2 m \in Q$ [1].

In the first section of this paper, we introduce weakly Coquasi- invertible submodule and study some basic properties of this concept. In the second section we introduce the concept, weakly Coquasi-Dedekind modules. In the third section, we introduce the concept of weakly coprime module and study a relation between it and weakly Coquasi-Dedekind module. In section four we introduce the concept of P-coquasi-invertible submodule and P-coquassi-Dedekind module and give some of their properties

§1: Weakly Coquasi-invertible submodules.

In this section we introduce weakly Coquasi-invertible submodule as a generalization of Coquasi-invertible submodule and obtain some of its basic properties.

Definition 1.1

Let N be a proper submodule of an R-module M. We say that N is weakly Coquasi-invertible submodule of M if, every $f \in \mathrm{Hom}(M, N)$, kerf is quasi-essential in M.

Example and Remarks 1.2

1- Every Coquasi-invertible submodule is weakly Coquasi-invertible submodule, but the converse is not true in general as the following example shows: In the Z-module Z_4 the submodule $\langle\bar{2}\rangle$ is a quasi-essential in Z_4, then for any homomorphism $f: Z_4 \to \langle\bar{2}\rangle$ kerf is either Z_4 or $\langle\bar{2}\rangle$. Thus $\langle\bar{2}\rangle$ is weakly Coquasi-invertible submodule, but $\langle\bar{2}\rangle$ is not Coquasi-invertible submodule because if $f: Z_4 \to \langle\bar{2}\rangle$ defined by $f(\bar{1}) = \bar{2}$ is non- zero homomorphism.

2- The zero submodule is weakly Coquasi-invertible submodule of any non-zero module.

Proposition 1.3

If K is weakly Coquasi-invertible submodule of an R-module M, then $\mathrm{ann}_M(s)$is quasi-essential in M for all $s \in [K: M]$.

Proof:

Let $s \in [K: M]$, then $sm \in K$ for all $m \in M$. Therefore, we can define a homomorphism $f: M \to K$ by $f(m) = sm$. But K is weakly Coquasi-invertible submodule, then $\ker f = \{m \in M: f(m) = 0\}$ is quasi essential in M. Thus $\{m \in M: sm = 0\}= \mathrm{ann}_M(s)$ is quasi-essential in M.

Proposition 1.4

Every non-zero quasi-prime weakly Coquasi-invertible submodule of an R-module M is not a direct summand of M.

Proof:

Let K be non-zero quasi-prime weakly Coquasi-invertible submodule of M and suppose that K is a direct summand of M. then there exists a proper submodule V of M such that $M = K \oplus V$. Hence the projection homomorphism from M onto K has kernel equal to V which is a quasi- essential in M which is contradiction.

In the following proposition, we give a characterization of weakly Coquasi-invertible submodule.

Proposition 1.5

Let K be a proper submodule of an R-module M and $\pi: M \to \frac{M}{K}$ be a natural epimorphism, then K is weakly Coquasi-invertible submodule if and only if $g \in \mathrm{End}(M)$ such that $\pi \circ g = \pi$, then the set of all fixed point of g is quasi-essential in M.

Proof:

Let $g: M \to M$ be such that $\pi \circ g = \pi$, then $\pi \circ g(m) = \pi(m)$. Hence $\pi(g(m) - m) = 0$ implies that $g(m) - m \in Ker\pi = K$. That is $g(m) - m \in K$. Thus $g - I$ is a homomorphism from M to K, where $I: M \to M$is the identity homomorphism. But K is weakly Coquasi-invertible submodule, then $ker(g - I)$ is quasi-essential in M. That is $ker(g - I) = \{m \in M: (g - I)(m) = 0\} = \{m \in M: g(m) = m\}$
is a quasi-essential in M.

Conversely, let $f \in Hom\,(M, K)$ and let $i: K \to M$ be the inclusion homomorphism. Put $g = (I - (i \circ f): M \to M$, where I is the identity homomorphism on M. Now, for each $m \in M$, we have $\pi \circ g(m) = \pi(I - (i \circ f)(m) = \pi\big(m - f(m)\big) = m - f(m) + K = m + K = f(m)$, thus by our assumption, the set of fixed points of g is quasi-essential in M. That is the set $\{m \in M: (I - (i \circ f))(m) = m\} = \{m \in M: m - f(m) = m\} = \{m \in M: f(m) = 0] = kerf$ is a quasi-essential in M. Thus, K is weaklyCoquasi-invertible submodule.

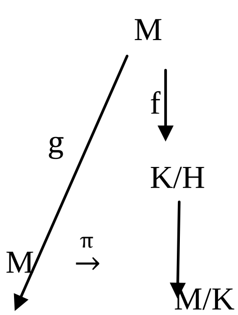
Recall that an R-module M is called quasi-projective if, every submodule N of M, any homomorphism $f: M \to M/N$ can be lifted to a homomorphism $g: M \to M$ [5].

Proposition 1.6

Let K be a proper submodule of quasi-projection module M, K is weakly Coquasi-invertible submodule if and only if for every submodule H of M such that $H \subseteq K$ and for every $f: M \to K/H$, kerf is quasi-essential in M.

Proof:

Suppose that K is weakly Coquasi-invertible submodule of M and let H be any submodule of M such that $\subseteq K$, and $f: M \to K/H$ be any homomorphism. Consider the following diagram where $i: K/H \to M/K$ is the inclusion homomorphism and π is the natural epimorphism.



Since M is quasi-projective, then there exists a homomorphism $g: M \to M$ such that $\pi \circ g = i \circ g$. For each m in M we have $\pi \circ g(m) = i \circ g(m)$ implise that $\pi(g(m)) = f(m)$, $g(m) + H = x + H$ where $x \in K$, that is $g(m) - x \in H$. But $H \subseteq K$ and $x \in K$ thus $g(m) \in K$. Therefore $g: M \to K$. Since N is weakly Coquasi-invertible submodule, then $ker\,g$ is quasi-essential in M. let $m \in ker\,g$, then $g(m) = 0$. But $\pi \circ g = i \circ g$, then $\pi(g(m)) = f(m)$, then $g(m) = f(m)$ implies that $f(m) = 0$. That is $ker\,g \subseteq kerf$. Hence by [2] kerf is quasi-essential in M.

The converse is trivial.

§2: Weakly Coquasi-Dedekind module.

In this section we introduce the concept of weakly Coquasi-Dedekind module as a generalization of Coquasi-Dedekind module and give some basic properties examples and characterization of this concept.

Definition 2.1

An R-module M is called weakly Coquasi-Dedekind module if, every proper submodule of M is weakly Coquasi-invertible submodule of M.

Every Coquasi-Dedekind module is weakly Coquasi-Dedekind module, but the converse is not true in general as the following example says;

The Z-module Z_4 is weakly Coquasi-Dedekind module, but not Coquasi-Dedekind module.

The following proposition gives a characterization of weakly Coquasi-Dedekind module.

Proposition 2.2

Let M be an R-module. Then M is weakly Coquasi-Dedekind module if and only if for every $f \in \mathrm{End}(M)$, either f is an epimorphism or $\ker f$ is quasi-essential in M.

Proof:

Suppose that M is weakly Coquasi-Dedekind module, and let $f \in \mathrm{End}(M)$. Suppose that f is not epimorphism. That is, $f(M)$ is submodule of M. But M is weakly Coquasi-Dedekind, then f(M) is weakly Coquasi-invertible submodule which implies that kerf is quasi-essential in M.

Conversely, Suppose that $N \subseteq M$ and $g \in \mathrm{Hom}\,(M, N)$. Put $f = i \circ g \in \mathrm{End}(M)$, where $i: N \to M$ is the inclusion homomorphism. It is clear that f is not epimorphism. Hence by hypothesis $\ker f$ is a quasi-essential in M. But $\ker f = \ker(i \circ g) = \ker g$. is a quasi-essential in M.

Proposition 2.3

Let M be an R-module over an integral domain R. If M is weakly Coquasi-Dedekind torsion free module, then M is Coquasi-Dedekind module.

Proof:

Suppose that M is weakly Coquasi-Dedekind torsion free module over integral domain R, and $f \in \mathrm{End}(M)$, then by prop.2.2 f is either an epimorphism or $\ker f$is a quasi-essential in M, then by [2, Prop.1.10] there exists $x \in Q$ for each quasi-prime submodule Q of M an there exists $0 \neq r \in R$ such that $0 \neq rx \in \ker f$. That is $f(rx) = rf(x) = 0$. But M is torsion free module, therefore f(x)=0. That is $\ker f = M$ and f=0. This implies that for each $f \in \mathrm{End}(M)$, f is epimorphism or f=0. That is by [6 ,Th.2.1.4] M is Coquasi-Dedekind

Recall that a non zero module M is uniform, if every non-zero submodule of M is essential in M [4].

Proposition 2.4

Every finite uniform module is weakly-coquasi-Dedekind module.

Proof:

Let K be a proper submodule of finite uniform R-module M, and let $f \in \mathrm{Hom}(M, K)$, we have kerf is a submodule of M. Since M is finite and K is proper submodule of M , then $\ker f \neq 0$. But M is uniform, then kerf is essential in M, hence kerf is quasi-essential in M by [2]. Then M is weakly-Coquasi Dedekind module.

Since every Coquasi Dedekind module is weakly-Coquasi Dedekind module, we have the following result.

Proposition 2.5

Every almost finitely generated R-module is weakly-Coquasi Dedekind module.

Proof:

Let M be almost finitely generated R-module. Hence by [5, prop.2.3] M is Coquasi Dedekind module. Hence M is weakly-Coquasi Dedekind module.

Proposition 2.6

Every anti-hopfain R-module is weakly-Coquasi Dedekind R-module.

Proof:

Let M be anti-hopfain R-module, then by [6, Prop.2.3.3] M is Coquasi-Dedekind module. Hence M is weakly-Coquasi Dedekind module.

§3: Weakly coprime Modules

In this section we introduce the definition of weakly coprime module as a generalization of coprime module and study its relation with weakly Coquasi-Dedekind module.

Definition 3.1

A non zero R-module M is called weakly coprime module, if for every $r \in R$ either $rM = M$ or $ann_M(r)$ is quasi-essential in M.

Example and Remark 3.2

1- The Z-module Z_4 is weakly coprime module because for each n in Z ether $nZ_4 = Z_4$ or $ann_M(n) = \{\bar{0}, \bar{2}\}$which is quasi-essential in Z_4.

2- If M is coprime module, then M is weakly coprime module.

Proof: since M is Coprime module, then for every r in R ether $rM = M$ or $rM = 0$, then $ann_M(r) = M$ is quasi-essential.

Proposition 3.3

If M is weakly Coquasi-Dedekind R-module, then M is weakly coprime module.

Proof:

Suppose that M is weakly Coquasi-Dedekind module and let r in R, define $f: M \to M$ by $f(m) = rm$, since M is weakly Coquasi Dedekind module, then f is either epimorphism or kerf is quasi-essential in M. If f is an epimorphism, then rM=M. if kerf is quasi essential in M, then $kerf = \{m \in M: f(m) = 0\} = \{m \in M: rM = 0\}$is quasi-essential in M. that is $ann_M(r)$ is a quasi-essential in M.

§4: P-Coquasi-Dedekind modules

In this section we introduce the concept of P-Coquasi-invertible submodule and P-Coquasi Dedekind module and give some of their properties.

Before we introduce the definition of P-Coquasi-invertible submodule, we recall the following definition.

A proper submodule N of an R-module M is called P-small in M, if $N + P \neq M$ for any prime submodule P of M [3].

Definition 4.1

A proper submodule N of an R-module M is said to be P-Coquasi-invertible submodule of M, if for every $f \in Hom(M, N)$, Imf is P-small in M.

Example and Remark 4.2

1. Every Coquasi-invertible submodule is P-Coquasi-invertible submodule.

2. In the Z-module Z_4 the submodule $\{\bar{0},\bar{2}\}$ is P-Coquasi-invertible submodule in Z_4, since for every homomorphism $f \in$ Hom $(Z_4,\{\bar{0},\bar{2}\})$, Imf is $\{\bar{0},\bar{2}\}$ is P-small in Z_4.

The following proposition is a characterization of P-Coquasi-invertible submodule.

Proposition 4.3

Let M be a quasi-projective R-module, and N be a submodule of M, then the following are equivalent:

1. N is P-Coquasi-invertible submodule of M.
2. For every submodule K of M such that $K \subseteq N$ and for every $\phi: M \to \frac{N}{K}$,Im$\emptyset$ is P-small in M/K

Proof:

$(1) \to (2)$

Suppose that N is P-Coquasi-invertible submodule and let K be any submodule of M such that $K \subseteq N$, and let $\phi: M \to M/K$be any homomorphism and let $i: N/K \to M/K$ be the inclusion homomorphism. Then $i \circ \phi: M \to M/K$. But $\pi: M \to M/K$ is an epimorphism, and since M is quasi-projective, then there exists $\alpha: M \to M$ such that $\pi \circ \alpha = i \circ \phi$. Let m in M, then $\phi(m) = x + K$, where x in N.

Now, $\pi \circ \alpha(m) = i \circ \phi(m)$. That is $\pi(\alpha(m)) = \phi(m)$ i.e. $\alpha(m) + K = x + K$, then $\alpha(m) - x \in K \subseteq N$. Hence $\alpha(m) \in N$. Therefore $\alpha: M \to N$ is homomorphism. But N is P-Coquasi-invertible submodule, then $\mathrm{Im}(\alpha) = \alpha(M)$ is P-small in M and therefore by [3] Im $(\pi \circ \alpha)$ is P-small in M/K. this implies that Im $(i \circ \phi) = \mathrm{Im}(\phi)$ is P-small in M/K.

$(2) \to (1)$ trivial.

Definition 4.4

A non zero R-module M is P-Coquasi-Dedekind module if every proper submodule of M is P-Coquasi-invertible submodule.

Every Coquasi-Dedekind module is P-Coquasi-Dedekind module, but the converse is not true as the following example: Z_4 as Z-module is P-Coquasi-Dedekind module, but not Coquasi-Dedekind module.

The following proposition is a characterization of P-Coquasi-Dedekind module.

Proposition 4.5

An R-module M is P-Coquasi-Dedekind module if and only if for every $f \in \mathrm{End}(M)$, either f is an epimorphism or Imf is P-small in M.

Proof:

Let M be P-Coquasi-Dedekind module, and let $f \in \mathrm{End}(M)$, if f is not epimorphism, then Imf $\neq$ M, let $g: M \to f(M)$ be defined by $g(m) = f(m)$ for all m in M. Since M is P-Coquasi-Dedekind module, then Img is P-Coquasi-invertible submodule, implies that, Img is P-small in M, that is Imf is P-small in M.

Conversely, Let N be a proper submodule of M, and let $f \in$ Hom (M, N) and $i: N \to M$ be the inclusion homomorphism. $\mathrm{Im}(i \circ f) = $ Imf. But f is not an epimorphism, therefore Imf is P-small in M, that is N is P-Coquasi-Dedekind module.

Now, we introduce the following definition.

Definition 4.6

Let M be an R-module, then M is called P-coprime, if every $r \in R$ either $rM =$ M or rM is P-small in M.

Proposition 4.7

If M is a P-Coquasi-Dedekind module, then M is P-coprime.

Proof:

let $r \in R$, if $rM \neq M$ then rM is a proper submodule of M. Define $f: M \rightarrow M$ by f(m)=rm for each m in M. Thus $Imf = rM$ is P-small submodule of M. therefore M P-coprime module.

Proposition 4.8

Any Homomorphic image of P-coprime module is P-coprime.

Proof:

Let M and M' be two R-modules, with M is P-coprime module. Suppose that $f: M \rightarrow M'$ is an epimorphism. Let $r \in R$, then $rM' = r(f(M) = f(rM)$. But M is P-coprime, thus either $rM = M$ or rM is P-small. If $rM = M$, then $rM' = M'$. If rM is P-small in M, then f(rM) is P-small in M' by [3 Prop. 1.3]. Thus M' is P-coprime module.

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