Extensibility of Eulerian Graphs and Digraphs

Akram B. Attar & Ahmed. J. Elawi

Faculty of Education for Pure Sciences, Department of Mathematics

Thi-Qar University, Iraq

akramattar70@yahoo.com   and   ahm999946@yahoo.com

Abstract

In this paper the concepts extension of graphs and the extensible class of graphs have been introduced. The extensibility number of graphs has been defined. Furthermore the regular graphs (digraphs) which have extensibility number $1, 2$ or 3 have been characterized.

Keywords: Extension of graphs(digraphs), Eulerian graphs(digraphs), Regular graphs(digraphs).

الملخص

في هذا البحث تم تقديم مفهوم توسيع البيانات والبيانات القابله للتوسيع وكذلك تم تعريف عدد التوسيع وتم تشخيص البيانات المنتظمه التي لها عدد توسيع 1 و2 او 3

1   Introduction

Kharat and Whaphare [2001] introduced the concept of reducibility number for posts in lattices theory. Attar [2005] introduced analogous concept in graph theory. In fact he studied the reducibility of graphs (digraphs) and he characterized the reducibility

72

number for some classes of graphs(digraphs). Attar [2007] introduced the concept of contractibility number of graphs. Further, Attar [2009] introduced the concept of extension graphs (digraphs), and he characterized the extensibility number for some graphs (digraphs). In this work, a new definition for extension graphs has been introduced. Furthermore, the extensible class of graphs and the extensibility number of regular graphs(digraphs) have been characterized.

If e is an edges of a graph G having end vertices v, w then e is said to join the vertices v and w and these vertices are said to be adjacent. In this case we also say that e is incident to v and w, and that w is a neighbor of v. An independent set of vertices in G is a set of vertices of G no two of which are adjacent. Let v be a vertex of the graph G, if v joined to itself by an edge, such an edge is called loop. The degree of v denoted by $d(v)$ is the number of edges of G incident with v, counting each loop twice. If two (or more) edges of G have the same end vertices then these edges are called parallel. A graph is called simple if it has no loops and parallel edges. We say that G is r-regular graph if the degree of every vertex is r. A simple graph in which every two vertices are adjacent is called a complete graph, the complete graph with n vertices is denoted by K_n.

A graph G is connected if there is a path joining each pair of vertices of G, a graph which is not connected is called disconnected. A connected graph which contains no cycle is called a tree. A graph G is Hamiltonian if it has a cycle which includes every vertex of G

A digraph D is said to be weakly connected (or connected) if its underlying graph is connected. A digraph D is called simple if, for any pair of vertices u and v of D, there is at most one arc from u to v and there is no arc from itself.

Let v be a vertex in the digraph D. The indegree $id(v)$ of v is the number of arcs of D that have v as its head, i. e., the number of arcs that go to v. Similarly the outdegree $od(v)$ of v is the number of arcs of D that have v as its tail, i. e., that go out of v. A digraph D is called k-regular if $id(v) = od(v) = k$ for each vertex v of D.

For the undefined concepts and terminology we refer the reader to [4, 5, 6].

All the graphs (digraph) through out this paper are simples.

2   Extension of Graphs

In this section, we introduced the concepts extension of graph, extensible class of graphs and the extensibility number of graphs.

Definition 1 [4]

Let G_1 and G_2 be two graphs with no vertex in common. We define the join of G_1 and G_2 denoted by $G_1 + G_2$ to be the graph with vertex set and edges set given as follows:

$V(G_1 + G_2) = V(G_1) \cup V(G_2)$,

$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup J$

where $J = \{x_1 x_2 : x_1 \in V(G_1), x_2 \in V(G_2)\}$.

Thus J consists of edges which join every vertex of G_1 to every vertex of G_2.

Attar[1] defined the extension of graphs as follows:

Definition 2 [1]

Let G be a nontrivial graph. The extension of G is a simple graph denoted by $G + S$ obtained from G by adding a nonempty set of independent vertices S such that every vertex in S is adjacent to every vertex in G .

In this work we define the extension of graphs as follows:

Definition 3

Let G be a nontrivial graph. The extension of G is a simple graph denoted by $G * S$ obtained from G by adding a nonempty set S of independent vertices different from the vertices of G to the graph G such that every vertex in S is adjacent to at least one vertex in G. In such a way S is called extension set of G. In particular if S consists of a single element v, then v is called extension vertex of G.

Here, we define the extensible class of graphs.

Definition 4

Let $\Im$ be the class of graphs with certain property. Then $\Im$ is called extensible class of graphs, if for every graph $G \in \Im$, there exists an extension vertex v different from the vertices of G such that $G * v \in \Im$.

Now, we introduce the following two propositions.

proposition 1

The class of connected graphs is extensible class of graphs.

proof

It follows from Definition 3, the extension vertex preserve the connectedness of graph.

proposition 2

1. The class of Hamiltonion graphs is not extensible class.

2. The class of trees is not extensible class.

3. The class of complete graphs is not extensible class.

4. The class of bipartite graphs is not extensible class.

5. The class of Eulerian graphs is not extensible class.

6. The class of regular graphs is not extensible class.

proof

1. Let G be a Hamiltonian graph with n vertices. It is clear that the extension vertex v which is adjacent to exactly one vertex in G gives $G * v$ is not Hamiltonian.

2. Let T be a tree. If the extension vertex v is adjacent to more than one vertex in T, then $T * v$ contains a cycle.

3. Let K_n be a complete graph. If the extension vertex v is adjacent to less than n vertices of K_n, then $K_n * v$ is not complete.

4. Let G be X, Y- bipartite graph, and v be extension vertex of G such that v is adjacent to a vertices in X and a vertices in Y. It is clear that the resulting graph $G * v$ is not bipartite.

5. Let $\mathcal{E}$ be an Eulerian graph. Then $\mathcal{E}$ is connected and every vertex of $\mathcal{E}$ has an even degree. Suppose that v_0 is an extension vertex of $\mathcal{E}$. If v_0 is adjacent to odd number of vertices in $\mathcal{E}$. Then $\mathcal{E} * v_0$ is not Eulerian.

6. Let R be a regular graph with n vertices, and v_0 be an extension vertex of R. If v_0 is adjacent to h vertices in R such that $h<n$. Then $R*v_0$ is clearly not regular.

Now, the question is that, what is the smallest extension set of vertices which make the non extensible graph is extensible. In order to answer this question we introduced the following definition:

Definition 5

Let $\Im$ be the class of graphs with certain property, and $G \in \Im$ be a nontrivial. The extensibility number of G with respect to $\Im$ is the smallest positive integer m, if exists such that there exists an extension set S of G with cardinality m in which the new graph $G * S \in \Im$. We write $m = ext_\Im(G)$. If such a number dose not exist for G then we say that the corresponding extensibility number is ∞.

One can see immediate, the class of graphs $\Im$ is extensible class if and only if the extensibility number of every graph $G \in \Im$ is one.

3   Extension of Regular Graphs

In this section, we characterized the regular graphs which have extensibility number equal to $1, 2$ or 3.

Theorem 1

Let $\mathcal{R}$ be the class of regular graphs, $R \in \mathcal{R}$.Then the $ext_{\mathcal{R}}(R) = 1$ if and only if R is a trivial or complete graph and there exists a vertex v_0 different from the vertices of R such that v_0 is adjacent to every vertex in R exactly once .

proof:

Let R be an r-regular graph with n vertices. Suppose that $ext_{\mathcal{R}}(R) = 1$. Then by Definition 5, there exists an extension set of vertices with single element v_0, such that $R * v_0 \in \mathcal{R}$. By Definition 3, we must have v_0 is adjacent to every vertex in R exactly once. Then $d(v_0) = n$ and the degree of every vertex of R in the graph $R * v_0$ is $r + 1$, but $R * v_0$ is regular. Then we must have $n = r + 1$. As $R * v_0$ is regular, then either $r = 0$, then $n = 1$ and R is a trivial graph, or $r = n - 1$ and R is complete graph.

Conversely, Let v_0 be a vertex different from the vertices of R such that v_0 is adjacent to every vertex in R exactly once, and R is trivial or complete. If R is trivial graph then it is not difficult to see that $ext_{\mathcal{R}}(R) = 1$.

Suppose that R is complete graph with n vertices. Then R is regular graph with regularity degree $n - 1$, we prove that $ext_{\mathcal{R}}(R) = 1$.

If v_0 is adjacent to every vertex in R exactly once. Then $d(v_0) = n$ and the degree of every vertex of R is $n - 1 + 1$. Then the degree of every vertex in $R * v_0$ is n. Thus the new graph $R * v_0$ is n-regular graph. As such v_0 is extension vertex of R with respect to $\mathcal{R}$. Hence $ext_{\mathcal{R}}(R) = 1$

Theorem 2

Let $\mathcal{R}$ be a class of regular graphs, $R \in \mathcal{R}$ be an r-regular simple graph with n

vertices. Then $ext_{\mathcal{R}}(R) = 2$ if and only if there exist two independent vertices u, v different from the vertices of R and one of the following holds:

(a) each of u and v is adjacent to every vertex in R exactly once, and $r = n - 2$.

(b) each of u and v is adjacent to $\frac{n}{2}$ vertices in R, n is even, $r = \frac{n}{2} - 1$ and $N(u) \cap N(v) = \phi$.

proof:

Let R be an r-regular simple graph with n vertices. Suppose that $ext_{\mathcal{R}}(R) = 2$. Then by Definition 5, there exist two extension vertices u, v such that $R * \{u, v\} \in \mathcal{R}$ and u, v are independent vertices. As $R * \{u, v\}$ is regular then $d(u) = d(v)$ in $R * \{u, v\}$. Since $R * \{u, v\}$ is simple then we have the following cases:

(i) $R * \{u, v\}$ is $(r + 2)$-regular graph.

(ii) $R * \{u, v\}$ is $(r + 1)$-regular graph.

If $R * \{u, v\}$ is $(r + 2)$-regular then we must have each of u, v is adjacent to every vertex in R exactly once, in this case $d(u) = d(v) = n$. As $R * \{u, v\}$ is$(r + 2)$-regular, then we must have $n = r + 2$, that is $r = n - 2$ in R and (a) holds.

If $R * \{u, v\}$ is $(r + 1)$-regular graph, then we must have each of u and v is adjacent to $\frac{n}{2}$ vertices in R and $N(u) \cap N(v) = \phi$. In this case n must be even and $d(u) = d(v) = \frac{n}{2}$. Thus every vertex in R must have regularity degree $\frac{n}{2} - 1$ (as $R * \{u, v\}$ is regular) and (b) holds.

Conversely, Suppose that (a) holds. That is there are two vertices u and v such that each of them is adjacent to every vertex in R and $r = n - 2$. Then $d(u) = d(v) = n$ in $R*\{u, v\}$ and every vertex in R has degree $r+2$ in $R*\{u, v\}$, but $r = n - 2$ in R, then every vertex in R has a degree $n - 2 + 2$ in $R * \{u, v\}$. That is every vertex in $R * \{u, v\}$ has a degree n, i.e., $R * \{u, v\}$ is regular. As such u, v are extension vertices and $ext_{\mathcal{R}}(R) \leq 2$.

If $ext_{\mathcal{R}}(R) = 1$, by Theorem 1, there exist a vertex v_0 different from the vertices of R such that v_0 is adjacent to every vertex in R exactly once and R is either trivial or complete graph. If R is trivial, then R has regularity degree 0 and $0 = n - 2 \Longrightarrow n = 2$ a contradiction. If R is a complete graph then $r = n - 1$ a contradiction to our assumption that $r = n - 2$. Thus $ext_{\mathcal{R}}(R) \neq 1$. Hence $ext_{\mathcal{R}}(R) = 2$

If (b) holds, that is there are two vertices u and v such that each of them is adjacent to $\frac{n}{2}$ vertices in R, $N(u) \cap N(v) = \phi$, n is even and $r = \frac{n}{2} - 1$. Then $d(u) = d(v) = \frac{n}{2}$ in $R * \{u, v\}$ and every vertex in R has a degree $r + 1$ in $R * \{u, v\}$, but $r = \frac{n}{2} - 1$ in R then $r = \frac{n}{2} - 1 + 1 = \frac{n}{2}$ in $R * \{u, v\}$. Thus every vertex in $R * \{u, v\}$ has a degree $\frac{n}{2}$. Hence $R * \{u, v\}$ is regular and u, v are extension vertices. Hence $ext_{\mathcal{R}}(R) \leq 2$.

If $ext_{\mathcal{R}}(R) = 1$, then by Theorem 1, R is either trivial or complete graph. If R is trivial graph then R has regularity degree 0 and $0 = \frac{n}{2} - 1 \Longrightarrow n = 2$ a contradiction. If R is a complete graph, then $r = n - 1$ which implies $n - 1 = \frac{n}{2} - 1$ a contradiction. Hence $ext_{\mathcal{R}}(R) = 2$.

Theorem 3

Let $\mathcal{R}$ be a class of regular graphs, $R \in \mathcal{R}$ be an r-regular graph with n vertices. Then $ext_{\mathcal{R}}(R) = 3$ if and only if, there exist three independent vertices u, v and w different from the vertices of R and one of the following holds:

(1) each of u, v, w is adjacent to every vertex in R exactly once, $r = n - 3$, and $n \neq 4$.

(2) every vertex in R is adjacent to exactly two vertices from u, v, w and $d(u) = d(v) = d(w)$ in $R * \{u, v, w\}$, and the regularity degree of R is $r = \frac{2n}{3} - 2$ with $n \neq 6$ and n divisor of 3.

(3) each of u, v, w is adjacent to $\frac{n}{3}$ vertices in R and $N(u), N(v), N(w)$ are mutually

disjoint, $r = \frac{n}{3} - 1$ and n divisor of 3.

proof:

Let R be an r-regular graph with n vertices. Suppose that $ext_{\mathcal{R}}(R) = 3$. By Definition 5, there exist three vertices u, v and w such that $R * \{u, v, w\} \in \mathcal{R}$ and u, v and w are independent vertices.

As $R * \{u, v, w\}$ is simple regular, then $d(u) = d(v) = d(w)$ in $R * \{u, v, w\}$, and we have the following cases:

(i)$R * \{u, v, w\}$ is $(r + 3)$-regular graph.

(ii)$R * \{u, v, w\}$ is $(r + 2)$-regular graph.

(iii)$R * \{u, v, w\}$ is $(r + 1)$-regular graph.

If (i) holds, then we must have each of u, v, w is adjacent to every vertex in R exactly once. In this case $d(u) = d(v) = d(w) = n$. As $R * \{u, v, w\}$ is simple regular we must have $r = n - 3$ in R. If $n = 4$, and $r = n - 3 \Rightarrow r = 1$, in this case R is isomorphic to two independent edges. In such away we can add two independent vertices u_0, v_0 such that each of them is adjacent to two vertices from R and $N(u_0) \cap N(v_0) = \phi$. Then we get $R * \{u, v\}$ is regular and u_0, v_0 are extension vertices. Thus by Theorem 2 (b), $ext_{\mathcal{R}}(R) = 2$ a contradiction, and (1) holds.

If(ii) holds, that is the regularity degree of $R * \{u, v, w\}$ is $r + 2$. In this case we must have each vertex of R is adjacent to exactly two vertices from u, v, w. As $R * \{u, v, w\}$ is regular graph, then we have $d(u) = d(v) = d(w) = \frac{2n}{3}$ in $R * \{u, v, w\}$. In this case n must be divisor of 3 and the regularity degree of R is equal to $\frac{2n}{3} - 2$. If $n = 6$, and $r = \frac{2n}{3} - 2 \Longrightarrow r = 2$, in this case R is C_6 and we can add two independent vertices x, y different from the vertices of C_6 such that each of them is adjacent to exactly three vertices of C_6 and $N(u) \cap N(v) = \phi$. In such away $R * \{u, v\}$ is a 3-regular graph and u, v extension vertices. By Theorem 2 (b), $ext_{\mathcal{R}}(R) = 2$ a contradiction to our assumption. Hence $ext_{\mathcal{R}}(R) = 3$ and (2)holds.

If(iii) holds, that is the regularity degree of $R * \{u, v, w\}$ is $r + 1$. Then we must have each of u, v, w is adjacent to $\frac{n}{3}$ vertices in R and $N(u), N(v), Nw)$ are mutually disjoint. In this case n must be divisible of 3 (as R is regular), in this case $d(u) = d(v) = d(w) = \frac{n}{3}$ in $R * \{u, v, w\}$. Hence we must have $r = \frac{n}{3} - 1$ in R and (3) holds.

Conversely, Suppose that one of $(1), (2)$ or (3) hold. Suppose that (1) holds that is there are three vertices u, v, w such that each of them is adjacent to every vertex in R and $r = n - 3$. Then $d(u) = d(v) = d(w) = n$ in $R * \{u, v, w\}$ and every vertex in R has a degree $r + 3$ in $R * \{u, v, w\}$. As $r = n - 3$ in R, then every vertex in R has a degree $n - 3 + 3 = n$ in $R * \{u, v, w\}$. That is every vertex in $R * \{u, v, w\}$ has a degree n, thus $R * \{u, v, w\}$ is regular and u, v, w are extension vertices. Hence $ext_{\mathcal{R}}(R) \leq 3$.

Suppose that $ext_{\mathcal{R}}(R) = 1$, by Theorem 1, R is either trivial or a complete graph and there exists a vertex v_0 different from the vertices of R such that v_0 is adjacent to every vertex of R exactly once. If R is trivial then R has regularity degree 0 and $0 = n - 3 \Longrightarrow n = 3$ a contradiction to definition of trivial graph. If R is a complete graph then $r = n - 1$. Thus $n - 3 = n - 1$ a contradiction. Hence $ext_{\mathcal{R}}(R) \neq 1$.

Suppose that $ext_{\mathcal{R}}(R) = 2$, then by Theorem 2 there exist two vertices u, v different from the vertices of R and one of following hold:

(a) each of u, v is adjacent to every vertex in R exactly once and $r = n - 2$. or

(b) each of u, v is adjacent to $\frac{n}{2}$ vertices in R, n is even, $r = \frac{n}{2} - 1$ and $N(u) \cap N(v) = \phi$.

If(a) holds, then $r = n - 2$, thus $n - 2 = n - 3$ a contradiction.

If (b) holds, then $\frac{n}{2} - 1 = n - 3 \Rightarrow n = 4$ a contradiction to our assumption. Thus $ext_{\mathcal{R}}(R) \neq 2$. Hence $ext_{\mathcal{R}}(R) = 3$.

Suppose that (2) holds. That is $r = \frac{2n}{3} - 2$, n divisor 3 with $n \neq 6$ and there exist three vertices u, v, w different from the vertices of R such that every vertex in R is adjacent to two vertices from u, v, w and $d(u) = d(v) = d(w)$. Then $d(u) = d(v) = d(w) = \frac{2n}{3}$ in $R * \{u, v, w\}$. Then the degree of every vertex of R increase by 2 in the graph

$R * \{u, v, w\}$. But $r = \frac{2n}{3} - 2$ in R.Thus every vertices in R has a degree $\frac{2n}{3} - 2 + 2$ in $R * \{u, v, w\}$. That is every vertex in $R * \{u, v, w\}$ has a degree $\frac{2n}{3}$. Thus $R * \{u, v, w\}$ is regular, u, v, w are extension vertices. Hence $ext_{\mathcal{R}}(R) \leq 3$. If $ext_{\mathcal{R}}(R) = 1$, then by Theorem 1 R is either trivial or complete graph. If R is trivial then R has regularity degree 0 and $0 = \frac{2n}{3} - 2 \Rightarrow n = 3$ a contradiction. If R is complete graph then $r = n - 1$ implies $\frac{2n}{3} - 2 = n - 1$ a contradiction. Thus $ext_{\mathcal{R}}(R) \neq 1$.

If $ext_{\mathcal{R}}(R) = 2$, then by Theorem 2 , one of the conditions (a) or (b) above holds:

If(a) holds then $r = n - 2$ and $\frac{2n}{3} - 2 = n - 2$ a contradiction.

If (b) holds then $\frac{2n}{3} - 2 = \frac{n}{2} - 1 \Rightarrow n = 6$ a contradiction to our assumption $n \neq 6$.

Thus $ext_{\mathcal{R}}(R) \neq 2$.Hence $ext_{\mathcal{R}}(R) = 3$.

Suppose that (3) holds, that is there are three vertices u, v, w such that each of them is adjacent to $\frac{n}{3}$ vertices in R and $N(u), N(v), N(w)$ are mutually disjoint and $r = \frac{n}{3} - 1$. Then $d(u) = d(v) = d(w) = \frac{n}{3}$ in $R * \{u, v, w\}$ and every vertex in R has a degree $r + 1$ in $R * \{u, v, w\}$. As $r = \frac{n}{3} - 1$ in R , thus every vertex in R has a degree $\frac{n}{3} - 1 + 1 = \frac{n}{3}$ in $R * \{u, v, w\}$. That is every vertex in $R * \{u, v, w\}$ has a degree $\frac{n}{3}$. Thus $R * \{u, v, w\}$ is regular graph and u, v, w are extension vertices. Hence $ext_{\mathcal{R}}(R) \leq 3$. Suppose that $ext_{\mathcal{R}}(R) = 1$ by Theorem 1, R is either trivial or complete graph. If R is trivial then R has regularity degree 0 then $0 = \frac{n}{3} - 1 \Rightarrow n = 3$ a contradiction. If R is a complete graph then $r = n - 1$. Thus $n - 1 = \frac{n}{3} - 1$ a contradiction.

Suppose that $ext_{\mathcal{R}}(R) = 2$, by Theorem 2 one of the conditions (a) or (b) above holds.

If(a) holds, then $\frac{n}{3} - 1 = n - 2$ a contradiction.

If(b) holds, then $\frac{n}{2} - 1 = \frac{n}{3} - 1$ a contradiction, thus $ext_{\mathcal{R}}(R) \neq 2$. Hence $ext_{\mathcal{R}}(R) = 3$, and the proof is completed.

4 Extension of Digraphs

In this section, we introduced the concepts extension of digraph, extensible class of digraphs and the extensibility number of digraphs.

Attar [1] defined the extension of digraphs as follows:

Definition 6 [1]

Let D be a nontrivial digraph. The extension of D is a digraphs denoted by $D + S$ obtained from D by adding a nonempty set of independent vertices S such that every vertex in S is adjacent or adjacent by but not both every vertex in D.

In this work we defined the extension of digraphs as follows :

Definition 7

Let D be a nontrivial digraph. The extension of D is a simple digraph denoted by $D * S$ obtained from D by adding a nonempty set of independent vertices S different from the vertices of D such that every vertex in S is adjacent or adjacent by but not both at least one vertex in D. In such away S is called extension set of D. In particular, If S consists of single element v, then v is called extension vertex of D. Now, we define the extensible class of digraph.

Definition 8

Let $\mathcal{D}$ be the class of digraphs with certain property. Then $\mathcal{D}$ is called extensible class, if for every digraph $D \in \mathcal{D}$, there exist an extension vertex v, such that $D * v \in \mathcal{D}$.

Here we introduce two propositions:

proposition 3

The class of connected digraphs is extensible class of digraphs with respect to connectedness.

proof

The proof is follows from Definition 7.

proposition 4

Each of the classes regular digraphs, Eulerian digraphs and Hamiltonian digraphs is not extensible class.

proof

The proof is similar to that in proposition 2.

The definition of extensibility number of digraph is analogous to that in Definition 5, only replace every graph G by a digraph D as following:

Definition 9

Let $\mathcal{D}$ be the class of digraphs with certain property, and $D \in \mathcal{D}$ be a nontrivial. The extensibility number of D with respect to $\mathcal{D}$ is the smallest positive integer m, if exists such that there exists an extension set of vertices of D with cardinality m in which the new digraph $D * S \in \mathcal{D}$. We write $m = ext_{\mathcal{D}}(D)$. If such a number dose not exist for D, then we say the corresponding extensibility number is ∞.

5   Extension of Regular Digraphs

In this section, we characterized the regular digraphs which have extensibility number equal to 2 or 3.

Remark 2

Let $\mathcal{R}$ be a class of regular digraphs and $D \in \mathcal{R}$. Then $ext_{\mathcal{R}}(D) \geq 2$.

Proof

The proof follows from Definition 7.

Theorem 4

Let $\mathcal{R}$ be the class of regular digraphs, $D \in \mathcal{R}$ be an r-regular digraph with even number of vertices n. Then $ext_{\mathcal{R}}(D) = 2$, if and only if $r = \frac{n}{2} - 1$, and there exist two independent vertices u, v different from the vertices of D such that each of them is adjacent to $\frac{n}{2}$ vertices and adjacent by the remaining $\frac{n}{2}$ vertices of D exactly once,

with $N^+(u) \cap N^+(v) = \phi$.

Proof

Let D be an r-regular digraph with even number of vertices n. Suppose that $ext_{\mathcal{R}}(D) = 2$. Then by Definition 9, there exist two extension vertices u, v such that $D * \{u, v\} \in \mathcal{R}$, and u, v are independent vertices. As $D * \{u, v\}$ is regular digraph then $id(u) = od(u) = id(v) = od(v)$ in $D * \{u, v\}$. By Definition 7, $D * \{u, v\}$ is simple digraph, then we must have $D * \{u, v\}$ is $(r + 1)$-regular digraph. That is $id(w) = od(w) = r + 1$, $\forall\ w \in D$ in the digraph $D * \{u, v\}$. In this case we must have u is adjacent to $\frac{n}{2}$ vertices in D and adjacent by the remaining $\frac{n}{2}$ vertices of D exactly once similarly for v. If u and v have a common neighbour in D, we get a contradiction to the regularity of $D * \{u, v\}$. Since the regularity degree of every vertex in D is $(r + 1)$ in $D * \{u, v\}$ and the degree of each of u, v is $\frac{n}{2}$. Then we must have $\frac{n}{2} = r + 1$ in $D * \{u, v\}$. This implies $r = \frac{n}{2} - 1$ in D.

Conversely, Suppose that $r = \frac{n}{2} - 1$ and there exist u, v different from the vertices of D such that each of them is adjacent to $\frac{n}{2}$ vertices and adjacent by $\frac{n}{2}$ of D exactly once and $N^+(u) \cap N^+(v) = \phi$. Then $id(u) = od(u) = id(v) = od(v) = \frac{n}{2}$ in $D * \{u, v\}$ also $id(w) = od(w) = \frac{n}{2} - 1 + 1$, $\forall w \in D$, which implies $id(w) = od(w) = \frac{n}{2}$. Hence $D * \{u, v\}$ is regular digraph and u, v are extension vertices. Thus $ext_{\mathcal{R}}(D) \leq 2$. By the Remark 2, above $ext_{\mathcal{R}}(D) \neq 1$. Hence $ext_{\mathcal{R}}(D) = 2$.

Theorem 5

Let $\mathcal{R}$ be the class of regular digraphs, $D \in \mathcal{R}$ be an r-regular digraph with n vertices such that n divisible of 3. Then $ext_{\mathcal{R}}(D) = 3$, if and only if $r = \frac{n}{3} - 1$, and there exist three independent vertices u, v, w different from the vertices of D such that each of u, v, w is adjacent to $\frac{n}{3}$ vertices of D exactly once, and adjacent by $\frac{n}{3}$ vertices from the remaining vertices of D exactly once, with $N^+(u), N^+(v), N^+(w)$ are mutually disjoint and every vertex in D is adjacent to exactly one vertex from u, v, w.

Proof

Let D be an r-regular digraph with n vertices and n divisible of 3. Suppose that $ext_{\mathcal{R}}(D) = 3$. Then by Definition 9, there exist three extension vertices u, v, w such that $D*\{u, v, w\} \in \mathcal{R}$, and u, v, w are independent vertices. As $D*\{u, v, w\}$ is regular digraph then $id(u) = od(u) = id(v) = od(v) = id(w) = od(w) = \frac{n}{3}$ in $D*\{u, v, w\}$. By Definition 7, $D*\{u, v, w\}$ is simple digraph, then we must have $D*\{u, v, w\}$ is $(r+1)$-regular digraph. That is $id(z) = od(z) = r+1$, $\forall z \in D$ in the digraph $D*\{u, v, w\}$. In this case we must have u is adjacent to $\frac{n}{3}$ vertices in D exactly once and adjacent by $\frac{n}{3}$ vertices from the remaining vertices of D exactly once, and every vertex in D is adjacent to exactly one vertex from u, v, w. Similarly for v and w. If two or three from u, v, w have a common neighbour, we get a contradiction to the regularity of $D*\{u, v, w\}$. If h is a vertex in D which is adjacent to two or three vertices from the vertices u, v, w. Then we get a contradiction to $D*\{u, v, w\}$ is regular. Since the regularity degree of every vertex in D is $r+1$ in $D*\{u, v, w\}$ and the regularity degree of u, v, w is $\frac{n}{3}$. Then $\frac{n}{3} = r+1$ in $D*\{u, v, w\}$. This implies $r = \frac{n}{3} - 1$ in D.

Conversely, Suppose that there exist three vertices u, v, w different from the vertices of D such that each of them is adjacent to $\frac{n}{3}$ vertices exactly once and adjacent by $\frac{n}{3}$ vertices from the remaining vertices of D exactly once with $N^+(u), N^+(v), N^+(w)$ are mutually disjoint and every vertex in D is adjacent to exactly one vertex from u, v, w, $r = \frac{n}{3} - 1$. Then $id(u) = od(u) = id(v) = od(v) = id(w) = od(w) = \frac{n}{3}$ in $D*\{u, v, w\}$, also $id(z) = od(z) = \frac{n}{3} - 1 + 1$ $\forall z \in D$ in the digraph $D*\{u, v, w\}$, which implies $id(z) = od(z) = \frac{n}{3}$. Hence $D*\{u, v, w\}$ is regular digraph and u, v, w are extension vertices. Thus $ext_{\mathcal{R}}(D) \leq 3$.

By Remark 2, $ext_{\mathcal{R}}(D) \neq 1$. If $ext_{\mathcal{R}}(D) = 2$, by Theorem 4, there exist two vertices u, v different from the vertices of D such that each of them is adjacent to $\frac{n}{2}$ vertices and adjacent by the remaining $\frac{n}{2}$ vertices of D exactly once and u, v have no common neighbour with $r = \frac{n}{2} - 1$ in D. Thus $\frac{n}{2} - 1 = \frac{n}{3} - 1$ a contradiction, thus $ext_{\mathcal{R}}(D) \neq 2$. Hence $ext_{\mathcal{R}}(D) = 3$ and the proof is completed.

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